Exam-style practice: Paper 2

1 $y = ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}$

Maximum at $(-2, 8)$

$\Rightarrow \frac{b}{2a} = 2 \Rightarrow b = 4a \quad (1)$

and $c - \frac{b^2}{4a} = 8 \quad (2)$

Passes through $(-4, 4)$

$\Rightarrow 16a - 4b + c = 4 \quad (3)$

$(3) - (2)$: $16a - 4b + \frac{b^2}{4a} = -4$

Substitute expression for b from (1) gives

$16a - 16a + 4a = -4$

$\Rightarrow a = -1 \Rightarrow b = 4a = -4$

From (3) $\Rightarrow c = 4 + 4b - 16a$

$= 4 + 4(-4) - 16(-1) = 4$

So $a = -1,\ b = -4,\ c = 4$

2 a l_1 passes through $P(6, 4)$ and $Q(0, 28)$

Gradient $= m_1 = \frac{28-4}{0-6} = -4$

So $(y-4) = (-4)(x-6)$

$\Rightarrow y = -4x + 28$

b Let l_2 have gradient m_2

Since l_1 and l_2 are perpendicular,

$m_1 m_2 = -1 \Rightarrow m_2 = \frac{1}{4}$

So $(y-4) = \frac{1}{4}(x-6)$

$\Rightarrow y = \frac{1}{4}x + \frac{5}{2}$

c R is positioned where l_2 crosses the x-axis

$\frac{1}{4}x + \frac{5}{2} = 0 \Rightarrow x = -10$

So $R(-10, 0)$

d ΔPQR is a right-angled triangle.

Area $= \frac{1}{2} \times (\text{base}) \times (\text{height})$

Using Pythagoras' theorem,

$|PQ| = \sqrt{(6)^2 + (24)^2} = \sqrt{612} = 2\sqrt{153}$

$|RP| = \sqrt{(16)^2 + (4)^2} = \sqrt{272} = 2\sqrt{68}$

Area $= \frac{1}{2}\left(2\sqrt{153}\right)\left(2\sqrt{68}\right)$

$= 2\sqrt{10\,404} = 204$

So the area of triangle PQR is 204 units^2

3 $f(x) = e^{3x} - 1, \quad x \in \mathbb{R}$

$y = e^{3x} - 1$

$\Rightarrow y + 1 = e^{3x}$

$\Rightarrow 3x = \ln(y+1)$

$\Rightarrow x = \frac{1}{3}\ln(y+1)$

So $f^{-1}(x) = \frac{1}{3}\ln(x+1), \quad x > -1$

4 a The student did not apply the laws of logarithms correctly in moving from the first line to the second line:

$\log_a x + \log_a y = \log_a xy$

b $\log_4(x+3) + \log_4(x+4) = \frac{1}{2}$

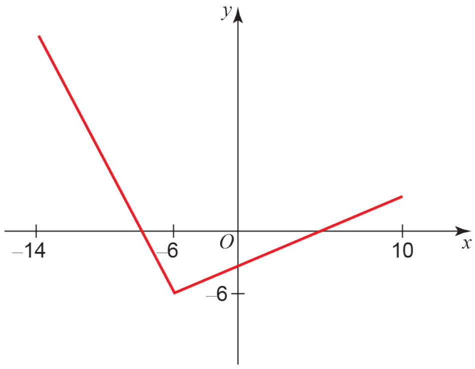
$\Rightarrow (x+3)(x+4) = 4^{\frac{1}{2}} = 2$

$\Rightarrow x^2 + 7x + 10 = (x+5)(x+2) = 0$

So $x = -2$

Note that $x = -5$ is not a solution since the function $\log_4 x$ is defined only on the domain $x > 0$, so $\log_4(x+4)$ is undefined when $x = -5$.

5 a



b $-6 \le y \le 18$

c Consider each section of the domain separately. There will be a solution in each section, because $18 > -3 > -6$ and $-6 < -3 < 2$.

First find the equation of the line when $-14 \leqslant x \leqslant -6$ and solve for $y = -3$

$$(y-(-6)) = \frac{-6-18}{-6-(-14)}(x-(-6))$$

$$\Rightarrow y+6 = -3(x+6)$$

$$\Rightarrow y = -3x-24$$

$$-3 = -3a-24 \Rightarrow a = -7$$

Now find the equation of the line when $-6 \leqslant x \leqslant 10$ and solve for $y = -3$

$$(y-(-6)) = \frac{2-(-6)}{10-(-6)}(x-(-6))$$

$$\Rightarrow y+6 = \frac{1}{2}(x+6)$$

$$\Rightarrow y = \frac{1}{2}x-3$$

$$-3 = \frac{1}{2}a-3 \Rightarrow a = 0$$

Solutions are $a = -7$, $a = 0$

6 a $f(x) = x^3 - kx^2 - 10x + k$

$(x+2)$ is a factor of $f(x)$ so $f(-2) = 0$

$$\Rightarrow f(-2) = (-2)^3 - k(-2)^2 - 10(-2) + k$$
$$= -8 - 4k + 20 + k = 0$$
$$\Rightarrow 3k = 12 \Rightarrow k = 4$$

b $x^3 - 4x^2 - 10x + 4 = 0$

First take out the known factor $(x + 2)$

$$\Rightarrow (x+2)(x^2-6x+2) = 0$$

So $x = -2$ or $x^2 - 6x + 2 = 0$

$$x^2 - 6x + 2 = 0$$

$$\Rightarrow (x-3)^2 - 9 + 2 = 0$$

$$\Rightarrow (x-3)^2 = y \Rightarrow$$

$$x = 3 \pm \sqrt{7}$$

Solutions are $x = -2$, $x = 3+\sqrt{7}$ and $x = 3-\sqrt{7}$

7 a Area $= \frac{1}{2}(x-3)(x-10)\sin 30° = 11$

$$\Rightarrow \frac{1}{2}(x-3)(x-10) \times \frac{1}{2} = 11$$

$$\Rightarrow (x-3)(x-10) = 44$$

$$\Rightarrow x^2 - 13x + 30 = 44$$

$$\Rightarrow x^2 - 13x - 14 = 0$$

b $x^2 - 13x - 14 = (x-14)(x+1) = 0$

So $x = 14$, $x = -1$, but $x > 3$ as the lengths of the sides of this triangle must be positive. So solution is $x = 14$.

8 a $x = 6\sin t + 5 \Rightarrow \sin t = \frac{x-5}{6}$

$$y = 6\cos t - 2 \Rightarrow \cos t = \frac{y+2}{6}$$

Since $\sin^2 t + \cos^2 t = 1$,

$$\left(\frac{x-5}{6}\right)^2 + \left(\frac{y+2}{6}\right)^2 = 1$$

$$\Rightarrow (x-5)^2 + (y+2)^2 = 36$$

So $h = -5$, $k = 2$, $c = 36$

8 b $c = 36 = (\text{radius})^2 \Rightarrow \text{radius} = 6$

t parameterises the circle and takes values

$-\frac{\pi}{3} \le t < \frac{3\pi}{4}$

So the angle that subtends the arc is

$\frac{3\pi}{4} - \left(-\frac{\pi}{3}\right) = \frac{13\pi}{12} = \theta$

So C is an arc of radius 6, and its length

is $\frac{\theta}{2\pi} \times (\text{circumference of circle radius 6})$

Length of $C = \frac{1}{2\pi} \times \frac{13\pi}{12} \times 2\pi \times 6 = \frac{13\pi}{2}$

9 a $\frac{4x^2 + 7x}{(x-2)(x+4)} = A + \frac{B}{x-2} + \frac{C}{x+4}$

So $4x^2 + 7x$

$= A(x-2)(x+4) + B(x+4) + C(x-2)$

Set $x = 2$: $30 = 6B \Rightarrow B = 5$

Set $x = -4$: $36 = -6C \Rightarrow C = -6$

Compare coefficients of $x^2 \Rightarrow A = 4$

So $A = 4,\ B = 5,\ C = -6$

b $\frac{4x^2 + 7x}{(x-2)(x+4)} = 4 + 5(x-2)^{-1} - 6(x+4)^{-1}$

So to find the expansion as far as the term in x^2, only need to find the expansions of $(x-2)^{-1}$ and $(x+4)^{-1}$ as far as the term in x^2

$$(x-2)^{-1} = -\frac{1}{2}\left(1 - \frac{x}{2}\right)^{-1}$$

$$= -\frac{1}{2}\left(1 + (-1)\left(-\frac{x}{2}\right) + \frac{(-1)(-2)}{2!}\left(-\frac{x}{2}\right)^2 + \ldots\right)$$

$$= -\frac{1}{2} - \frac{1}{4}x - \frac{1}{8}x^2 + \ldots$$

$$(x+4)^{-1} = \frac{1}{4}\left(1 + \frac{x}{4}\right)^{-1}$$

$$= \frac{1}{4}\left(1 + (-1)\left(\frac{x}{4}\right) + \frac{(-1)(-2)}{2!}\left(\frac{x}{4}\right)^2 + \ldots\right)$$

$$= \frac{1}{4} - \frac{1}{16}x + \frac{1}{64}x^2 + \ldots$$

So $\frac{4x^2 + 7x}{(x-2)(x+4)}$

$$= 4 + 5\left(-\frac{1}{2} - \frac{1}{4}x - \frac{1}{8}x^2 + \ldots\right) - 6\left(= \frac{1}{4} - \frac{1}{16}x + \frac{1}{64}x^2 + ..\right)$$

$$= \left(4 - \frac{5}{2} - \frac{3}{2}\right) + \left(-\frac{5}{4} + \frac{3}{8}\right)x + \left(-\frac{5}{8} - \frac{3}{32}\right)x^2 + \ldots$$

$$= -\frac{7}{8}x - \frac{23}{32}x^2 \ldots$$

10 $\overrightarrow{MN} = \overrightarrow{MB} + \overrightarrow{BN} = \frac{1}{2}\overrightarrow{OB} + \frac{1}{2}\overrightarrow{BA}$

So $\overrightarrow{MN} = \frac{1}{2}\mathbf{b} + \frac{1}{2}(-\mathbf{b} + \mathbf{a}) = \frac{1}{2}\mathbf{a}$

Therefore $\overrightarrow{OA}$ and $\overrightarrow{MN}$ are parallel

and $\overrightarrow{MN} = \frac{1}{2}\overrightarrow{OA}$ as required

11 a At $x = \frac{\pi}{2}$,

$$y = \left(\frac{\pi}{2}\right)^2\left(\sin\frac{\pi}{2} + \cos\frac{\pi}{2}\right)$$
$$= 2.46740 \text{ (5 d.p.)}$$

b $\int_a^b y\,dx \approx \frac{h}{2}(y_0 + 2(y_1 + y_2 + \ldots + y_{n-1}) + y_n)$

So $\int_0^{\frac{3\pi}{4}} y\,dx$

$$\approx \frac{h}{2}(y_0 + 2(y_1 + y_2 + \ldots + y_5) + y_6)$$
$$= \frac{1}{2}\left(\frac{\frac{3\pi}{4} - 0}{6}\right)2(0.20149 + 0.87239 + 1.81340 + 2.46740 + 2.08648)$$
$$= 2.922 \quad \text{(3d.p.)}$$

c Use integration by parts twice. First let

$u = x^2, \frac{dv}{dx} = \sin x + \cos x$

$\Rightarrow v = \sin x - \cos x$

$$\int_0^{\frac{3\pi}{4}} x^2(\sin x + \cos x)$$
$$= \left[x^2(\sin x - \cos x)\right]_0^{\frac{3\pi}{4}} - 2\int_0^{\frac{3\pi}{4}} x(\sin x - \cos x)dx$$

Use integration by parts again, letting

$u = x, \frac{dv}{dx} = \sin x - \cos x$

$\Rightarrow v = \sin x + \cos x$

This gives

$$\int_0^{\frac{3\pi}{4}} x^2(\sin x + \cos x)$$
$$= \left(\frac{3\pi}{4}\right)^2\left(\sqrt{2}\right) - 2\left\{-\left[x(\sin x + \cos x)\right]_0^{\frac{3\pi}{4}} + \int_0^{\frac{3\pi}{4}}(\sin x + \cos x)dx\right\}$$
$$= \left(\frac{9\pi^2\sqrt{2}}{16}\right) - 2\left\{0 + \left[\sin x - \cos x\right]_0^{\frac{3\pi}{4}}\right\}$$
$$= \left(\frac{9\pi^2\sqrt{2}}{16}\right) - 2\left(\sqrt{2} + 1\right)$$
$$= 3.023 \quad \text{(3d.p.)}$$

d $\frac{3.023 - 2.922}{3.023} \times 100 = 3.3\% \quad \text{(1d.p.)}$

12 a $u_n = a + (n-1)d$

$a = 1000,\ d = 150$

So $u_{18} = 1000 + (18-1)(150) = 3550$

In the 18th year Ruth saves £3550

b $S_n = \frac{n}{2}(2a + (n-1)d)$

$S_{18} = \frac{18}{2}(2(1000) + (18-1)(150)) = 40950$

So in 18 years, the total amount that Ruth will have saved is £40 950

12 c The sequence is now geometric with $a = 1000,\ r = 1.1$

$$S_{18} = \frac{1000(1-(1.1)^{18})}{1-1.1} = 45599.17313$$

So after 18 years, Ruth will have saved £45 599.17 (2 d.p.) under this new scheme

13 a $R\cos(x-\alpha) = R\cos x\cos\alpha + R\sin x\sin\alpha$

$R\cos(x-\alpha) = 0.09\cos x + 0.4\sin x$

$R\cos\alpha = 0.09,\ R\sin\alpha = 0.4$

$\Rightarrow R^2 = (0.09)^2 + (0.4)^2$

(as $\sin^2\alpha + \cos^2\alpha \equiv 1$)

So $R = \sqrt{(0.09)^2 + (0.4)^2} = 0.41 \quad (R > 0)$

$$\frac{R\sin\alpha}{R\cos\alpha} = \tan\alpha = \frac{0.4}{0.09} = \frac{40}{9}$$

$$\Rightarrow \alpha = \tan^{-1}\left(\frac{40}{9}\right) = 1.3495 \text{ rad (4 d.p.)}$$

So $R = 0.41,\ \alpha = 1.3495$

b Use part **a** to write equation as

$$h = \frac{16.4}{0.41\cos\left(\frac{t}{2}-\alpha\right)}$$

$$\Rightarrow h = \frac{40}{\cos\left(\frac{t}{2}-\alpha\right)}$$

So the minimum value of h occurs when

$$\frac{t}{2} - \alpha = 0 \Rightarrow t = 2\alpha$$

$\Rightarrow t = 2\times 1.3495 = 2.70$ seconds (2 d.p.)

$$h = \frac{40}{\cos 0} = 40\text{ cm}$$

c $$h = \frac{40}{\cos\left(\frac{t}{2}-\alpha\right)} = 100$$

$$\Rightarrow \cos\left(\frac{t}{2}-\alpha\right) = \frac{2}{5}$$

This has two solutions in the interval

$$-1.3495 \leqslant \frac{t}{2} - 1.3496 \leqslant 1.3505$$

$$\frac{t}{2} - \alpha = 1.1593, -1.1593$$

$t = 2\times(1.1593 + 1.3495) = 5.02$ seconds

$t = 2\times(-1.1593 + 1.3495) = 0.38$ seconds

14 a $\mathrm{h}(t) = -10e^{-0.3(t-6.4)} - 10e^{0.8(t-6.4)} + 70$

$\mathrm{h}'(t) = -10(-0.3)e^{-0.3(t-6.4)} - 10(0.8)e^{0.8(t-6.4)}$

$\Rightarrow \mathrm{h}'(t) = 3e^{-0.3(t-6.4)} - 8e^{0.8(t-6.4)}$

b From part **a**, when $\mathrm{h}'(t) = 0$

$$\frac{3}{8}e^{-0.3(t-6.4)} = e^{0.8(t-6.4)}$$

$$\Rightarrow \ln\left(\frac{3}{8}e^{-0.3(t-6.4)}\right) = 0.8(t-6.4)$$

$$\Rightarrow \frac{5}{4}\ln\left(\frac{3}{8}e^{-0.3(t-6.4)}\right) = t - 6.4$$

$$\Rightarrow t = \frac{5}{4}\ln\left(\frac{3}{8}e^{-0.3(t-6.4)}\right) + 6.4$$

c $$t_{n+1} = \tfrac{5}{4}\ln\left(\frac{3e^{-0.3(t_n-6.4)}}{8}\right) + 6.4$$

$$t_1 = \tfrac{5}{4}\ln\left(\frac{3e^{-0.3(5-6.4)}}{8}\right) + 6.4 = 5.6990$$

$$t_2 = \tfrac{5}{4}\ln\left(\frac{3e^{-0.3(t_1-6.4)}}{8}\right) + 6.4 = 5.4369$$

$$t_3 = \tfrac{5}{4}\ln\left(\frac{3e^{-0.3(t_2-6.4)}}{8}\right) + 6.4 = 5.5351$$

$$t_4 = \tfrac{5}{4}\ln\left(\frac{3e^{-0.3(t_3-6.4)}}{8}\right) + 6.4 = 5.4983$$

All answers are to 4 decimal places.

14 d $h'(5.5075) = 0.000360$ (6 d.p.)
$h'(5.5085) = -0.000702$ (6 d.p.)
The sign change implies slope change, which implies a turning point at $t = 5.508$ (3 d.p.)