KING EDWARD VI
HANDSWORTH GRAMMAR
SCHOOL FOR BOYS

KING EDWARD VI
ACADEMY TRUST
BIRMINGHAM

Year 12
Applied Mathematics
P2 12 Vectors Booklet

HGS Maths

Dr Frost Course

Name: ______________________________

Class: ______________________________

Contents

Prior knowledge check

Prior knowledge check

1 Given that $\mathbf{p} = 3\mathbf{i} - \mathbf{j}$ and $\mathbf{q} = -\mathbf{i} + 2\mathbf{j}$, calculate:

a $2\mathbf{p} + \mathbf{q}$ **b** $-3\mathbf{p} + 4\mathbf{q}$

← Year 1, Section 11.2

2 Given that $\mathbf{a} = 5\mathbf{i} - 3\mathbf{j}$, work out:

a the magnitude of **a**

b the unit vector that is parallel to **a**.

← Year 1, Section 11.3

3 M is the midpoint of the line segment AB.

Given that $\overrightarrow{AB} = 4\mathbf{i} + \mathbf{j}$,

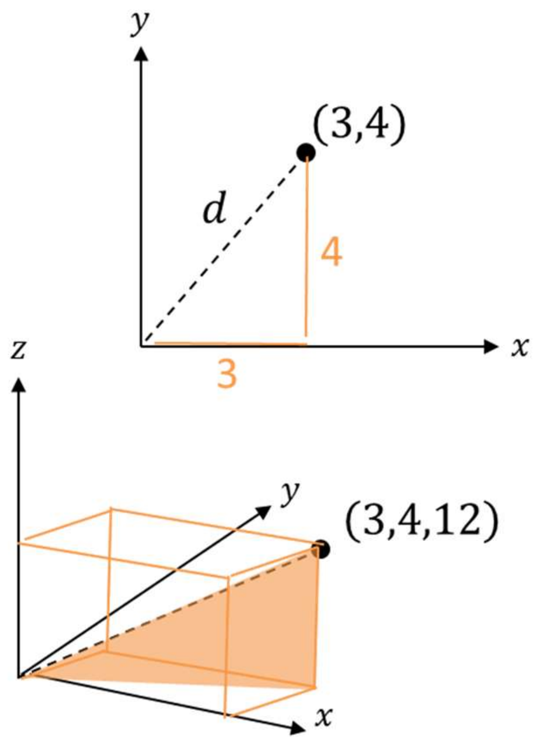
a find $\overrightarrow{BM}$ in terms of **i** and **j**.

The point P lies on AB such that $AP : PB = 3 : 1$.

b Find $\overrightarrow{AP}$ in terms of **i** and **j**.

← Year 1, Section 11.5

12.1) 3D coordinates



In 2D, how did we find the distance from a point to the origin?

Using Pythagoras:

$$\boldsymbol{d = \sqrt{3^2 + 4^2} = 5}$$

How about in 3D then?

You may be familiar with this method from GCSE.

Using Pythagoras on the base of the cuboid:

$$\boldsymbol{\sqrt{3^2 + 4^2} = 5}$$

Then using the highlighted triangle:

$$\boldsymbol{\sqrt{5^2 + 12^2} = 13}$$

We could have similarly done this is one go using:

$$\boldsymbol{\sqrt{3^2 + 4^2 + 12^2} = 13}$$

From Year 1 you will be familiar with the magnitude $|\boldsymbol{a}|$ of a vector $\boldsymbol{a}$ being its length.
We can see from above that this nicely extends to 3D:

The magnitude of a vector $\boldsymbol{a} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$:

$$|\boldsymbol{a}| = \sqrt{x^2 + y^2 + z^2}$$

And the distance of (x, y, z) from the origin is $\sqrt{x^2 + y^2 + z^2}$

Notes

Worked Example

505a: **Determine a 2D or 3D position vector given another position vector and the vector between them.**

Given that:

$$\overrightarrow{AB} = \begin{pmatrix} 9 \\ -5 \end{pmatrix}$$
$$\overrightarrow{OB} = \begin{pmatrix} 8 \\ -7 \end{pmatrix}$$

Find the position vector of A.

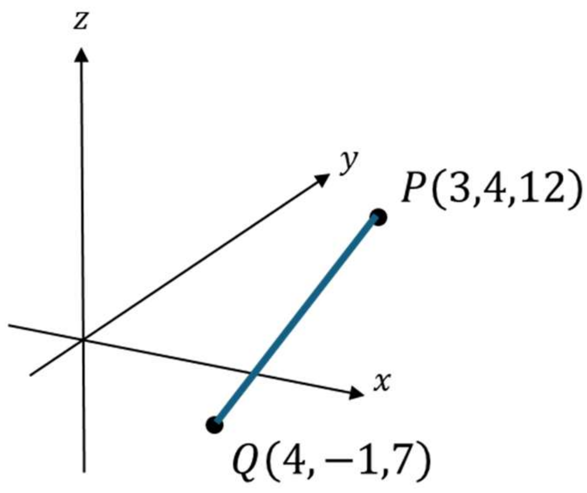
$$A = \begin{pmatrix} \square \\ \square \end{pmatrix}$$

Worked Example

Find the distance from the origin to the point with coordinates $(6, 8, 24)$

Find the distance from the origin to the point with coordinates $(-6, 0, -2)$

Distance between two 3D points



How do we find the distance between P and Q?

It's just the magnitude/length of the vector between them.

i.e.

$$|\overrightarrow{PQ}| = \left|\begin{pmatrix} 1 \\ -5 \\ -5 \end{pmatrix}\right|$$

$$= \sqrt{1^2 + (-5)^2 + (-5)^2} = \sqrt{51}$$

✏ The distance between two points is:

$$d = \sqrt{(dx)^2 + (dy)^2 + (dz)^2}$$

dx means "change in

Quickfire Questions:

Distance of $(4,0,-2)$ from the origin:

$$\sqrt{4^2 + 0^2 + (-2)^2} = \sqrt{20}$$

$$\left|\begin{pmatrix} 5 \\ 4 \\ -1 \end{pmatrix}\right| = \sqrt{5^2 + 4^2 + (-1)^2} = \sqrt{42}$$

Distance between $(0,4,3)$ and $(5,2,3)$.

$$d = \sqrt{5^2 + (-2)^2 + 0^2} = \sqrt{29}$$

Distance between $(1,1,1)$ and $(2,1,0)$.

$$d = \sqrt{1^2 + 0^2 + 1^2} = \sqrt{2}$$

Distance between $(-5,2,0)$ and $(-2,-3,-3)$.

$$d = \sqrt{3^2 + 5^2 + 3^2} = \sqrt{43}$$

Tip: Because we're squaring, it doesn't matter whether the change is negative or

Worked Example

Find the distance between the points:

a) $A(1, 3, 5)$ and $B(-6, 0, -4)$ $\qquad$ $b)$ $C(-1, 0, 1)$ and $D(0, 0, -3)$

Worked Example

The coordinates of A and B are $(3,5,-2)$ and $(3,k,-1)$ respectively. Given that the distance from A to B is $\sqrt{2}$ units, find the possible values of k.

12.2) Vectors in 3D

In 2D you were previously introduced to $\boldsymbol{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\boldsymbol{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ as unit vectors in each of the x and y directions.

It meant for example that $\begin{pmatrix} 8 \\ -2 \end{pmatrix}$ could be written as $8\boldsymbol{i} - 2\boldsymbol{j}$ since $8\begin{pmatrix} 1 \\ 0 \end{pmatrix} - 2\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 8 \\ -2 \end{pmatrix}$

Unsurprisingly, in 3D:

$$\boldsymbol{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \boldsymbol{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \boldsymbol{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Notes

Worked Example

Consider the points $A(-1, -5, 2)$ and $B(-7, 3, 0)$.

a) Find the position vectors of A and B in ijk notation

b) Find the vector $\overrightarrow{AB}$ as a column vector.

Worked Example

The vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are given by:

$$\boldsymbol{a} = \begin{pmatrix} 3 \\ -2 \\ -5 \end{pmatrix} \text{ and } \boldsymbol{b} = \begin{pmatrix} 2 \\ 0 \\ -4 \end{pmatrix}$$

a) Find:

i) $\boldsymbol{a} + 3\boldsymbol{b}$

ii) $4\boldsymbol{a} - 5\boldsymbol{b}$

b) State whether these vectors are parallel to $-4\boldsymbol{i} + 16\boldsymbol{j}$

Worked Example

Find the magnitude of the vector $\begin{pmatrix} 6 \\ 8 \\ 24 \end{pmatrix}$

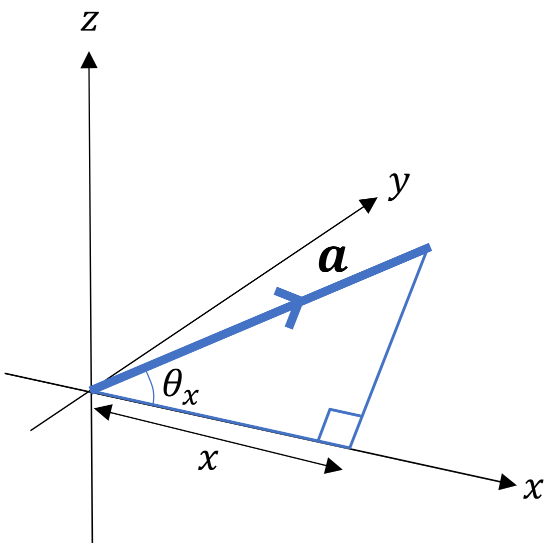
Worked Example

Find the magnitude of the vector

$$\boldsymbol{a} = 6\boldsymbol{i} - 8\boldsymbol{j} + 24\boldsymbol{k}$$

and hence find $\hat{\boldsymbol{a}}$, the unit vector in the direction of $\boldsymbol{a}$.

Angles between vectors and an axis



How could you work out the angle between a vector and the x-axis?

Just form a right-angle triangle!

The angle between $\boldsymbol{a} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and the x-axis is:

$$\cos\theta_x = \frac{x}{|\boldsymbol{a}|}$$

and similarly for the y and z axes.

Worked Example

Find the angles that the vector

$$\boldsymbol{a} = \boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k}$$

makes with each of the positive coordinate axes. Give your answers to 1 decimal place.

Worked Example

The points A and B have position vectors
$\boldsymbol{i} + 5\boldsymbol{j} + 3\boldsymbol{k}$ and $-2\boldsymbol{i} + 4\boldsymbol{j} + 8\boldsymbol{k}$ relative to a fixed origin, O.
Show that ΔOAB is isosceles.

Worked Example

$$\boldsymbol{a} = 3\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k} \text{ and } \boldsymbol{b} = \boldsymbol{i} + 3\boldsymbol{j} + 5\boldsymbol{k}$$

By considering the angles that $\boldsymbol{a}$ and $\boldsymbol{b}$ make with the x-axis, determine the area of OAB where $\overrightarrow{OA} = \boldsymbol{a}$ and $\overrightarrow{OB} = \boldsymbol{b}$.

Worked Example

A triangle PQR is such that
$\overrightarrow{PQ} = -2\boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k}$ and $\overrightarrow{QR} = 4\boldsymbol{i} - 3\boldsymbol{j} - 2\boldsymbol{k}$
Find $< PQR$ to 1 decimal place

12.3) Solving geometric problems

Worked Example

A, B, C and D are the points $(3, -4, -9)$, $(1, -7, -3)$, $(1, 0, -15)$ and $(7, 9, -33)$ respectively.

a) Find $\overrightarrow{AB}$ and $\overrightarrow{DC}$, giving your answers in the form $p\boldsymbol{i} + q\boldsymbol{j} + r\boldsymbol{k}$.

b) Show that the lines AB and DC are parallel and that $\overrightarrow{DC} = 3\overrightarrow{AB}$.

c) Hence describe the quadrilateral $ABCD$.

Worked Example

P, Q and R are the points $(9,3,-4)$, $(-5,5,5)$ and $(0,2,-8)$ respectively.
Find the coordinates of the point S so that $PQRS$ forms a parallelogram.

Comparing Coefficients

There are many contexts in maths where we can 'compare coefficients', e.g.

$3x^2 + 5x \equiv A(x^2 + 1) + Bx + C$
Comparing x^2 terms: $3 = A$

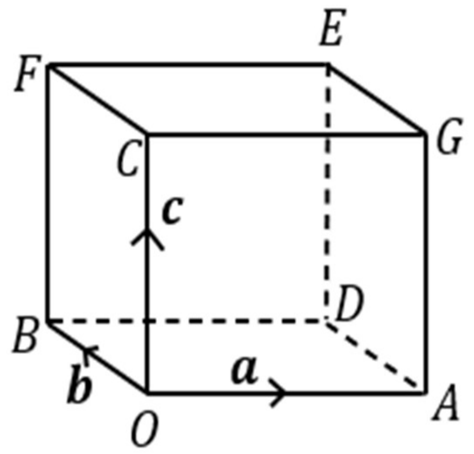
We can do the same with vectors:

Given that
$3\boldsymbol{i} + (p+2)\boldsymbol{j} + 120\boldsymbol{k} = p\boldsymbol{i} - q\boldsymbol{j} + 4pqr\boldsymbol{k}$,
find the values of p, q and r.

Comparing i: $3 = p$
Comparing j: $p + 2 = -q \quad \therefore q = -5$
Comparing k: $120 = 4pqr \quad \therefore \ r = -2$

Worked Example

The diagram shows a cuboid whose vertices are O, A, B, C, D, E, F and G.
Vectors a, b and c are the position vectors of the vertices A, B and C respectively.
Prove that the diagonals OE and AF bisect each other.



12.4) Application to mechanics

Out of displacement, speed, acceleration, force, mass and time, all but mass and time are vectors. Clearly these can act in 3D space.

	Vector		Scalar	
Force	$\begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix} N$	➡	$\sqrt{3^2 + 4^2 + (-1)^2}$ $= 5.10\ N$	
Acceleration	$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} ms^{-2}$	➡	$1.41\ ms^{-2}$	
Displacement	$\begin{pmatrix} 12 \\ 3 \\ 4 \end{pmatrix} m$	➡	$13\ m$	**Distance**
Velocity	$\begin{pmatrix} 0 \\ 4 \\ 3 \end{pmatrix} ms^{-1}$	➡	$5\ m$	**Speed**

Notes

Worked Example

Convert these vectors to scalar form:

- A force of $\begin{pmatrix} 1 \\ -3 \\ 4 \end{pmatrix} N$

- An acceleration of $\begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} ms^{-2}$

- A displacement of $\begin{pmatrix} -6 \\ 8 \\ -24 \end{pmatrix} m$

- A velocity of $\begin{pmatrix} 8 \\ -6 \\ 0 \end{pmatrix} ms^{-1}$

Worked Example

A particle of mass 0.25 kg is acted on by three forces.

$$F_1 = (\boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k})\ N$$
$$F_2 = (2\boldsymbol{i} - 4\boldsymbol{k})\ N$$
$$F_3 = (-5\boldsymbol{i} + 3\boldsymbol{j} + 4\boldsymbol{k})\ N$$

a) Find the resultant force R acting on the particle.
b) Find the acceleration of the particle, giving your answer in the form $(p\boldsymbol{i} + q\boldsymbol{j} + r\boldsymbol{k})$ ms^{-2}.
c) Find the magnitude of the acceleration.

Given that the particle starts at rest,
d) Find the distance travelled by the particle in the first 3 seconds of its motion.

7.

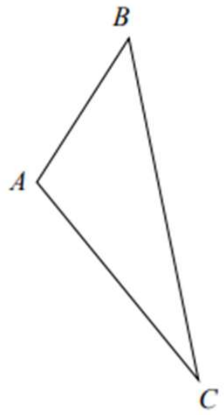


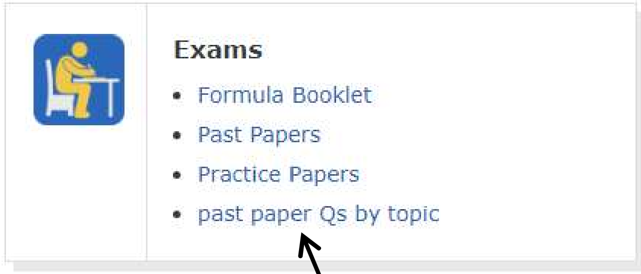
Figure 2

Figure 2 shows a sketch of a triangle ABC.

Given $\overrightarrow{AB} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$ and $\overrightarrow{BC} = \mathbf{i} - 9\mathbf{j} + 3\mathbf{k}$,

show that $\angle BAC = 105.9°$ to one decimal place.

(5)



Past paper practice by topic. Both new and old specification can be found via this link on hgsmaths.com

Attempts $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} = 2\mathbf{i}+3\mathbf{j}+\mathbf{k}+\mathbf{i}-9\mathbf{j}+3\mathbf{k} = 3\mathbf{i}-6\mathbf{j}+4\mathbf{k}$	M1	3.1a
Attempts to find any one length using 3-d Pythagoras	M1	2.1
Finds all of $\lvert AB\rvert = \sqrt{14}$, $\lvert AC\rvert = \sqrt{61}$, $\lvert BC\rvert = \sqrt{91}$	A1ft	1.1b
$\cos BAC = \frac{14+61-91}{2\sqrt{14}\sqrt{61}}$	M1	2.1
angle $BAC = 105.9°$ *	A1*	1.1b
	(5)	

(5 marks)

Summary of Key Points

Summary of key points

1 The distance from the origin to the point (x, y, z) is $\sqrt{x^2 + y^2 + z^2}$

2 The distance between the points (x_1, y_1, z_1) and (x_2, y_2, z_2) is $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$

3 The unit vectors along the x-, y- and z-axes are denoted by **i**, **j** and **k** respectively.

$\mathbf{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $\mathbf{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ $\mathbf{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

Any 3D vector can be written in column form as $p\mathbf{i} + q\mathbf{j} + r\mathbf{k} = \begin{pmatrix} p \\ q \\ r \end{pmatrix}$

4 If the vector $\mathbf{a} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ makes an angle θ_x with the positive x-axis then $\cos\theta_x = \frac{x}{|\mathbf{a}|}$ and similarly for the angles θ_y and θ_z.

5 If **a**, **b** and **c** are vectors in three dimensions which do not all lie in the same plane then you can compare their coefficients on both sides of an equation.