KING EDWARD VI HANDSWORTH GRAMMAR SCHOOL FOR BOYS

KING EDWARD VI ACADEMY TRUST BIRMINGHAM

Year 13
Applied Mathematics
P2 7 Trigonometry and modelling Booklet

HGS Maths

Dr Frost Course

Name: ______________________________

Class: ______________________________

Contents

Prior knowledge check

Prior knowledge check

1 Find the exact values of:

a $\sin 45°$ **b** $\cos \frac{\pi}{6}$ **c** $\tan \frac{\pi}{3}$ ← **Section 5.4**

2 Solve the following equations in the interval $0 \leqslant x < 360°$.

a $\sin(x + 50°) = -0.9$ **b** $\cos(2x - 30°) = \frac{1}{2}$

c $2\sin^2 x - \sin x - 3 = 0$ ← **Year 1, Chapter 10**

3 Prove the following:

a $\cos x + \sin x \tan x \equiv \sec x$ **b** $\cot x \sec x \sin x \equiv 1$

c $\dfrac{\cos^2 x + \sin^2 x}{1 + \cot^2 x} \equiv \sin^2 x$ ← **Section 6.4**

7.1) Addition formulae

Addition Formulae allow us to deal with a **sum or difference of angles**.

$$\sin(A+B) = \sin A\cos B + \cos A\sin B$$
$$\sin(A-B) = \sin A\cos B - \cos A\sin B$$

$$\cos(A+B) = \cos A\cos B - \sin A\sin B$$
$$\cos(A-B) = \cos A\cos B + \sin A\sin B$$

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A\tan B}$$

How to memorise:

First notice that for all of these the first thing on the RHS is the same as the first thing on the LHS!

- For sin, the operator in the middle is the same as on the LHS.
- For cos, it's the opposite.
- For tan, it's the same in the numerator, opposite in the denominator.

- For sin, we mix sin and cos.
- For cos, we keep the cos's and sin's together.

Do I need to memorise these?
They're all technically in the formula booklet, but you REALLY want to eventually memorise these (particularly the sin and cos ones).

Why is $\sin(A+B)$ not just $\sin(A)+\sin(B)$?

Because sin is a <u>function</u>, not a quantity that can be expanded out like this. It's a bit like how $(a+b)^2 \not\equiv a^2+b^2$.
We can easily disprove it with a counterexample.

Notes

Worked Example

590m: Use the compound angle formulae leading to $\tan x = a$

Given that

$$\sin(2x + 45^\circ) = 2\cos(2x - 30^\circ)$$

show that

$$\tan 2x = a$$

where a is a constant to be found.

T.169 7A: Qs 7-8, P.49 7.1 Qs 2

Worked Example

Express the following as a single sine, cosine or tangent, and evaluate:

a) $\sin 30° \cos 60° + \cos 30° \sin 60°$

b) $\cos 20° \cos 25° - \sin 20° \sin 25°$

$c)$ $\dfrac{\tan \frac{\pi}{18} + \tan \frac{\pi}{9}}{1 - \tan \frac{\pi}{18} \tan \frac{\pi}{9}}$

T.169 7A: Qs 9

Worked Example

Write in the form $\sin(x \pm \theta)$ or $\cos(x \pm \theta)$ where $0 < \theta < \frac{\pi}{2}$:

$\frac{1}{2}(\sqrt{3}\sin x + \cos x)$

$\frac{1}{2}(\sqrt{3}\cos x - \sin x)$

Worked Example

Given that $\tan(x + \frac{\pi}{6}) = \frac{1}{2}$ evaluate $\tan x$

Given that $\tan(x - \frac{\pi}{3}) = \frac{1}{2}$ evaluate $\tan x$

T.169 7A: Qs 10+, P.49 7.1 Qs 3+

7.2) Using the angle addition formulae

Notes

Worked Example

Using the trigonometric angle addition formulae find:

$\sin 75°$

$\tan 75°$

Worked Example

Using the trigonometric angle addition formulae find:

$\sin 15°$

$\tan 15°$

T.172 7B: Qs 1-2, P.49 7.3 Qs 1-2

Worked Example

Given that: $\sin A = \frac{8}{17}$ and $0° < A < 90°$, and $\cos B = -\frac{4}{5}$, B is obtuse, find the value of $\cos(A + B)$

Worked Example

Given that:

$\sin A = \frac{8}{17}$ and $0° < A < 90°$, and $\cos B = -\frac{4}{5}$, B is obtuse, find the value of $\tan(A - B)$

Worked Example

Given that: $\sin A = \frac{8}{17}$ and $0° < A < 90°$, and $\cos B = -\frac{4}{5}$, B is obtuse, find the value of $\sec(A - B)$

T.172 7B: Qs 3+, P.49 7.3 Qs 3+

Worked Example

Given that $2\cos(x-40)° = \sin(x-50)°$,show that $\tan x = 3\tan 50°$

T.172 7B: Qs 3+, P.49 7.3 Qs 3+

7.3) Double-angle formulae

Double-angle formula allow you to halve the angle within a trig function. **NOT IN FORMULAE BOOKLET**

$$\sin(2A) \equiv 2\sin A\cos A$$

$$\cos(2A) \equiv \cos^2 A - \sin^2 A$$

$$\equiv 2\cos^2 A - 1$$

$$\equiv 1 - 2\sin^2 A$$

This first form is relatively rare.

Fro Tip: The way I remember what way round these go is that the cos on the RHS is 'attracted' to the cos on the LHS, whereas the sin is pushed away.

$$\tan(2A) = \frac{2\tan A}{1 - \tan^2 A}$$

These are all easily derivable by just setting $A = B$ in the compound angle formulae. e.g.

$$\sin(2A) = \sin(A + A)$$

$$= \sin A\cos A + \cos A\sin A$$

$$= 2\sin A\cos A$$

TASK: derive other two

Notes

Worked Example

Use the double-angle formulae to write as a single trigonometric ratio:

a) $\cos^2 50° - \sin^2 50°$ b) $2\cos^2 \frac{2\pi}{9} - 1$ c) $1 - 2\sin^2 30°$

Worked Example

Use the double-angle formulae to write as a single trigonometric ratio:

a) $\cos^2 2x - \sin^2 2x$ b) $4\cos^2 3x - 2$ c) $3 - 6\sin^2 4x$

Worked Example

Use the double-angle formulae to write as a single trigonometric ratio:

a) $2\sin 45° \cos 45°$ b) $4\sin\frac{\pi}{12}\cos\frac{\pi}{12}$ c) $7\sin 5x \cos 5x$

Worked Example

Use the double-angle formulae to write as a single trigonometric ratio:

a) $\frac{2\tan 30°}{1-\text{ta}\ \ ^{2}\,30°}$ $\quad b)$ $\frac{2\tan\frac{\pi}{12}}{1-\tan^2\frac{\pi}{12}}$ c) $\frac{4\tan 6}{1-\tan^2 6x}$

Worked Example

Use the double-angle formulae to write as a single trigonometric ratio:

a) $\dfrac{8\sin 2\ \ .5°}{\sec 22.5°}$ b) $\dfrac{6\cos\frac{\pi}{4}}{\operatorname{cosec}\frac{\pi}{4}}$

T.175 7C: Qs 4-8, P.50 7.3 Qs 1-5

Worked Example

Given that $x = 2\sin\theta$ and $y = 4 - 3cos2\theta$, eliminate θ and express y in terms of x.

Worked Example

Given that $\cos x = \frac{5}{8}$ and x is acute, find the exact value of

(a) $\sin 2x$ (b) $\tan 2x$

Worked Example

Using the double-angle formulae, evaluate:

a) $\left(\sin\frac{\pi}{3}+\cos\frac{\pi}{3}\right)^2$ b) $\left(\sin\frac{\pi}{4}-\cos\frac{\pi}{4}\right)^2$

Worked Example

Given that $0 < \theta < \pi$, find the value of $\tan\frac{\theta}{2}$ when $\tan\theta = -\frac{3}{4}$

T.175 7C: Qs 9+, P.50 7.3 Qs 6+

7.4) Solving trigonometric equations

Notes

Worked Example

Solve in the interval $0 \leq x \leq 360°$: $8\sin(\theta + 60°) = 4\sqrt{2}\cos\theta$

Worked Example

Solve in the interval $0 \leq x \leq 360°$: $8\cos(\theta - 60°) = 4\sqrt{2}\sin\theta$

T.180 7D: Qs 1-4, P.51 7.4 Qs 1-5

Worked Example

Solve in the interval $0 \leq x \leq 360°$: $3\cos 2x + \cos x + 2 = 0$

Worked Example

Solve in the interval $0 \leq x \leq 360°$: $3\cos 2x - \sin x - 2 = 0$

Worked Example

Solve in the interval $0 \leq x \leq 360°$: $4\sin 2x - 5\cos x = 0$

Worked Example

Solve in the interval $0 \leq y \leq 2\pi$: $3\tan 2y \tan y = 2$

T.180 7D: Qs 5-10, P.51 7.4 Qs 6

Worked Example

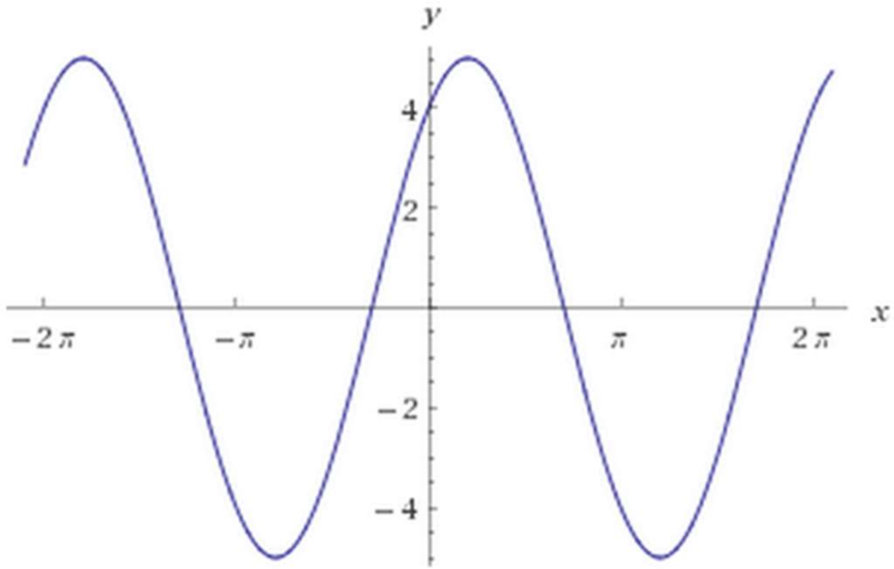
a) Show that $\cos(3A) = 4\cos^3 A - 3\cos A$.

b) Hence or otherwise, solve, for $0 < \theta < 2\pi$, the equation $12\cos\theta - 16\cos^3\theta - 2\sqrt{3} = 0$

T.180 7D: Qs 11+, P.51 7.4 Qs 7+

7.5) Simplifying $a\cos x \pm b\sin x$

Here's a sketch of $y = 3\sin x + 4\cos x$.



It's a sin graph that seems to be translated on the x-axis and stretched on the y axis. This suggests we can represent it as $y = R\sin(x + \alpha)$, where α is the horizontal translation and R the stretch on the y-axis.

Notes

Put $3\sin x + 4\cos x$ in the form $R\sin(x+\alpha)$ giving α in degrees to 1dp.

STEP 1: Expanding:

$$R\sin(x+\alpha) = R\sin x\cos\alpha + R\cos x\sin\alpha$$

STEP 2: Comparing coefficients:

$$R\cos\alpha = 3 \qquad R\sin\alpha = 4$$

STEP 3: Using the fact that $R^2\sin^2\alpha + R^2\cos^2\alpha = R^2$:

$$R = \sqrt{3^2+4^2} = 5$$

If $R\cos\alpha = 3$ and $R\sin\alpha = 4$ then $R^2\cos^2\alpha = 3^2$ and $R^2\sin^2\alpha = 4^2$.

$$R^2\sin^2\alpha + R^2\cos^2\alpha = 3^2+4^2$$
$$R^2(\sin^2\alpha + \cos^2\alpha) = 3^2+4^2$$
$$R^2 = 3^2+4^2$$
$$R = \sqrt{3^2+4^2}$$

(You can write just the last line in exams)

STEP 4: Using the fact that $\frac{R\sin\alpha}{R\cos\alpha} = \tan\alpha$:

$$\tan\alpha = \frac{4}{3}$$
$$\alpha = 53.1°$$

STEP 5: Put values back into original expression.

$$3\sin x + 4\cos x \equiv 5\sin(x+53.1°)$$

Notes

Worked Example

591a: Write $a\cos x + b\sin x$ in the form $R\cos(x+\alpha)$

Write $9\sin\theta + 3\cos\theta$ in the form $R\sin(\theta+\alpha)$ leaving R as an exact value and α in degrees correct to 1 decimal place.

You may use these formulae:

$\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$
$\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$

Worked Example

Express $5 \sin x + 12 \cos x$ in the form: $R \cos(x - \alpha), R > 0, 0 < \alpha < 90°$

Worked Example

Express $5\sin x + 12\cos x$ in the form: $R\sin(x - \alpha), R > 0, 0 < \alpha < 180°$

Worked Example

Express $5\sin x + 12\cos x$ in the form: $R\cos(x+\alpha), R > 0, 0 < \alpha < 180°$

T.184 7E: Qs 1-4, P.52 7.5 Qs 1-3

Worked Example

Solve in the interval $0 < \theta < 360°$:

$$5\cos\theta + 2\sin\theta = 3$$

Worked Example

Solve in the interval $0 \leq \theta < 180°$:

$$5 \sin 3\theta - 12 \cos 3\theta = 1$$

Worked Example

Solve in the interval $0 \leq \theta < 360°$:

$$\cot\theta + 4 = cosec\ \theta$$

T.184 7E: Qs 5-8, P.52 7.5 Qs 4-5

Worked Example

Find the maximum value and the smallest positive value of θ at which the maximum occurs for:

$$4\cos\theta + 3\sin\theta$$

Worked Example

Find the maximum value and the smallest positive value of θ at which the maximum occurs for:

$$\frac{5}{7 + 4\cos\theta - 3\sin\theta}$$

Worked Example

Find the minimum value and the smallest positive value of θ at which the minimum occurs for:

$$\frac{5}{7 + 4\cos\theta - 3\sin\theta}$$

T.184 7E: Qs 9+, P.52 7.5 Qs 6+

7.6) Proving trigonometric identities

Notes

Worked Example

Prove that:

$$\cot 2\theta \equiv \frac{\cot\theta - \tan\theta}{2}$$

Worked Example

Prove that:

$$\frac{-\sin 2\theta}{\cos 2\theta - 1} \equiv \cot\theta$$

Worked Example

Prove that:

$$\cot 2x - cosec\ 2x \equiv -\tan x$$

Worked Example

Prove, starting with the left-hand side:

$$\tan 2x + sec\ 2x \equiv \frac{\cos x + \sin x}{\cos x - \sin x}$$

T.187 7F: Qs 1-5, P.53 7.6 Qs 1-6

Worked Example

Show that:

$$\sin^4 \theta = \frac{3}{8} - \frac{1}{2}\cos 2\theta + \frac{1}{8}\cos 4\theta$$

Worked Example

By writing $\cos x = \cos\left(2 \times \frac{x}{2}\right)$, prove the identity

$$\frac{1 + \cos x}{1 - \cos x} \equiv \cot^2\left(\frac{x}{2}\right)$$

T.187 7F: Qs 6+, P.53 7.6 Qs 7

7.7) Modelling with trigonometric functions

Notes

Worked Example

The cabin pressure, P (psi) on an aeroplane at cruising altitude can be modelled by the equation

$$P = 14.5 - 0.2\sin(t - 3)$$

where t is the time in hours since cruising altitude was first reached, and angles are in radians. Find:

a) The maximum and minimum cabin pressure
b) The time after reaching cruising altitude that the cabin first reaches a maximum pressure
c) The cabin pressure after 3 hours at cruising altitude
d) All the times within the first 10 hours of cruising that the cabin pressure would be exactly 14.42 psi

Worked Example

a) Express $7\cos\theta - 5\sin\theta$ in the form $R\cos(\theta + \alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$. State the exact value of R and give α to four decimal places.

b) State the maximum value of $7\cos\theta - 5\sin\theta$ and the value of θ, for $0 < \theta < 2\pi$ at which this maximum occurs.

The height H above ground of a passenger on a Ferris wheel is modelled by the equation

$$H = 12 - 7\cos(\frac{\pi t}{4}) + 5\sin(\frac{\pi t}{4})$$

where H is measured in metres, and t is the time in minutes after the wheel starts turning.

c) Calculate the maximum value of H predicted by this model, and the value of t when this maximum first occurs

d) Determine the time for the Ferris wheel to complete five revolutions

Your Turn

a) Express $9\cos\theta - 2\sin\theta$ in the form $R\cos(\theta + \alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$. State the exact value of R and give α to four decimal places.

b) State the maximum value of $9\cos\theta - 2\sin\theta$ and the value of θ, for $0 < \theta < 2\pi$ at which this maximum occurs.

The height H above ground of a passenger on a Ferris wheel is modelled by the equation

$$H = 10 - 9\cos(\frac{\pi t}{5}) + 2\sin(\frac{\pi t}{5})$$

where H is measured in metres, and t is the time in minutes after the wheel starts turning.

c) Calculate the maximum value of H predicted by this model, and the value of t when this maximum first occurs

d) Determine the time for the Ferris wheel to complete two revolutions

a) $R = \sqrt{85}, \alpha = 0.2187$

b) Maximum $= \sqrt{85}$ when $\theta = 6.06$

c) Maximum $H = 19.22m$ at $t = 4.65$

d) 20 minutes

Worked Example

a) Express $1.5 \sin\theta - 2\cos\theta$ in the form $R\sin(\theta - \alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$. State the exact value of R and give α to four decimal places.

b) State the maximum value of$1.5 \sin\theta - 2\cos\theta$and the value of θ, for $0 < \theta < \pi$ at which this maximum occurs.

The height H of sea water on a particular day can be modelled by the equation

$$H = 8 + 1.5 \sin\left(\frac{2\pi t}{25}\right) - 2\cos\left(\frac{2\pi t}{25}\right), 0 \leq t < 12$$

where H is measured in metres, and t is the number of hours after midnight.

c) Calculate the maximum value of H predicted by this model, and the value of t when this maximum occurs

d) Calculate, to the nearest minute, the times when the height of sea water is predicted, by this model, to be 7 metres.

T.190 7G: Qs all, P.54 7.7 Qs all

Extract from Formulae book

Trigonometric identities

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B} \quad (A \pm B \neq (k + \tfrac{1}{2})\pi)$$

$$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$$

$$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$$

$$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$$

$$\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$$

Small angle approximations

$$\sin\theta \approx \theta$$

$$\cos\theta \approx 1 - \frac{\theta^2}{2}$$

$$\tan\theta \approx \theta$$

where θ is measured in radians

13. (a) Show that

$$\operatorname{cosec} 2x + \cot 2x \equiv \cot x, \qquad x \neq 90n°, n \in \mathbb{Z}$$

(5)

(b) Hence, or otherwise, solve, for $0 \leqslant \theta < 180°$,

$$\operatorname{cosec}(4\theta + 10°) + \cot(4\theta + 10°) = \sqrt{3}$$

You must show your working.
(*Solutions based entirely on graphical or numerical methods are not acceptable.*)

(5)

Exams

- Formula Booklet
- Past Papers
- Practice Papers
- past paper Qs by topic

Past paper practice by topic. Both new and old specification can be found via this link on hgsmaths.com

Question	Scheme	Marks	AOs
13(a)	$\operatorname{cosec}2x + \cot 2x \equiv \cot x,\ x \neq 90n°,\ n \in ¢$		
	$\operatorname{cosec}2x + \cot 2x = \frac{1}{\sin 2x} + \frac{\cos 2x}{\sin 2x}$	M1	1.2
	$= \frac{1+\cos 2x}{\sin 2x}$	M1	1.1b
	$= \frac{1+2\cos^2 x - 1}{2\sin x\cos x} = \frac{2\cos^2 x}{2\sin x \cos x}$	M1	2.1
		A1	1.1b
	$= \frac{\cos x}{\sin x} = \cot x$ *	A1*	2.1
		(5)	
(b)	$\operatorname{cosec}(4\theta + 10°) + \cot(4\theta + 10°) = \sqrt{3};\ 0 „ \theta < 180°,$		
	$\cot(2\theta \pm ...°) = \sqrt{3}$	M1	2.2a
	$2\theta \pm ... = 30° \Rightarrow \theta = 12.5°$	M1	1.1b
		A1	1.1b
	$2\theta + 5° = 180° + PV° \Rightarrow \theta = ...°$	M1	2.1
	$\theta = 102.5°$	A1	1.1b
		(5)	
(10 marks)			

Summary of Key Points

Summary of key points

1 The **addition** (or compound-angle) formulae are:

- $\sin(A+B) \equiv \sin A\cos B + \cos A\sin B$ $\qquad$ $\sin(A-B) \equiv \sin A\cos B - \cos A\sin B$
- $\cos(A+B) \equiv \cos A\cos B - \sin A\sin B$ $\qquad$ $\cos(A-B) \equiv \cos A\cos B + \sin A\sin B$
- $\tan(A+B) \equiv \dfrac{\tan A + \tan B}{1 - \tan A\tan B}$ $\qquad$ $\tan(A-B) \equiv \dfrac{\tan A - \tan B}{1 + \tan A\tan B}$

2 The **double-angle** formulae are:

- $\sin 2A \equiv 2\sin A\cos A$
- $\cos 2A \equiv \cos^2 A - \sin^2 A \equiv 2\cos^2 A - 1 \equiv 1 - 2\sin^2 A$
- $\tan 2A \equiv \dfrac{2\tan A}{1 - \tan^2 A}$

3 For positive values of a and b,

- $a\sin x \pm b\cos x$ can be expressed in the form $R\sin(x \pm \alpha)$
- $a\cos x \pm b\sin x$ can be expressed in the form $R\cos(x \mp \alpha)$

with $R > 0$ and $0 < \alpha < 90°$ $\left(\text{or } \frac{\pi}{2}\right)$

where $R\cos\alpha = a$ and $R\sin\alpha = b$ and $R = \sqrt{a^2 + b^2}$.

T.192 mixed ex, P.57 BSG