KING EDWARD VI
HANDSWORTH GRAMMAR
SCHOOL FOR BOYS

KING EDWARD VI
ACADEMY TRUST
BIRMINGHAM

Year 12
Pure Mathematics
Proof

HGS Maths

Dr Frost Course

Name: ______________________________

Class: ______________________________

Contents

This booklet will cover all aspects of proof required for A level Mathematics.

This corresponds to the following textbook chapters

P1 7.4) Mathematical Proof
P1 7.5) Methods of Proof
P2 1.1) Proof by Contradiction

Extra notation used in academic mathematics (university)
Past Paper Practice
Map of proof
Summary

Prior knowledge check

1. How do you define and express a/an:
    a) Even number
    b) Odd number
    c) Square of an odd number
    d) Three consecutive numbers
    e) A multiple of 5
    f) Two consecutive even numbers
    g) A rational number

2. What is the difference between an equation and an identity? Give examples of each.

Terminology

Conjecture

A **conjecture** is a mathematical statement that has yet to be proven.

One famous conjecture is **Goldbach's Conjecture**.
It states "*Every even integer greater than* 2 *can be expressed as the sum of two primes*".
It has been verified up to 4×10^{18} (that's big!); this provides evidence that it is true, but does not prove it is true!

Theorem

A **theorem** is a mathematical statement that has been proven.

One famous misnomer was **Fermat's Last Theorem**, which states "*If* n is an integer where $n > 2$, *then* $a^n + b^n = c^n$ *has no non-zero integer solutions for* a, b, c". It was 300 years until this was proven in 1995. Only then was the 'Theorem' in the name then correct!

Useful Things to Remember

Even Numbers:
Even numbers have a factor of two. This means:
If an expression can be factorised into $\mathbf{2n}$ or $\mathbf{2}(\ldots)$ it must be even.

Odd Numbers:
Odd numbers are one more (or one less) than an even number. This means:
If an expression can be written as $\mathbf{2n \pm 1}$ or $\mathbf{2}(\ldots) \pm \mathbf{1}$ it must be odd.

Consecutive Integers:
If the starting integer is n, then the next integer will be $n + 1$ and the one after it $n + 2$, and so on. Thus a list of consecutive integers would be written as: $\mathbf{n, n + 1, n + 2, n + 3, n + 4}, \ldots$

Consecutive Even Numbers
An even number is a multiple of two so I will choose $2n$ to be the starting even number. This means that the list of consecutive even numbers would be written as: $\mathbf{2n, 2n + 2, 2n + 4, 2n + 6}, \ldots$

Consecutive Odd Numbers
An odd number is one more (or one less) than a multiple of two so I will choose $2n + 1$ as my starting odd number. Thus a list of consecutive odd numbers would be written as: $\mathbf{2n + 1, 2n + 3, 2n + 5, 2n + 7}, \ldots$

Multiples
If an expression can be written as $3n$ or $3(\ldots)$ then it must be a multiple of 3. The same applies for $4, 5, 6$ etc.
In general, if an expression can be written as $\mathbf{kn}$ or $\mathbf{k}(\ldots)$ then it must be a **multiple of $\mathbf{k}$**.

Types of Proof

A proof must show all **assumptions** you are using, have a clear **sequential list of steps** that logically follow, and must cover **all possible cases**.

You should usually make a **concluding statement**, e.g. restating the original conjecture that you have proven.

There are four types of proofs that we will cover:

a) Proof by Deduction
b) Proof by Exhaustion
c) Disproof by Counter-Example
d) Proof by Contradiction

Proof by Deduction

This is the simplest type, where you start from known facts and reach the desired conclusion via deductive steps.

e.g. Prove that the product of two odd numbers is odd.

Proof:

For integers n and m we have odd integers $2n + 1$ and $2m + 1$.

The product of two odd integers is:

$$\begin{aligned} &(2n+1)(2m+1) \\ &= 2nm + 2n + 2m + 1 \\ &= 2(nm + n + m) + 1 \end{aligned}$$

Which is also an odd integer.

Worked Example

Worked Example

Prove that $(3x + 2)(x - 5)(x + 7) \equiv 3x^3 + 8x^2 - 101x - 70$

Non Example

Proof by Deduction requires you to **start from known facts** and end up at the conclusion. It is **not** acceptable to start with the conclusion, and verify it works, **because you are assuming the thing you are trying to prove**.

Incorrect Proof:
Let the lengths be $3, 4, 5$.
Therefore:

$$3^2 + 4^2 = 5^2$$
$$25 = 25$$

We are assuming the thing we are trying to prove!

This satisfies Pythagoras' Theorem, and the numbers are consecutive.

The underlying problem is that this technique doesn't prove there can't be **other** consecutive integers that work – we have only verified $3, 4, 5$ is one such solution.

Correct Proof:
If the sides are consecutive, then let the sides be:

$$x, x+1, x+2$$

Then by Pythagoras' Theorem:

$$x^2 + (x+1)^2 = (x+2)^2$$
$$\ldots$$
$$x^2 - 2x - 3 = 0$$
$$(x+1)(x-3) = 0$$

We are only assuming things in the 'if' bit. This is fine!

$x = 3$ (as x can't be negative)
Thus the sides are 3, 4, 5.

Worked Example

Worked Example

Prove that $\frac{x-y}{\sqrt{x}-\sqrt{y}} = \sqrt{x} + \sqrt{y}$

T.149 7D Qs 1-8, P.58 7.4 Qs 1,6

Worked Example

Worked Example

Prove that $A(1,1), \mathrm{B}(3,3)$ and $C(4,2)$ are the vertices of a right-angled triangle.

T.149 7D Qs 14-17, P.58 7.4 Qs 2,4

Worked Example

Worked Example

(a) Prove that the difference of the squares of two consecutive even numbers is always divisible by 4.

(b) Is this statement true for odd numbers? Give a reason for your answer.

Worked Example

Worked Example

Prove that $x^2 + 4x + 5$ is positive for all values of x.

Worked Example

Worked Example

The equation $kx^2 + 3kx + 2 = 0$, where k is a constant, has no real roots. Prove that k satisfies the inequality $0 \leq k < \frac{8}{9}$

Worked Example

Worked Example

Show that $a^2 + b^2 \geq 2ab$ for all values of a and b.

T.149 7D Qs 9-13,18-20, P.58 7.4 Qs 3,5,7-9

Proof by Exhaustion

This means breaking down the statement into **all possible smaller cases**, where we prove each individual case.
(This technique is sometimes known as ‘case analysis’)

For example, if you are proving something about integers, then all integers can be split into

- Odd and even integers (2 cases)
- Primes and non-primes (2 cases)
- Multiples of 3, one more than a multiple of 3 and one less than a multiple of 3 (3 cases)

Worked Example

Prove, for integers between 10 and 40, that reversing the digits of a multiple of 3 gives a number that is also a multiple of 3.

Worked Example

Let n be an integer. Prove by exhaustion that $n^2 - 5n + 4$ is positive for $6 \leq n \leq 8$.

Worked Example

Prove that $n^2 + n$ is even for all integers n.

Worked Example

Prove by exhaustion that if n is not divisible by 3, then $n^2 = 3k + 1$ for some integer k.

T.152 7E Qs 1-4, 7-9 P.59 7.5 Qs 6,8,9

Disproof by Counter-Example

While to prove a statement is true, we need to prove every possible case (potentially infinitely many!), **we only need one example to disprove** a statement.
This is known as a **counterexample**.

The fact that we only need one example to disprove any statement may seem counter-intuitive. So to help appreciate this, imagine your friend has a basket full of apples. Your friend then tells you that all of the apples in his basket are red apples. To his surprise, you pull out a green apple from his basket. This green apple in his basket is a counter example to his statement; he might have many red apples in his basket, but your counter example proved that not all of them are.

Worked Example

Disprove the statement: "$n^2 - n + 41$ is prime for all integers n."

Worked Example

Prove that the following statement is not true:
"The sum of two consecutive prime numbers is always even."

Worked Example

Use a counter example to show that the statement
$\sqrt{a+b} = \sqrt{a} + \sqrt{b}$ for all a and b is false.

Worked Example

(a) Prove that for all positive values of x and y:

$$\frac{x}{y} + \frac{y}{x} \geq 2$$

(b) Use a counter-example to show that this is not true when x and y are not both positive.

T.152 7E Qs 5-6, 10 P.59 7.5 Qs 1-5

Proof by Contradiction

To prove a statement is true by contradiction:

- **Assume** that the statement is in fact **false**.
- Prove that this would **lead to a contradiction**.
- Therefore, we were wrong in assuming the statement was false, and therefore it must be true.

Example:

Prove that there is no greatest odd integer

Proof:

Assume there is a greatest integer, call this n

But $n+1>n$ *and* $n+1$ *is an integer.*
This contradicts n is the greatest integer

Therefore there is no greatest integer

How to structure/word proof:

1. "Assume that *[negation of statement]*."
2. [*Reasoning followed by…*] *"This contradicts the assumption that…"* or *"This is a contradiction"*.
3. "Therefore [*restate original statement*]."

Negating the Original Statement

The first part of a proof by contradiction requires you to negate the original statement.

What is the negation of each of these statements?

1) "There are infinitely many prime numbers."
A) "There are infinitely many non-prime (i.e. composite) numbers."
B) "There are finitely many prime numbers."
C) "There are finitely many non-composite numbers."

2) "All Popes are Catholic."
A) "There exists a Pope who is not Catholic."
B) "No Popes are Catholic."
C) "Dr Frost is the Pope."

Comments: The negation of "all are" is not "none are". So the negation of "everyone likes green" wouldn't be "no one likes green", but: "not everyone likes green". Do not confuse a 'negation' with the 'opposite'.

Negating the Original Statement

3) "If it is raining, my garden is wet."
A) "It is not raining and my garden is dry."
B) "It is not raining and my garden is wet."
C) "It is raining and my garden is not wet."

Comments: If you have a <u>conditional</u> statement like "*If A then B*", then the negation is "*A and not B*", i.e. the condition is true, but the conclusion is false/negated.

Pure 2: T. 4 1A Qs 1, 2 , P.1 1.1 Q1

Worked Example

Prove by contradiction that if n^2 is even, then n must be even.

Worked Example

Prove by contradiction that $\sqrt{2}$ is an irrational number.

Worked Example

Prove by contradiction that there are infinitely many prime numbers.

This proof is courtesy of Euclid, and is one of the earliest known proofs.

Pure 2: T. 4 1A Qs 3+, P.1 1.1 Qs2+

10.4 Common sets of numbers

The commonly used sets of numbers are:

- The set of natural numbers, $\mathbb{N} = \{1, 2, 3, 4...\}$. Careful! Some mathematicians include 0 in $\mathbb{N}$,
- The set of integers, $\mathbb{Z} = \{-3, -2, -1, 0, 1, 2, 3, 4, ...\}$;
- The set of rational numbers $\mathbb{Q} = \{\frac{m}{n} : m, n \in \mathbb{Z}, n > 0\}$;
- The set of real numbers $\mathbb{R}$, which is the union[6] of both rational $\mathbb{Q}$ and irrational numbers (which cannot be expressed as a fraction, for example $\log 2, \sqrt{2}, \pi, e$).

Notice that one set is a subset of another, in the following order: $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$.

Extra notation used in academic mathematics (university)

- Quantifiers

$\forall$ (universal quantifier)	for all
$\exists$ (existential quantifier)	there exists

- Symbols in set theory

$\cup$	union
$\cap$	intersection
$\subseteq$	subset
$\subset, \subsetneq$ or $\subsetneqq$	proper subset
$\circ$	composition of functions

- Common symbols used when writing proofs and definitions

$\implies$	implies
$\iff$	if and only if
$:=$	is defined as
$\equiv$	is equivalent to
$:$ or $\mid$	such that
$\therefore$	therefore
$\lightning$ or ※	contradiction
$\blacksquare$ or $\square$	end of proof

A2 Specimen Paper 2

Algebraic Methods

14. (i) Kayden claims that

$$3^x \geqslant 2^x$$

Determine whether Kayden's claim is always true, sometimes true or never true, justifying your answer.

(2)

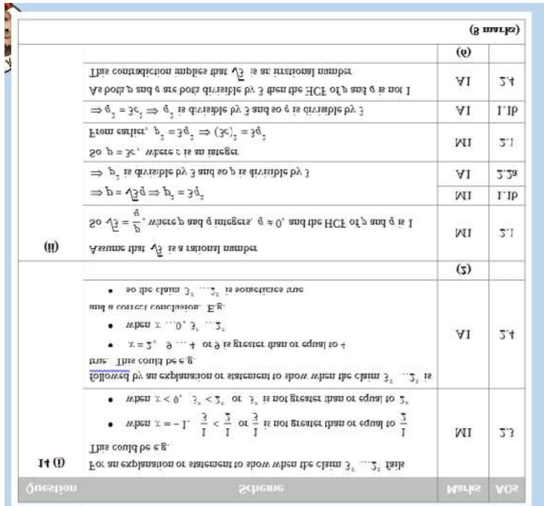
(ii) Prove that $\sqrt{3}$ is an irrational number.

(6)

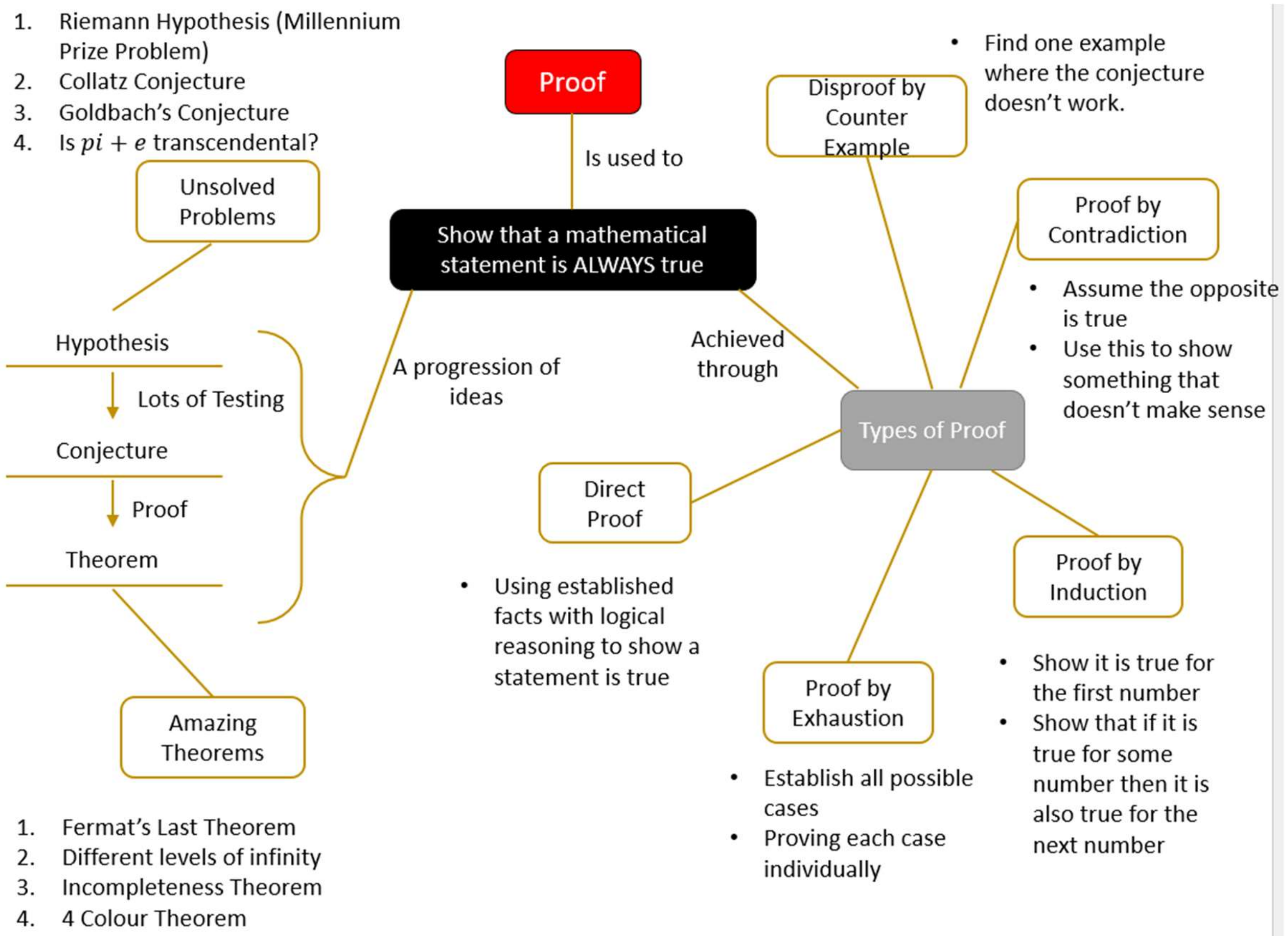
Exams

- Formula Booklet
- Past Papers
- Practice Papers
- past paper Qs by topic

Past paper practice by topic. Both new and old specification can be found via this link on hgsmaths.com



Map of proof


1. Riemann Hypothesis (Millennium Prize Problem)
2. Collatz Conjecture
3. Goldbach's Conjecture
4. Is $pi + e$ transcendental?
Unsolved Problems
Proof
Is used to
Show that a mathematical statement is ALWAYS true
A progression of ideas
Achieved through
Hypothesis
Lots of Testing
Conjecture
Proof
Theorem
Amazing Theorems
1. Fermat's Last Theorem
2. Different levels of infinity
3. Incompleteness Theorem
4. 4 Colour Theorem
Types of Proof
Disproof by Counter Example
• Find one example where the conjecture doesn't work.
Proof by Contradiction
• Assume the opposite is true
• Use this to show something that doesn't make sense
Direct Proof
• Using established facts with logical reasoning to show a statement is true
Proof by Exhaustion
• Establish all possible cases
• Proving each case individually
Proof by Induction
• Show it is true for the first number
• Show that if it is true for some number then it is also true for the next number

Summary of Key Points

Summary of key points

1. When simplifying an algebraic fraction, factorise the numerator and denominator where possible and then cancel common factors.
2. You can use long division to divide a polynomial by $(x \pm p)$, where p is a constant.
3. The **factor theorem** states that if $f(x)$ is a polynomial then:
   - If $f(p) = 0$, then $(x - p)$ is a factor of $f(x)$
   - If $(x - p)$ is a factor of $f(x)$, then $f(p) = 0$
4. You can prove a mathematical statement is true by **deduction**. This means starting from known factors or definitions, then using logical steps to reach the desired conclusion.
5. In a mathematical proof you must
   - State any information or assumptions you are using
   - Show every step of your proof clearly
   - Make sure that every step follows logically from the previous step
   - Make sure you have covered all possible cases
   - Write a statement of proof at the end of your working
6. To prove an identity you should
   - Start with the expression on one side of the identity
   - Manipulate that expression algebraically until it matches the other side
   - Show every step of your algebraic working
7. You can prove a mathematical statement is true by **exhaustion**. This means breaking the statement into smaller cases and proving each case separately.
8. You can prove a mathematical statement is not true by a **counter-example**. A counter-example is one example that does not work for the statement. You do not need to give more than one example, as one is sufficient to disprove a statement.

Pure 1: T. 154 mixed ex Qs15+

Pure 2: T.19 mixed ex Qs1,2