KING EDWARD VI
HANDSWORTH GRAMMAR
SCHOOL FOR BOYS

KING EDWARD VI
ACADEMY TRUST
BIRMINGHAM

Year 13
Pure Mathematics
P2 10 Numerical Methods

HGS Maths

Dr Frost Course

Name: ______________________________

Class: ______________________________

Contents

Prior knowledge check

Prior knowledge check

1 $f(x) = x^2 - 6x + 10$. Evaluate:

a $f(1.5)$ **b** $f(-0.2)$ ← **GCSE Mathematics**

2 Find $f'(x)$ given that:

a $f(x) = 3\sqrt{x} + 4x^2 - \frac{5}{x^3}$ ← **Year 1, Chapter 12**

b $f(x) = 5\ln(x + 2) + 7e^{-x}$ ← **Section 9.3**

c $f(x) = x^2 \sin x - 4\cos x$ ← **Section 9.4**

3 Given that $u_{n+1} = u_n + \frac{1}{u_n}$ and that $u_0 = 1$,

find the values of u_1, u_2 and u_3. ← **Section 3.7**

10.1) Locating roots

Why do we need numerical methods?

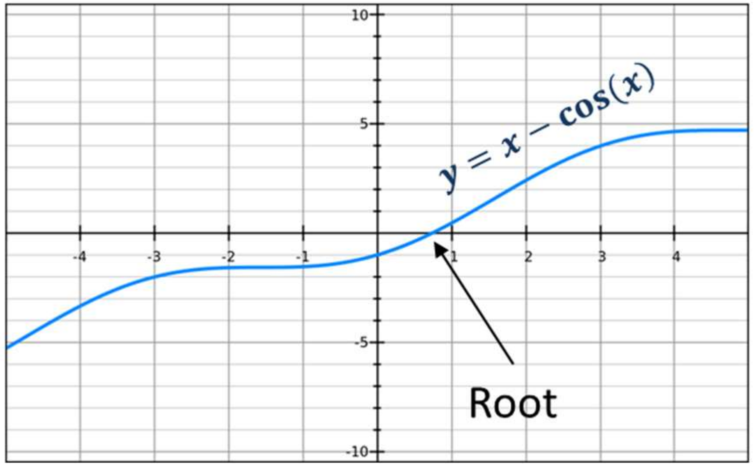
Finding the root of a function $f(x)$ is to: **solve the equation $f(x) = 0$**
(i.e. the inputs such that the output of the function is 0)

However, for some functions, the 'exact' root is either complicated and difficult to calculate:

$$x^3 + 2x^2 - 3x + 4 = 0 \quad \Rightarrow \quad x = \frac{1}{3}\left(-2 - \frac{13}{\sqrt[3]{89 - 6\sqrt{159}}} - \sqrt[3]{89 - 6\sqrt{159}}\right)$$

or there's no 'algebraic' expression at all! (involving roots, logs, sin, cos, etc.)

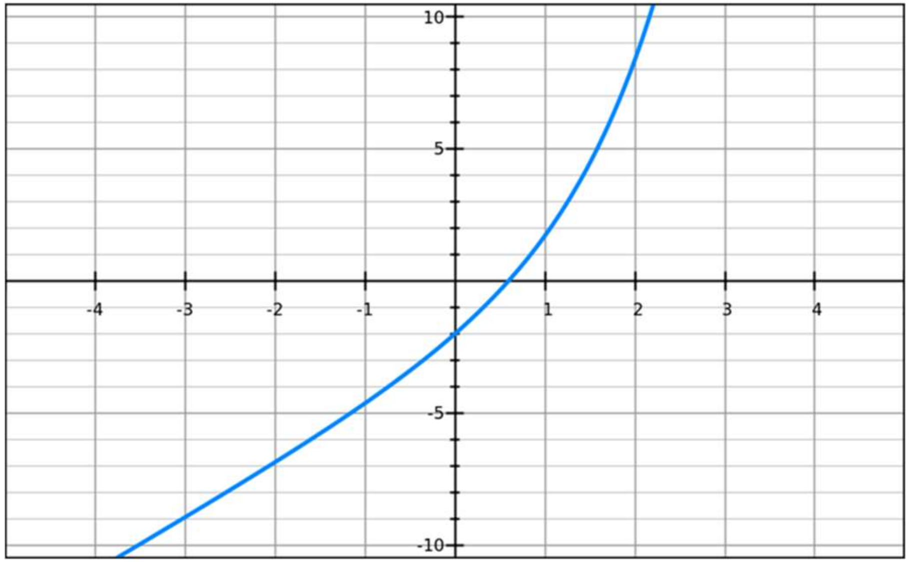
$$x - \cos(x) = 0 \quad \Rightarrow$$ **Exact solution not expressible ☹**



But there are a variety of 'numerical methods' which get progressively better solutions to an equation in the form $f(x) = 0$.
You have already seen 'iteration' at GCSE as one such method.

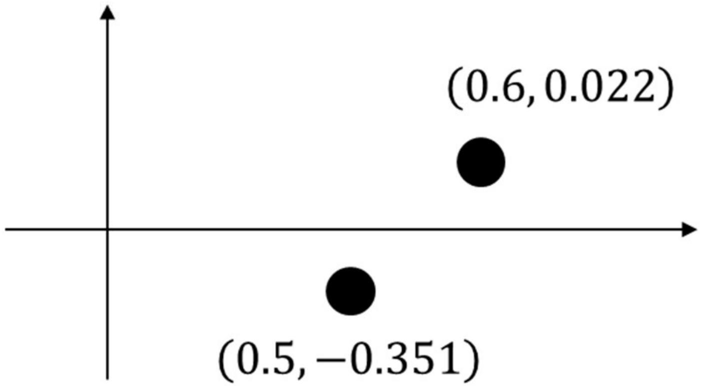
Proving a solution lies in a range

Show that $f(x) = e^x + 2x - 3$ has a root between $x = 0.5$ and $x = 0.6$



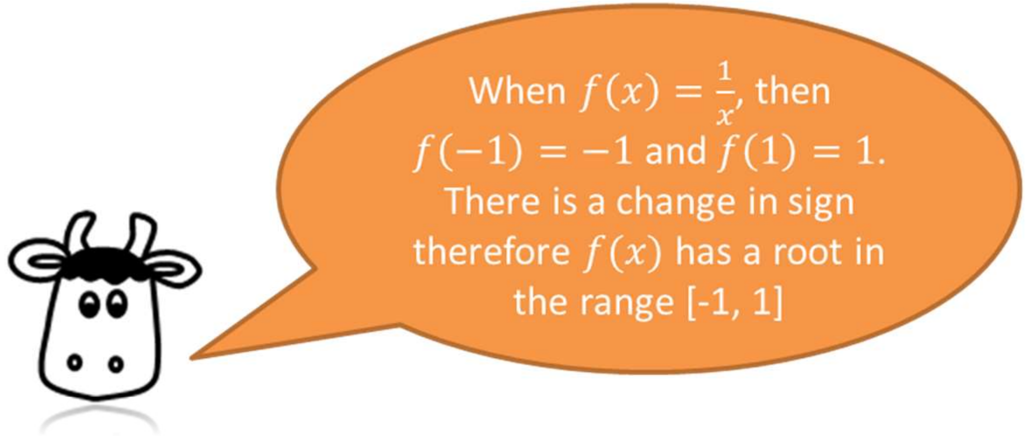
$$f(0.5) = -0.351$$
$$f(0.6) = 0.022 \ldots$$

There is a **change in sign**, and $\boldsymbol{f(x)}$ **is continuous**, so root must lie between 0.5 and 0.6

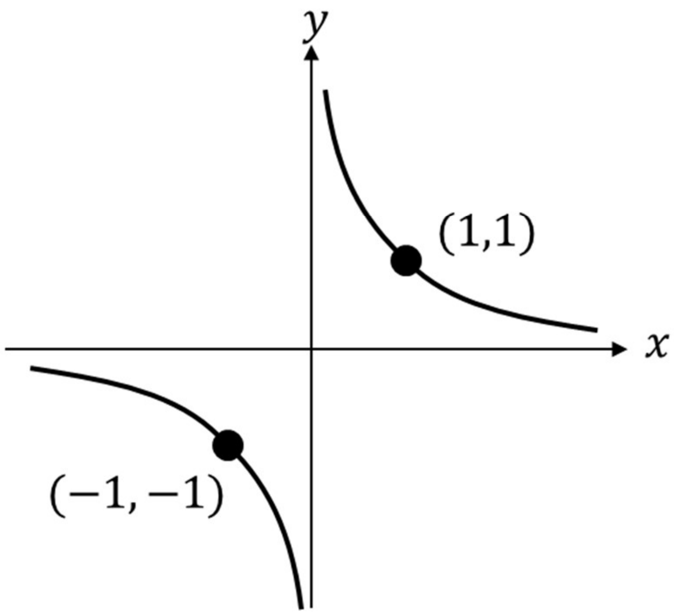


If the y value goes from negative to positive or vice versa, then clearly the y values must pass 0 somewhere in between.

Stupid Steve says:

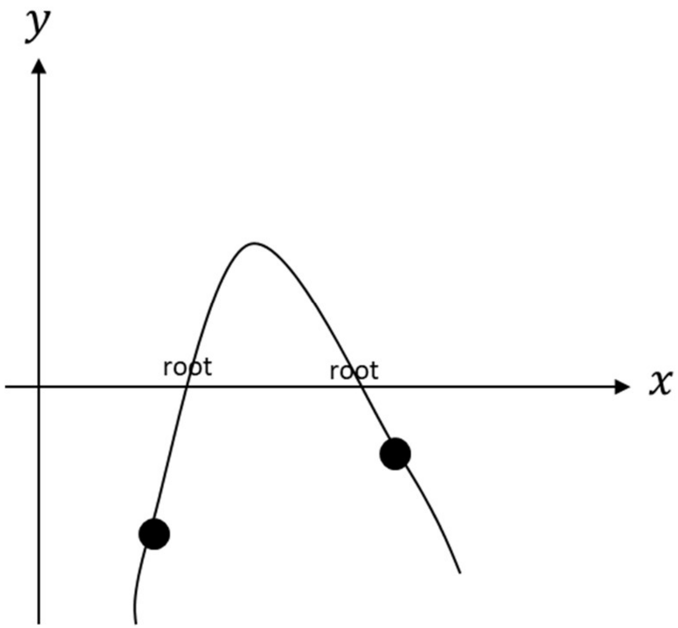


Why is Steve wrong?



A function is **continuous** if the line **does not ‘jump’**. A root is only guaranteed with a sign change if the function is continuous, as otherwise the line can skip past 0 (in this case due to a vertical asymptote.

…and no sign change doesn't mean there isn't a root



Beware! Just because there isn't a sign change, doesn't mean there's no root in that interval.

The sign change method fails to detect a root if there were an **even number of roots** in that interval.

Worked Example

Explain why there are no real roots to $f(x) = \frac{1}{x-2}$ between $x = 1$ and $x = 3$

Worked Example

Using the same axes, sketch the graphs of

$y = e^x$ and $y = \frac{1}{x}$

a) Explain how your diagram shows that the function $f(x) = e^x - \frac{1}{x}$ has only one root

b) Show that this root lies in the interval $0.5 < x < 0.6$

c) Show that the root is 0.567 to 3 decimal places

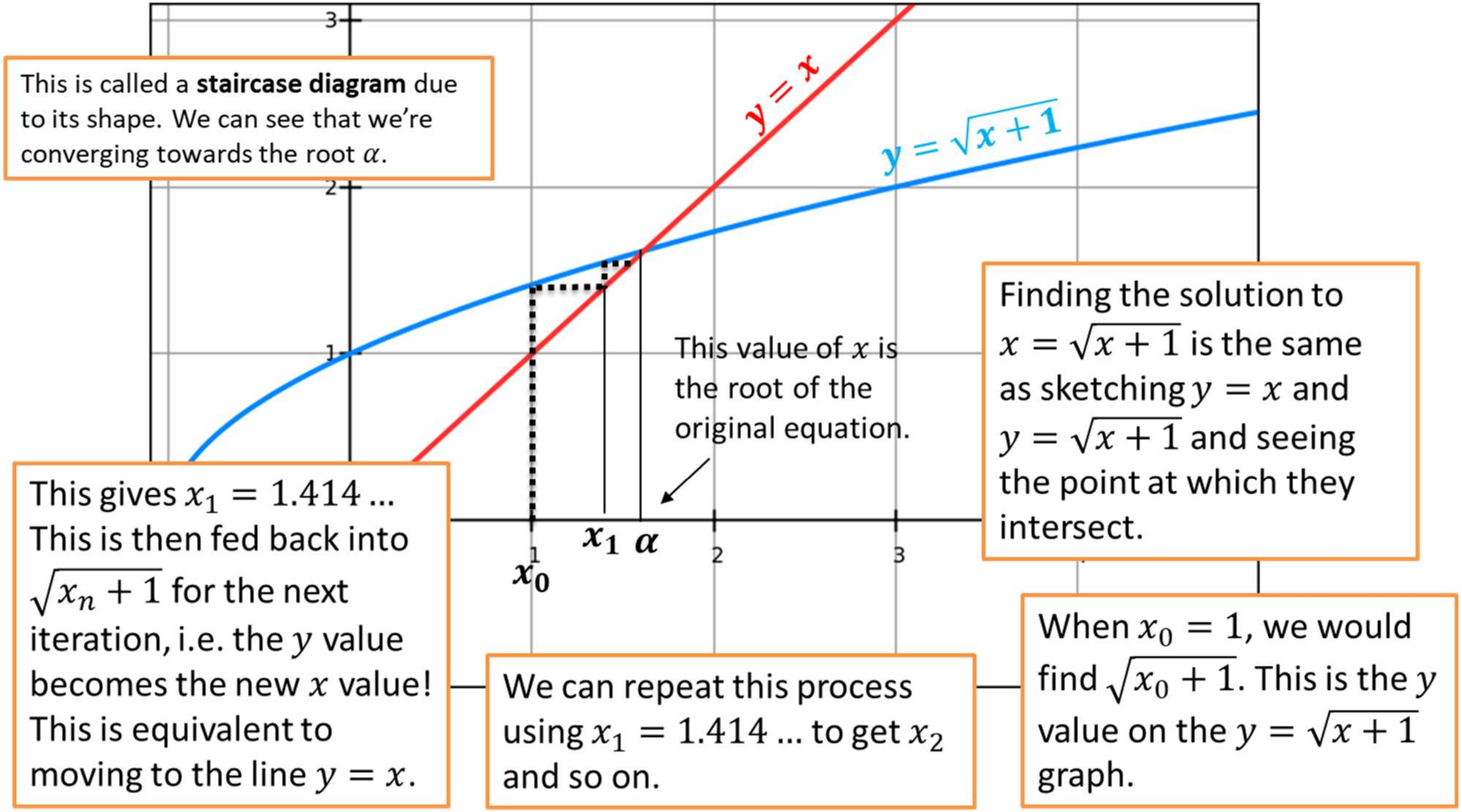
T.277 : 10A Qs 8, 11, 12

10.2) Iteration

Why does this method work?

Solve $x^2 - x - 1 = 0$

Recall we put in the form $x = g(x)$: in this case $x = \sqrt{x+1}$ is one possible rearrangement.
We can then use the recurrence $x_{n+1} = \sqrt{x_n + 1}$. Why does this recurrence work?

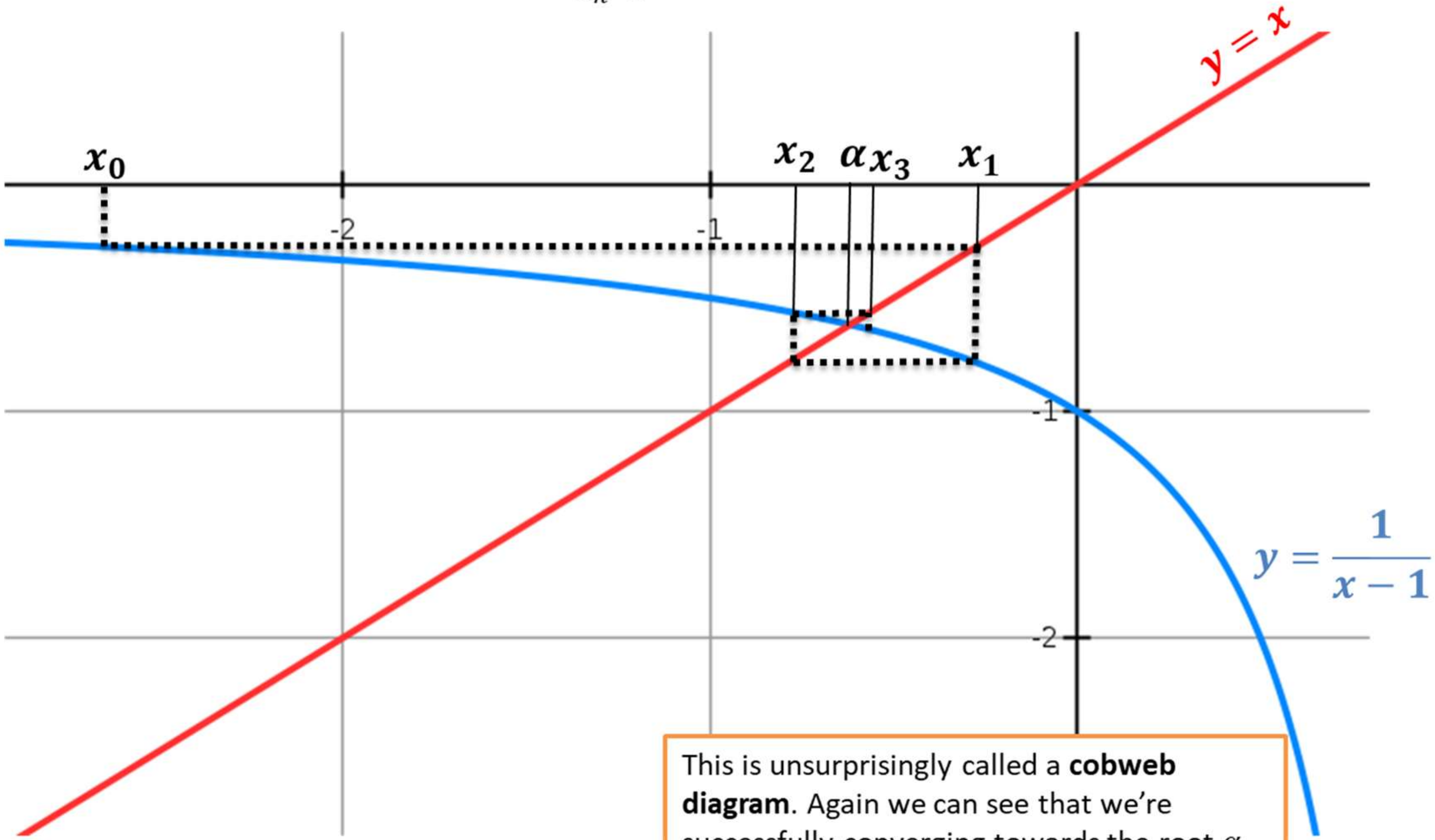


Cobweb Diagrams

Solve $x^2 - x - 1 = 0$

We could also have rearranged differently to $x = \frac{1}{x-1}$

Therefore we use the recurrence $x_{n+1} = \frac{1}{x_n - 1}$. What happens this time?

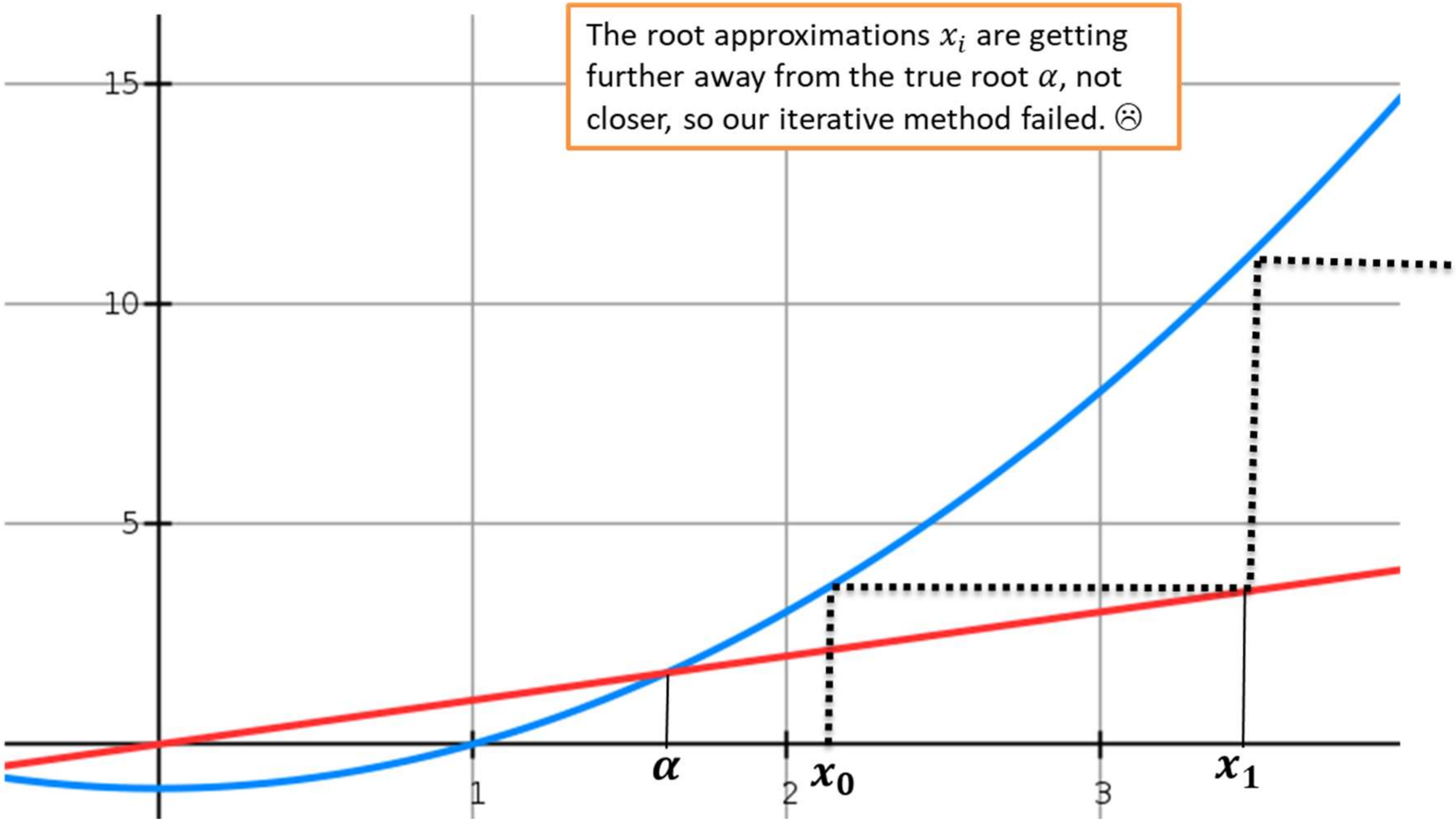


This is unsurprisingly called a **cobweb diagram**. Again we can see that we're successfully converging towards the root α.

And when iteration fails...

Solve $x^2 - x - 1 = 0$

But again, we could have rearranged differently! $x = x^2 - 1$
Therefore we use the recurrence $x_{n+1} = x_n^2 - 1$. What happens this time?



The root approximations x_i are getting further away from the true root α, not closer, so our iterative method failed. ☹

Notes

Worked Example

$$f(x) = x^2 - 5x - 3$$

a) Show that $f(x) = 0$ can be written as:

i) $x = \frac{x^2-3}{5}$ ii) $x = \sqrt{5x+3}$ iii) $x = 5 + \frac{3}{x}$

b) Starting with $x_0 = 3$ use each iterative formula to find a root of the equation $f(x) = 0$, rounding your answers to 3 decimal places

Worked Example

$$f(x) = e^{x-2} + x - 5$$

a) Show that $f(x) = 0$ can be written as:

$$x = \ln(5 - x) + 2, \qquad x < 5$$

The root of $f(x) = 0$ is α.
The iterative formula

$$x_{n+1} = \ln(5 - x_n) + 2, \qquad x_0 = 3$$

is used to find an approximate value for α

b) Calculate the values of x_1, x_2 and x_3 to four decimal places.

c) By choosing a suitable interval, show that $\alpha = 2.792$ correct to 3 decimal places.

Worked Example

$$f(x) = x^3 + 4x^2 + 3x - 12$$

(a) Show that the equation can be written as

$$x = \sqrt{\frac{3(4-x)}{4+x}}, x \neq -4$$

The equation $f(x) = 0$ has a single root between 1 and 2.

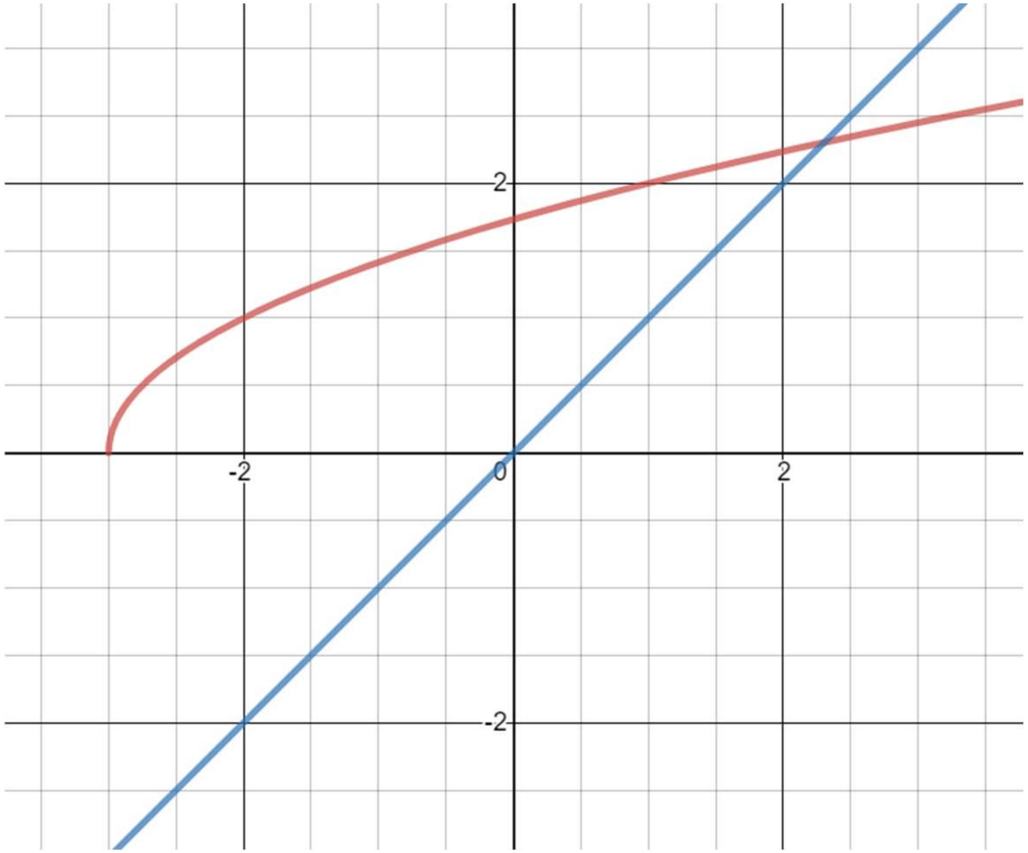
(b) Use the iterative formula

$$x_{n+1} = \sqrt{\frac{4(3-x_n)}{3+x_n}}, n \geq 0, x_0 = 1$$

to calculate the values of x_1, x_2 and x_3, giving your answers to 2 decimal places.

(c) The root of $f(x) = 0$ is α. By choosing a suitable interval, prove that $\alpha = 1.253$ (3 dp)

T.280: 10B Qs 1,2, P.82: 10.2 Qs 1-3

Worked Example
Use the graph of $y = x$ and $y = \sqrt{x+3}$ to solve the equation $$x^2 - x - 3 = 0$$ using the recurrence relation: $$x_{n+1} = \sqrt{x_n + 3}, x_0 = 1$$ 

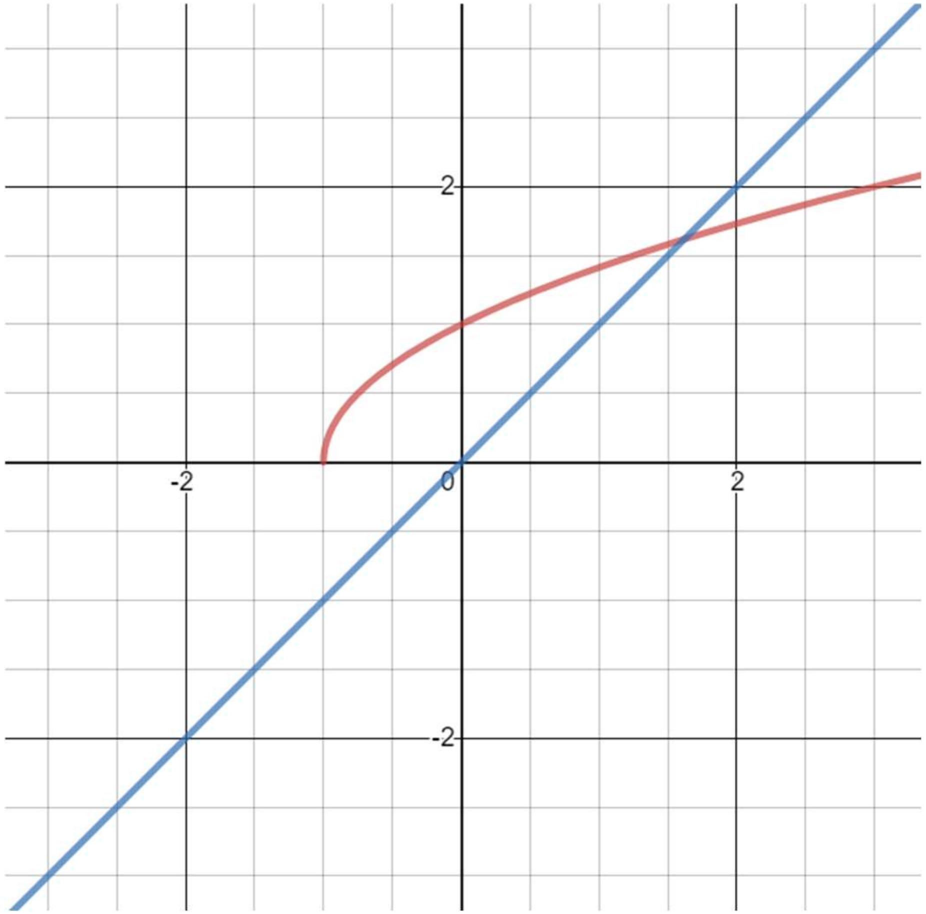
Your Turn

Use the graph of $y = x$ and $y = \sqrt{x + 1}$ to solve the equation

$$x^2 - x - 1 = 0$$

using the recurrence relation:

$$x_{n+1} = \sqrt{x_n + 1}, x_0 = 1$$



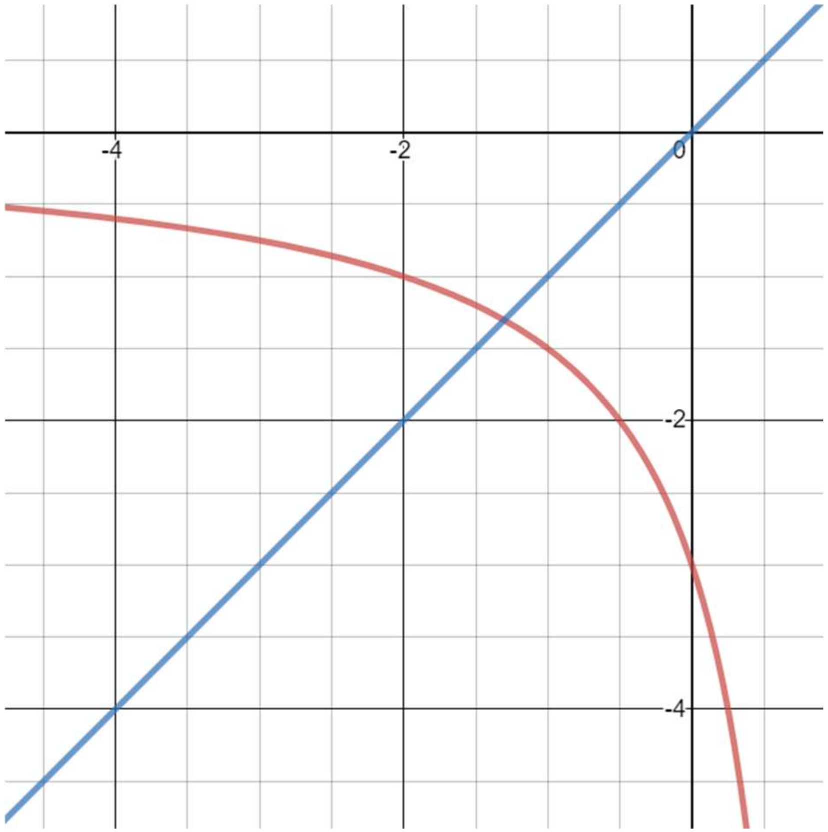
Worked Example

Use the graph of $y = x$ and $y = \frac{3}{x-1}$ to solve the equation

$$x^2 - x - 3 = 0$$

using the recurrence relation:

$$x_{n+1} = \frac{3}{x_n - 1}, x_0 = -4.5$$



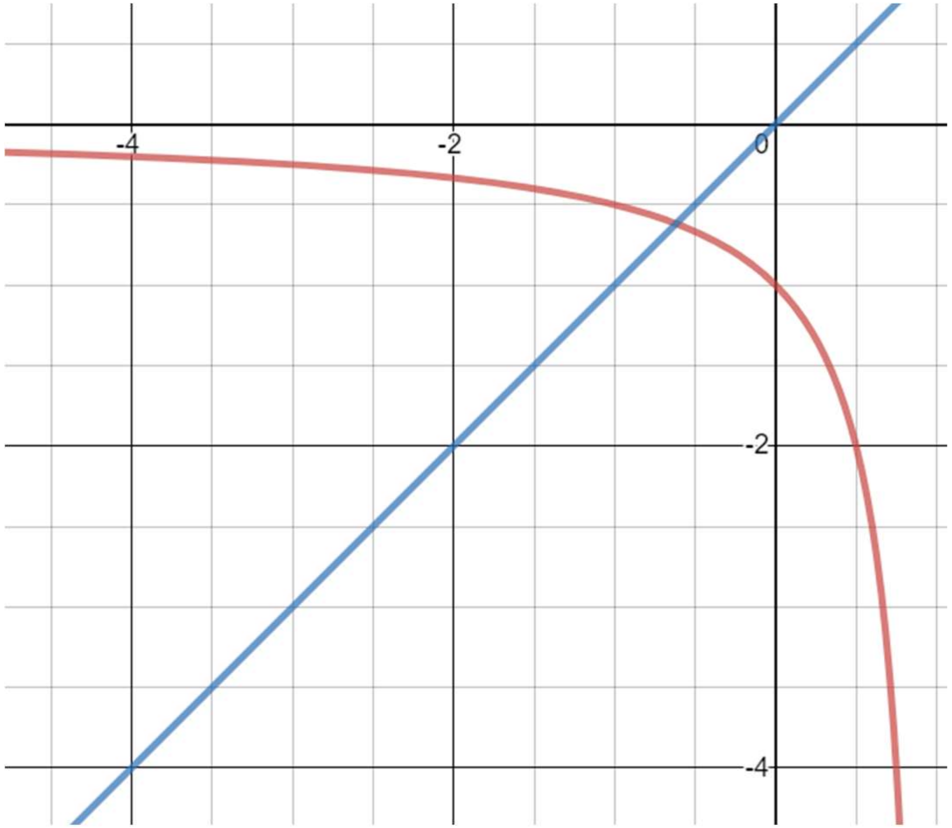
Your Turn

Use the graph of $y = x$ and $y = \frac{1}{x-1}$ to solve the equation

$$x^2 - x - 1 = 0$$

using the recurrence relation:

$$x_{n+1} = \frac{1}{x_n - 1}, x_0 = -2.5$$



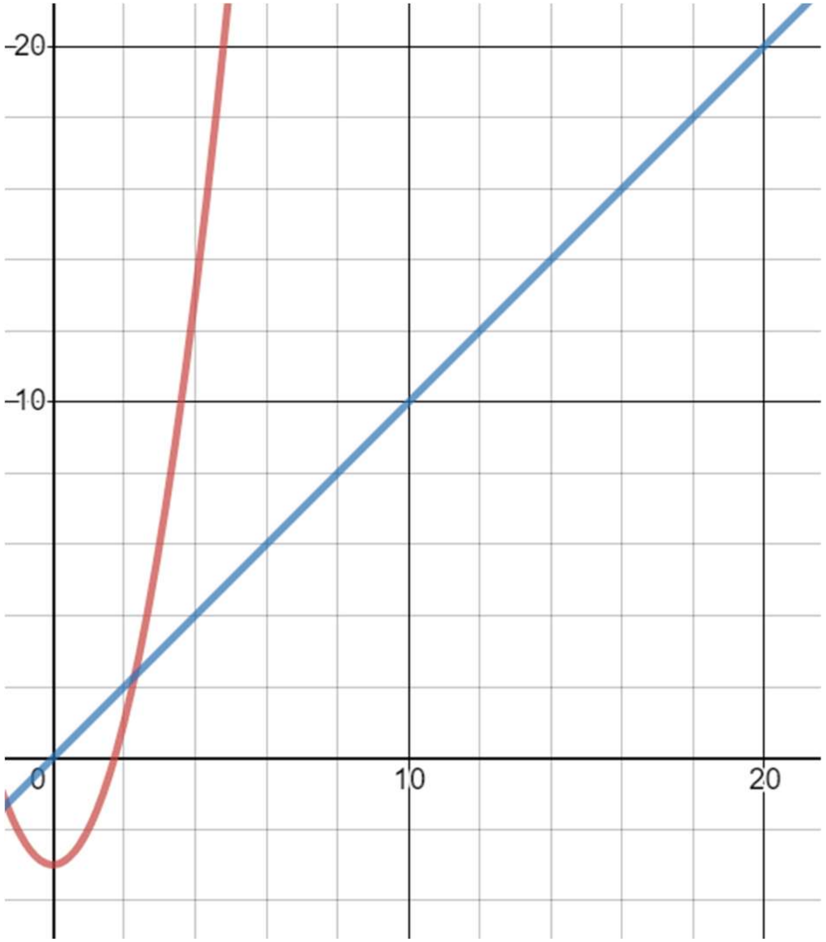
Worked Example

Use the graph of $y = x$ and $y = x^2 - 3$ to solve the equation

$$x^2 - x - 3 = 0$$

using the recurrence relation:

$$x_{n+1} = x_n^2 - 3, x_0 = 3$$



T.280: 10B Qs 3+, P.82: 10.2 Qs 4+

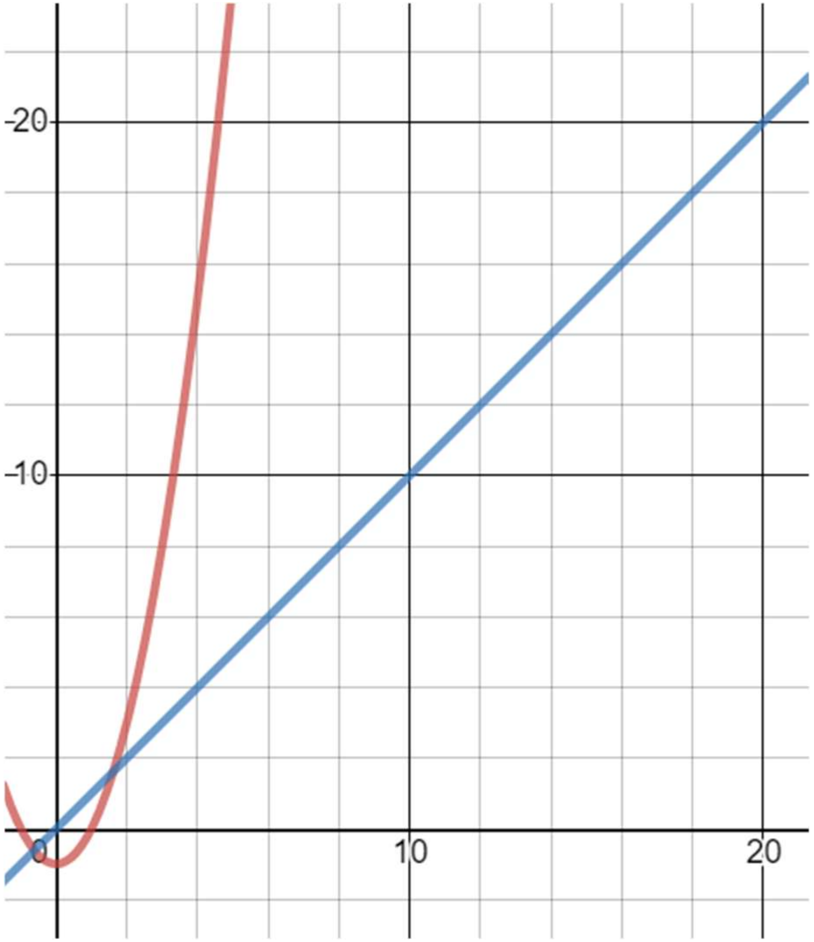
Your Turn

Use the graph of $y = x$ and $y = x^2 - 1$ to solve the equation

$$x^2 - x - 1 = 0$$

using the recurrence relation:

$$x_{n+1} = x_n^2 - 1, x_0 = 2$$



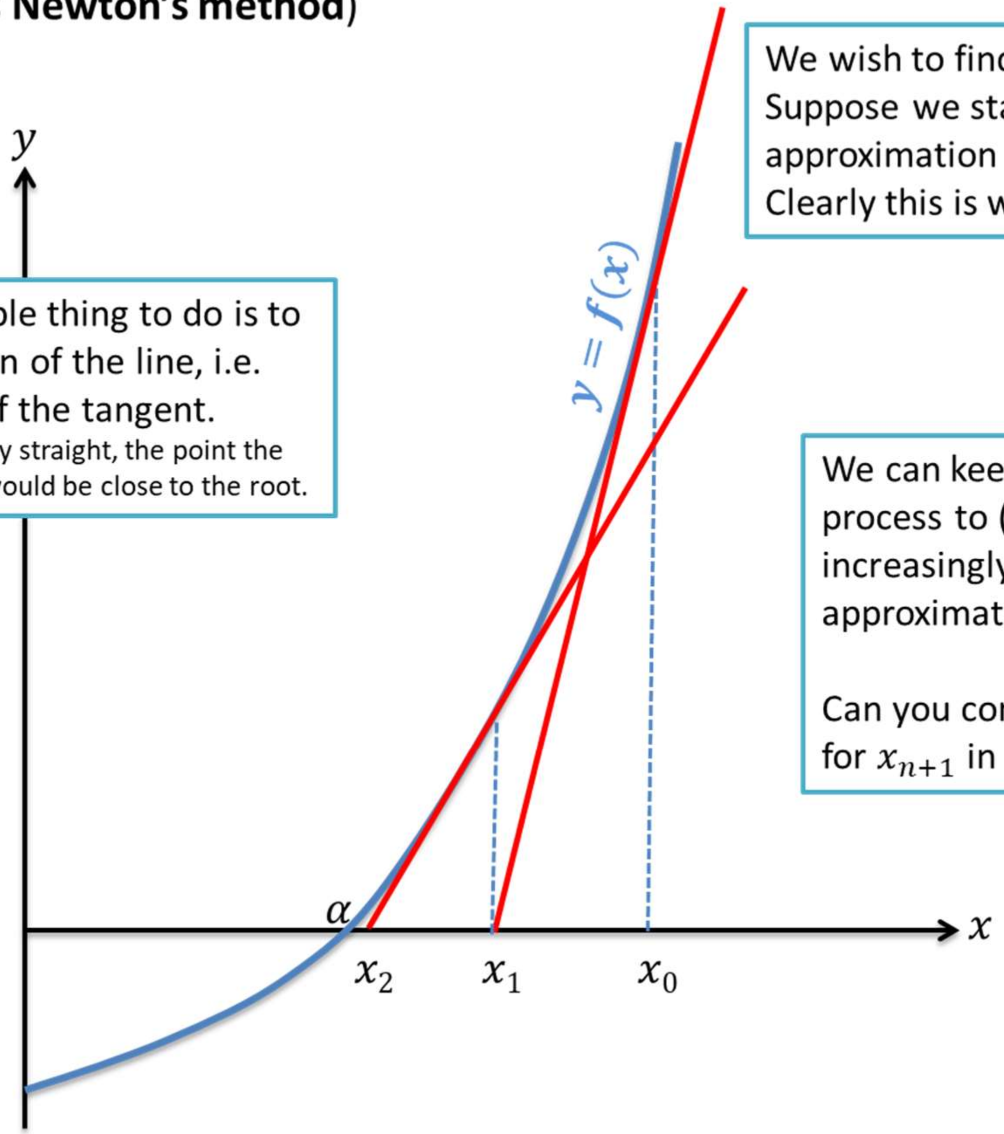
10.3) The Newton-Raphson method

The Newton-Raphson Process

(also known as **Newton's method**)

We wish to find the root α.
Suppose we start with the indicated approximation of the root, x_0.
Clearly this is well off the mark!

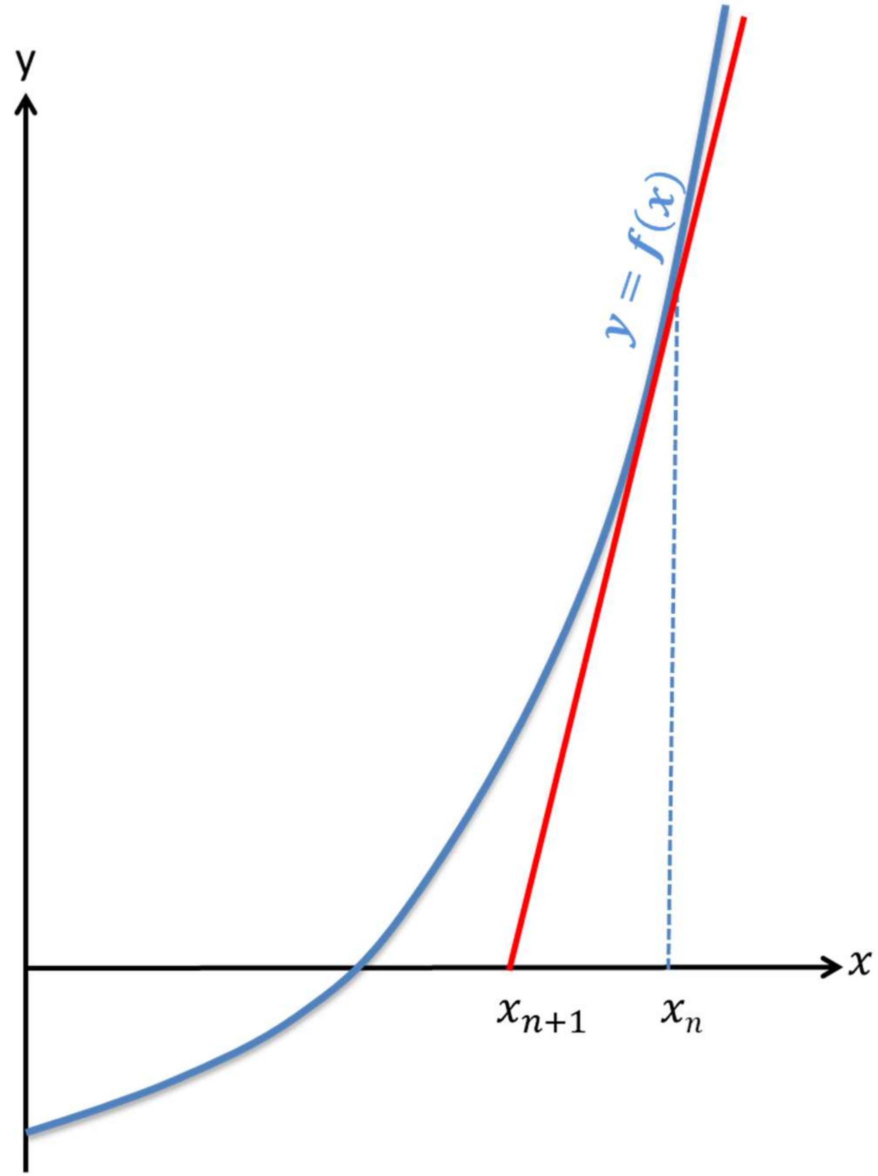
A seemingly sensible thing to do is to follow the direction of the line, i.e. use the gradient of the tangent.
If the line was reasonably straight, the point the tangent hits the x-axis would be close to the root.



We can keep repeating this process to (hopefully) get increasingly accurate approximations.

Can you come up with a formula for x_{n+1} in terms of x_n?

The Newton-Raphson Process



Using Year 1 coordinate geometry:

$$\boldsymbol{y - f(x_n) = f'(x_n)(x - x_n)}$$

But we're interested when $x = x_{n+1}$ and $y = 0$

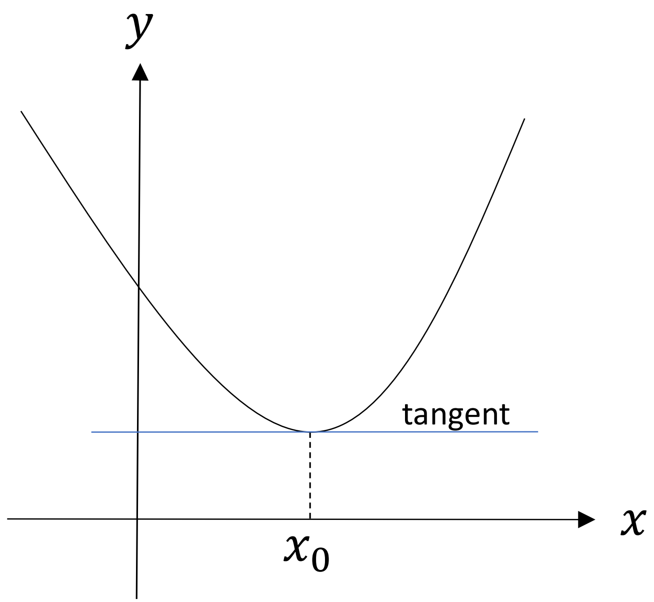
$$\boldsymbol{-f(x_n) = f'(x_n)(x_{n+1} - x_n)}$$

which rearranges to give:

Newton-Raphson Process:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

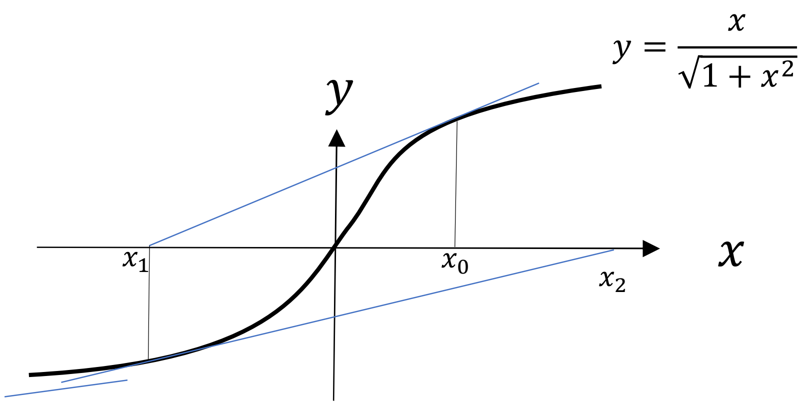
When does Newton-Raphson fail?



$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

If the starting value x_0 was the stationary point, then $f'(x_0) = 0$, resulting in a division by 0 in the above formula.

Graphically, it is because the tangent will never reach the x-axis.



Newton-Raphson also suffers from the same drawbacks as solving by iteration, in that it's possible for the values of x_i to **diverge**.

In this example, the x_i oscillate either side of 0, but gradually getting further away from $\alpha = 0$.

Worked example

Using the Newton-Raphson process, state the recurrence relation for the following functions:

$$f(x) = x^4 - 3$$

$$g(x) = \sec x$$

$$h(x) = x^2 + x + 3$$

Your turn

Using the Newton-Raphson process, state the recurrence relation for the following functions:

$$f(x) = x^3 - 2$$

$$g(x) = \tan x$$

$$h(x) = x^2 - x - 1$$

Worked example

Using three iterations of the Newton-Raphson process, starting with $x_0 = 0.5$, solve the equation

$$x = \sin x$$

Your turn

Using three iterations of the Newton-Raphson process, starting with $x_0 = 0.5$, solve the equation

$$x = \cos x$$

Worked example

$$f(x) = \frac{1}{3}x^4 - x^2 + 3x - 1$$

The equation $f(x) = 0$ has a root α in the interval $[-2, -3]$

Taking -2.5 as a first approximation to α, apply the Newton-Raphson process once to $f(x)$ to obtain a second approximation to α.

Give your answer to 2 decimal places.

Your turn

$$g(x) = \frac{1}{2}x^4 - x^3 + x - 3$$

The equation $g(x) = 0$ has a root β in the interval $[-2, -1]$

Taking -1.5 as a first approximation to β, apply the Newton-Raphson process once to $g(x)$ to obtain a second approximation to β.

Give your answer to 2 decimal places.

Worked example

$$f(x) = 11x^2 - \frac{3}{x^2}$$

The equation $f(x) = 0$ has a root α in the interval $[0, 1]$
Taking 0.4 as a first approximation to α, apply the Newton-Raphson process once to $f(x)$ to obtain a second approximation to α.
Give your answer to 3 decimal places.

Your turn

$$g(x) = 3x^2 - \frac{11}{x^2}$$

The equation $g(x) = 0$ has a root β in the interval $[1, 2]$
Taking 1.4 as a first approximation to β, apply the Newton-Raphson process once to $g(x)$ to obtain a second approximation to β.
Give your answer to 3 decimal places.

Worked example

$$f(x) = x^2 - 5x + 8$$

State why $x_0 = 2.5$ is not suitable to use as a first approximation to the roots of $f(x)$ when applying the Newton-Raphson method.

Your turn

$$f(x) = x^2 + 7x + 8$$

State why $x_0 = -3.5$ is not suitable to use as a first approximation to the roots of $f(x)$ when applying the Newton-Raphson method.

T.284: 10C al Qs, P.84: 10.3 all Qs

10.4) Applications to modelling

Notes

Worked Example

The price of a car in £s, x years after purchase, is modelled by the function

$$f(x) = 5000\,(0.58)^x - 100\sin x, \qquad x > 0$$

(a) Use the model to find the value, to the nearest hundred £s, of the car 5 years after purchase.

(b) Show that $f(x)$ has a root between 7 and 8.

(c) Taking 7.5 as a first approximation, apply the Newton-Raphson method once to $f(x)$ to obtain a second approximation for the time when the value of the car is zero. Give your answer to 3 decimal places.

(d) Criticise this model with respect to the value of the car as it gets older.

T.287: 10D all Qs, P.86: 10.4 all Qs

Extract from Formulae book

Numerical Methods

The trapezium rule: $\int_a^b y \, dx \approx \frac{1}{2} h\{(y_0 + y_n) + 2(y_1 + y_2 + \ldots + y_{n-1})\}$, where $h = \dfrac{b - a}{n}$

The Newton-Raphson iteration for solving $f(x) = 0 : x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

Past Paper Questions

Figure 8 shows a sketch of the curve C with equation $y = x^x,\ x > 0$

(a) Find, by firstly taking logarithms, the x coordinate of the turning point of C.

(Solutions based entirely on graphical or numerical methods are not acceptable.) **(5)**

The point $P(\alpha, 2)$ lies on C.

(b) Show that $1.5 < \alpha < 1.6$ **(2)**

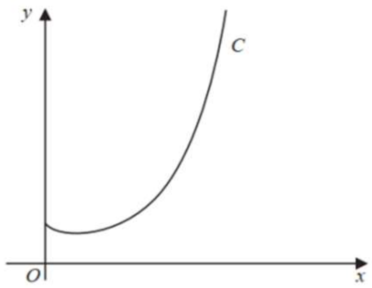


Figure 8

A possible iteration formula that could be used in an attempt to find α is

$$x_{n+1} = 2x_n^{\,1-x_n}$$

Using this formula with $x_1 = 1.5$

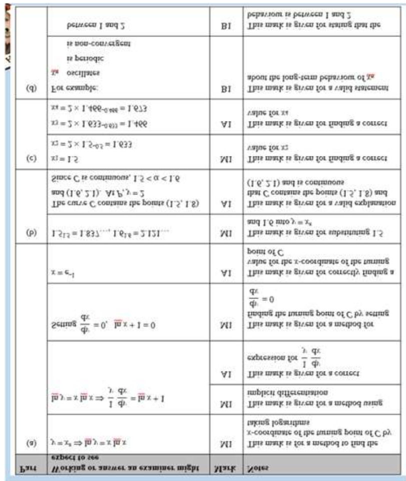
(c) find x_4 to 3 decimal places, **(2)**

(d) describe the long-term behaviour of x_n **(2)**

Exams

- Formula Booklet
- Past Papers
- Practice Papers
- past paper Qs by topic

Past paper practice by topic. Both new and old specification can be found via this link on hgsmaths.com



Summary of Key Points

Summary of key points

1 If the function $f(x)$ is continuous on the interval $[a, b]$ and $f(a)$ and $f(b)$ have opposite signs, then $f(x)$ has at least one root, x, which satisfies $a < x < b$.

2 To solve an equation of the form $f(x) = 0$ by an iterative method, rearrange $f(x) = 0$ into the form $x = g(x)$ and use the iterative formula $x_{n+1} = g(x_n)$.

3 The Newton–Raphson formula for approximating the roots of a function $f(x)$ is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

T.289: mixed ex, P.88: BSG