KING EDWARD VI
HANDSWORTH GRAMMAR
SCHOOL FOR BOYS

KING EDWARD VI
ACADEMY TRUST
BIRMINGHAM

Year 13
Pure Mathematics
P2 11 integration Booklet

HGS Maths

Dr Frost Course

Name: ______________________________

Class: ______________________________

Contents

11.1) Integrating standard functions

11.2) Integrating $f(ax+b)$

11.3) Using trigonometric identities

11.4) Reverse chain rule

11.5) Integration by substitution

11.6) Integration by parts

11.7) Partial fractions

11.8) Finding areas

11.8+) Finding areas: Areas under parametric curves

11.9) The trapezium rule

11.10) Solving differential equations

11.11) Modelling with differential equations

11.12) Integration as the limit of a sum

Extract from Formulae booklet
Past Paper Practice
Summary

Prior knowledge check

Prior knowledge check

1 Differentiate:

a $(2x - 7)^6$ **b** $\sin 5x$

c $e^{\frac{x}{3}}$ ← **Sections 9.1, 9.2, 9.3**

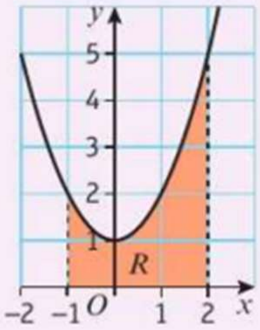
2 Given $f(x) = 8x^{\frac{1}{2}} - 6x^{-\frac{1}{2}}$

a find $\int f(x)\,dx$

b find $\int_4^9 f(x)\,dx$ ← **Year 1, Chapter 13**

3 Write $\dfrac{3x + 22}{(4x - 1)(x + 3)}$ as partial fractions. ← **Section 1.3**

4 Find the area of the region R bounded by the curve $y = x^2 + 1$, the x-axis and the lines $x = -1$ and $x = 2$.



← **Year 1, Chapter 13**

11.1) Integrating standard functions

There's certain results you should be able to integrate straight off, by just thinking about the opposite of differentiation. Have a go at these by thinking about what would differentiate to the function on the left:

y	$\int y\,dx$
x^n	
e^x	
$\frac{1}{x}$	
$\cos x$	
$\sin x$	
$\sec^2 x$	
$cosec\ x \cot x$	
$cosec^2 x$	
$\sec x \tan x$	

It's vital you remember this one.

Notes

Worked Example

Find:

$$\int 3\sin x - \frac{4}{x^2} + \sqrt[3]{x}\, dx$$

Worked Example

Find:

$$\int \frac{\sin x}{\cos^2 x}\ dx$$

Worked Example

Given that

$$\int_{a}^{5a} \frac{3x-1}{x}\, dx = \ln 2,$$

find the exact value of a.

11.2) Integrating $f(ax+b)$

For any expression where inner function is $ax + b$, integrate as before and $\div\ a$.

$$\int f'(ax+b)dx = \frac{1}{a}f(ax+b) + C$$

Notes

Worked Example

Find:

$$\int (6x + 1)^2 \, dx$$

$$\int (5x - 2)^3 \, dx$$

$$\int (4x + 3)^4 \, dx$$

Worked Example

Find:

$$\int \frac{1}{2(3x-4)^4}\,dx$$

$$\int \frac{1}{4(2-3x)^3}\,dx$$

Worked Example

Find:

$$\int (3x-1)^2$$

$$\int (3x-1)\,dx$$

$$\int \frac{1}{3x-1}\,dx$$

$$\int \frac{1}{(3x-1)^2}\,dx$$

Worked Example

Find:

$$\int \sin(6x + 1)\ dx$$

$$\int -\sin\left(\frac{x}{5} - 2\right)\ dx$$

$$\int \sin(3 - 4x)\ dx$$

Worked Example

Find:

$$\int \cos(6x + 1)\ dx$$

$$\int -\cos\left(\frac{x}{5} - 2\right)\ dx$$

$$\int \cos(3 - 4x)\ dx$$

Worked Example

Find:

$$\int \frac{1}{6x-1}\,dx$$

$$\int \frac{1}{\frac{1}{5}x+2}\,dx$$

$$\int \frac{1}{3-4x}\,dx$$

Worked Example

Find:

$$\int \sec^2(2x - 3)\, dx$$

$$\int 6\sec^2(5 - 4x)\, dx$$

Worked Example

Find:

$$\int \sec 5x \tan 5x \, dx$$

$$\int \sec \frac{x}{4} \tan \frac{x}{4} \, dx$$

Worked Example

Find:

$$\int e^{6x+1}\,dx$$

$$\int e^{\frac{1}{5}x-2}\,dx$$

$$\int e^{4-3x}\,dx$$

Worked Example

Find:

$$\int (e^x - 1)^3 \, dx$$

Exercise 11B

1 a $\int \sin(2x+1)\,dx =$?

c $\int 4e^{x+5}\,dx =$?

e $\int cosec^2 3x\,dx =$?

f $\int \sec 4x \tan 4x\,\,dx =$?

g $\int 3\sin\left(\frac{1}{2}x+1\right)dx =$?

h $\int cosec\,2x\cot 2x\,\,dx =$?

2 a $\int e^{2x} - \frac{1}{2}\sin(2x-1)\,dx$

$=$?

b $\int (e^x+1)^2\,dx =$?

c $\int \sec^2 2x\,(1+\sin 2x)\,dx =$?

d $\int \frac{3-2\,\text{co}\,\left(\frac{1}{2}x\right)}{\sin^2\left(\frac{1}{2}x\right)}dx =$?

e $\int e^{3-x} + \sin(3-x) + \cos(3-x)\,dx$

$=$?

3 a $\int \frac{1}{2x+1}\,dx =$?

b $\int \frac{1}{(2x+1)^2}\,dx =$?

c $\int (2x+1)^2\,dx =$?

d $\int \frac{3}{4x-1}\,dx =$?

f $\int \frac{3}{(1-4x)^2}\,dx =$?

h $\int \frac{3}{(1-2x)^3}\,dx =$?

j $\int \frac{5}{3-2x}\,dx =$?

4 a $\int 3\sin(2x+1) + \frac{4}{2x+1}\,\,dx$

$=$?

c $\int \frac{1}{\sin^2 2x} + \frac{1}{1+2x} + \frac{1}{(1+2x)^2}\,dx$

$=$?

11.3) Using trigonometric identities

Given:

Trigonometric identities

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

$\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B} \quad (A \pm B \neq (k + \tfrac{1}{2})\pi)$

$\sin A + \sin B = 2\sin\dfrac{A+B}{2}\cos\dfrac{A-B}{2}$

$\sin A - \sin B = 2\cos\dfrac{A+B}{2}\sin\dfrac{A-B}{2}$

$\cos A + \cos B = 2\cos\dfrac{A+B}{2}\cos\dfrac{A-B}{2}$

$\cos A - \cos B = -2\sin\dfrac{A+B}{2}\sin\dfrac{A-B}{2}$

Small angle approximations

$\sin\theta \approx \theta$

$\cos\theta \approx 1 - \dfrac{\theta^2}{2}$

$\tan\theta \approx \theta$

where θ is measured in radians

NOT given:

$$\begin{aligned}
\sin(2A) &= 2\sin(A)\cos(A) \\
\cos(2A) &= \cos^2(A) - \sin^2(A) \\
&= 1 - 2\sin^2(A) \\
&= 2\cos^2(A) - 1 \\
\tan(2A) &= \frac{2\tan(A)}{1 - \tan^2(A)}
\end{aligned}$$

Can derive from (the addition formulae above)

$$\sin^2\theta + \cos^2\theta = 1$$

$$1 + \tan^2\theta = \sec^2\theta$$

$$1 + \cot^2\theta = \operatorname{cosec}^2\theta$$

Notes

Worked Example

Find:

$$\int \sin^2 x \ dx$$

Worked Example

Find:

$$\int \cot^2 x \ \ dx$$

Worked Example

Find:

$$\int (\sec x - \tan x)^2 \, dx$$

Worked Example

Find:

$$\int \sin 5x \cos 5x \; dx$$

$$\int \sin\frac{x}{4}\cos\frac{x}{4}\; dx$$

Worked Example

Find:

$$\int (\sin x - \cos x)^2 dx$$

Worked Example

Find:

$$\int (\cos 2x + 1)^2 \, dx$$

Worked Example

Find:

$$\int \frac{(1+\sin x)^2}{\cos^2 x}\,dx$$

Worked Example

Find:

$$\int \frac{\cos 2x}{\sin^2 x}\, dx$$

Worked Example

Show that:

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \cos^2 x \ dx = \frac{\pi}{24} + \frac{2-\sqrt{3}}{8}$$

Worked Example

Find:

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \cos^2 3x \ dx$$

Worked Example

Find:

$$\int \sin^4 x \ dx$$

11.4) Reverse chain rule

There's certain more complicated expressions which look like the result of having applied the chain rule. I call this process '**<u>consider then scale</u>**':

1. **<u>Consider</u> some expression that will differentiate to something similar to it.**
2. **Differentiate, and adjust for any <u>scale</u> difference.**

$$\int x(x^2+5)^3\,dx \qquad \int \cos x \sin^2 x\;dx \qquad \int \frac{2x}{x^2+1}\,dx$$

Notes

Quickfire

In your head!

Tip: If there's as power around the whole denominator, DON'T use ln: re-express the expression as a product.
e.g. $x(x^2+5)^{-3}$

$$\int \frac{4x^3}{x^4-1}\,dx =$$

$$\int \frac{\cos x}{\sin x + 2}\,dx =$$

$$\int \cos x\ e^{\sin x}\,dx =$$

$$\int \cos x\,(\sin x - 5)^7\,dx =$$

$$\int x^2(x^3+5)^7 =$$

Not in your head…

$$\int \frac{x}{(x^2+5)^3}\,dx =$$

Worked Example

618d: Integrate other standard functions given in the form $f(ax+b)$ including $\frac{1}{ax+b}$

Find the following integral.

$$\int 4 \sec x \tan x \, \mathrm{d}x$$

Worked Example

618e: Integrate expressions given in the form $kf'(x)[f(x)]^n$ by inspection.

Find the following integral.

$$\int x^5(3x^6+5)^9dx$$

Worked Example

618h: **Integrate** $\sin^2 x$, $\cos^2 x$, $\tan^2 x$, $\cot^2 x$

Find $\int 19\cos^2 x\,\mathrm{d}x$

Worked Example

Find:

$$\int \frac{3x^2 + 5}{\sqrt{x^3 + 5x - 2}}\, dx$$

Worked Example

Find:

$$\int \frac{\sec^2 x}{\tan x - 3}\, dx$$

Worked Example

Find:

$$\int \sin x \ e^{\cos x} \, dx$$

$$\int \sec^2 x \ e^{\tan} \quad dx$$

Worked Example

Find:

$$\int \sin x \, (\cos x - 1)^5 \, dx$$

$$\int \sec^2 x \, (\tan x + 3)^6 \, dx$$

Worked Example

Find:

$$\int \sec^2 x \tan x \ dx$$

Worked Example

Find:

$$\int \frac{cosec^2 x}{(3-\cot x)^4}\,dx$$

Worked Example

Find:

$$\int \frac{cosec^2 x}{(3-\cot x)^4}\,dx$$

Worked Example

Find:

$$\int \tan x \, dx$$

11.5) Integration by substitution

For some integrations involving a complicated expression, we can make a substitution to turn it into an equivalent integration that is simpler. We wouldn't be able to use 'reverse chain rule' on the following:

Q Use the substitution $u = 2x + 5$ to find $\int x\sqrt{2x+5}\ dx$

The aim is to completely remove any reference to x, and replace it with u. We'll have to work out x and dx so that we can replace them.

STEP 1: Using substitution, work out x and dx (or variant)

$$\frac{du}{dx} = 2 \quad \rightarrow \quad dx = \frac{1}{2}du$$

$$x = \frac{u-5}{2}$$

Tip: Be careful about ensuring you reciprocate when rearranging.

STEP 2: Substitute these into expression.

$$\int x\sqrt{2x+5}\ dx = \int \frac{u-5}{2}\sqrt{u}\frac{1}{2}du = \frac{1}{4}\int \sqrt{u}(u-5)\ du$$

$$= \frac{1}{4}\int u^{\frac{3}{2}} - 5u^{\frac{1}{2}}$$

Tip: If you have a constant factor, factor it out of the integral.

STEP 3: Integrate simplified expression.

$$= \frac{1}{4}\left(\frac{2}{5}u^{\frac{5}{2}} - \frac{10}{3}u^{\frac{3}{2}}\right) + C$$

STEP 4: Write answer in terms of x.

$$= \frac{(2x+5)^{\frac{5}{2}}}{10} - \frac{5(2x+5)^{\frac{3}{2}}}{6} + C$$

Notes

Worked Example

619a: Integrate expressions given in the form $x(ax+b)^n$ using a substitution.

Using $u = 5x + 1$, find an expression in terms of x for:

$$\int 6x(5x+1)^3 dx$$

How can we tell what substitution to use?

In Edexcel you will usually be given the substitution!
However in some other exam boards, and in STEP, you often aren't.
There's no hard and fast rule, but it's often helpful to replace to replace expressions inside roots, powers or the denominator of a fraction.

Sensible substitution:

$$\int \cos x \sqrt{1 + \sin x}\, dx \qquad \boldsymbol{u} =$$

$$\int 6x\, e^{x^2} dx \qquad \boldsymbol{u} =$$

$$\int \frac{xe^x}{1+x} dx \qquad \boldsymbol{u} =$$

$$\int e^{\frac{1-x}{1+x}} dx \qquad \boldsymbol{u} =$$

Worked Example

Find:

$$\int_0^{\frac{\pi}{2}} \cos x \sin x \,(2+\sin x)^4 \, dx$$

using the substitution $u = \sin x + 2$

Worked Example

Find:

$$\int \frac{3\sin 2x}{2+\sin x}\,dx$$

using the substitution $u = 2 + \sin x$

Worked Example

Calculate:

$$\int_{0}^{\frac{\pi}{2}} \sin x \sqrt{2 + \cos x} \; dx$$

using the substitution $u = \cos x + 2$

Worked Example

Use the substitution $u = \sqrt{x} - 1$ to evaluate:

$$\int_{36}^{49} \frac{1}{\sqrt{x} - 1}\, dx$$

Worked Example

A finite region is bounded by the curve with equation $y = x^3 \ln(x^2+3)$, the x-axis and the lines $x = 0$ and $x = \sqrt{5}$.

Use the substitution $u = x^2 + 3$ to show that the area of R is $\frac{1}{2}\int_3^8 (u-3)\ln u \, du$

Worked Example

Using integration by substitution, prove that:

$$-\int \frac{1}{\sqrt{1-x^2}}\,dx = \arccos x + c$$

Worked Example

Use the substitution $u = \cos x$ to evaluate

$$\int_0^{\frac{\pi}{3}} \sin^3 x \cos^2 x \, dx$$

Worked Example

Use the substitution $x = \cos u$ to evaluate

$$\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} x^2\sqrt{1-x^2}\,dx$$

11.6) Integration by parts

$$\int x \cos x \ dx = ?$$

Just as the Product Rule was used to **differentiate the product** of two expressions, we can often use 'Integration by Parts' to **integrate a product.**

To integrate by parts:

$$\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} \ dx$$

$$\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx} \quad \rightarrow \quad \int \frac{d}{dx}(uv).\,dx = \int v\frac{du}{dx}.\,dx + \int u\frac{dv}{dx}.\,dx$$

$$uv = \int v\frac{du}{dx}.\,dx + \int u\frac{dv}{dx}.\,dx$$

Notes

Worked Example

Find:

$$\int x \sin x \; dx$$

Worked Example

Find:

$$\int x^2 \cos x \ dx$$

Worked Example

Find:

$$\int x^2 e^{-x} \, dx$$

By parts twice (or even thrice...)

Find $\int x^2 e^x \, dx$

Worked Example

Find:

$$\int x^3 \ln x \ dx$$

Worked Example

Evaluate:

$$\int_1^4 \ln x \, dx$$

Worked Example

Find:

$$\int e^x \cos x \; dx$$

Worked Example

Evaluate:

$$\int_0^{\frac{\pi}{2}} x \cos x \; dx$$

11.7) Partial fractions

Notes

Worked Example

Find:

$$\int \frac{x+5}{(x-1)(x+2)}\, dx$$

Worked Example

Find:

$$\int \frac{3x + 15 - 4x^2}{(2x + 1)(x - 2)^2}\ dx$$

Worked Example

Evaluate:

$$\int_0^2 \frac{8x^2 + 34x + 20}{(2x+1)(x+1)(x+3)}$$

Worked Example

Find:

$$\int \frac{4}{x^2 - 4}\, dx$$

Worked Example

Find:

$$\int \frac{x^2}{x-1}\,dx$$

Worked Example

Find:

$$\int \frac{4x^2 - 2x - 18}{4x^2 - 9}\, dx$$

11.8) Finding areas

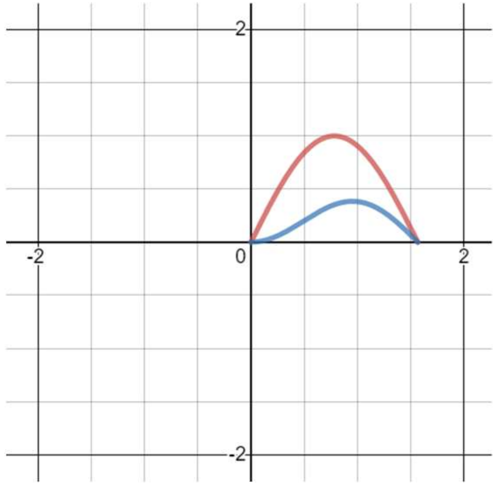
Notes

Worked Example

A finite region is bound by the curve
$y = \frac{3}{\sqrt{9+4x}}$, the x-axis, and the lines $x = 0$ and $x = 4$. Use integration to find the area of the region.

Worked Example

A finite region is bound between the curves $y = \sin 2x$ and $y = cos\, x \sin^2 x$ where $0 \leq x \leq \frac{\pi}{2}$. Use integration to find the area of the region.



11.8+) Finding areas: Areas under parametric curves

Area: $$\int y\, dx = \int y \frac{dx}{dt}\, dt$$

Notes

Worked Example

Determine the area bound between the curve with parametric equations
$x = t^2$ and $y = t - 1$, the x-axis, and the lines $x = 0$ and $x = 5$.

Worked Example

The curve C has parametric equations

$$x = t(2+t), \qquad y = \frac{1}{2+t}, \qquad t \geq 0$$

Find the exact area of the region, bounded by C, the x-axis and the lines $x = 0$ and $x = 8$.

Worked Example

The curve C has parametric equations

$$x = 1 - \frac{1}{4}t, \qquad y = 4^t - 1, \qquad t \geq 0$$

A finite region is bounded by the curve C, the x-axis and the line $x = -1$. Find the exact area of this region.

Exercise 11.X

This exercise is not in the current version of the Pearson textbooks as the content was added later. I have temporarily included the exercise subsequently produced by Pearson.

ANSWERS

1 ?

2 ?

3 ?

4 ?

5 ?

6 ?

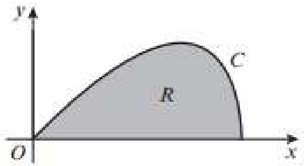
7 ?

(P) **1** The curve C has parametric equations $x = t^3$, $y = t^2$, $t \geqslant 0$. Show that the exact area of the region bounded by the curve, the x-axis and the lines $x = 0$ and $x = 4$ is $k\sqrt[3]{2}$, where k is a rational constant to be found.

(E/P) **2** The curve C has parametric equations

$$x = \sin t,\ y = \sin 2t,\ 0 \geqslant t \geqslant \frac{\pi}{2}$$

The finite region R is bounded by the curve and the x-axis. Find the exact area of R. **(6 marks)**



(E/P) **3** This graph shows part of the curve C with parametric equations $x = (t+1)^2$, $y = \frac{1}{2}t^3 + 3$, $t \geqslant -1$

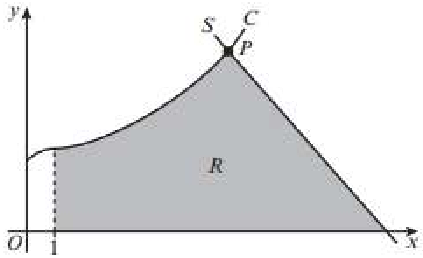
P is the point on the curve where $t = 2$.

The line S is the normal to C at P.

a Find an equation of S. **(5 marks)**

The shaded region R is bounded by C, S, the x-axis and the line with equation $x = 1$.

b Using integration, find the area of R. **(5 marks)**

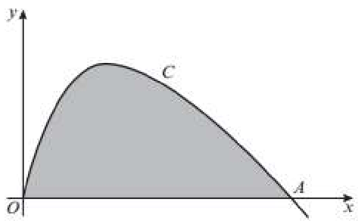


(E/P) **4** The diagram shows the curve C with parametric equations

$$x = 3t^2,\ y = \sin 2t,\ t \geqslant 0.$$

a Write down the value of t at the point A where the curve crosses the x-axis. **(1 mark)**

b Find, in terms of π, the exact area of the shaded region bounded by C and the x-axis. **(6 marks)**



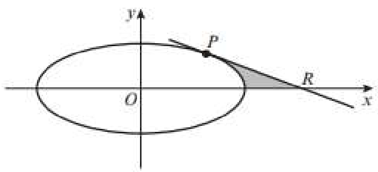
(E/P) **5** The curve shown has parametric equations

$$x = 5\cos\theta,\ y = 4\sin\theta,\ 0 \leqslant \theta < 2\pi$$

a Find the gradient of the curve at the point P at which $\theta = \frac{\pi}{4}$ **(3 marks)**

b Find an equation of the tangent to the curve at the point P. **(3 marks)**

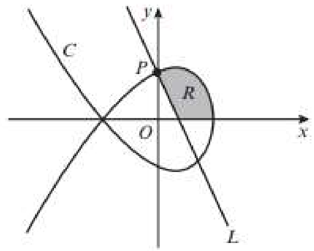
c Find the exact area of the shaded region bounded by the tangent PR, the curve and the x-axis. **(6 marks)**



(E/P) **6** The curve C has parametric equations

$$x = 1 - t^2,\ y = 2t - t^3,\ t \in \mathbb{R}$$

The line L is a normal to the curve at the point P where the curve intersects the positive y-axis. Find the exact area of the region R bounded by the curve C, the line L and the x-axis, as shown on the diagram. **(7 marks)**



(E/P) **7** The curve shown in the diagram has parametric equations

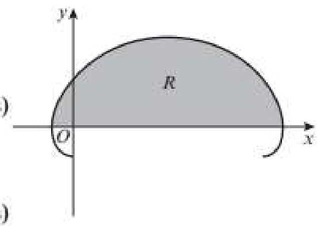
$$x = t - 2\sin t,\ y = 1 - 2\cos t,\ 0 \leqslant t \leqslant 2\pi$$

a Show that the curve crosses the x-axis where $t = \frac{\pi}{3}$ and $t = \frac{5\pi}{3}$ **(3 marks)**

The finite region R is enclosed by the curve and the x-axis, as shown shaded in the diagram.

b Show that the area R is given by $\int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} (1 - 2\cos t)^2\, dt$ **(3 marks)**

c Use this integral to find the exact value of the shaded area. **(4 marks)**



11.9) The trapezium rule

In general:

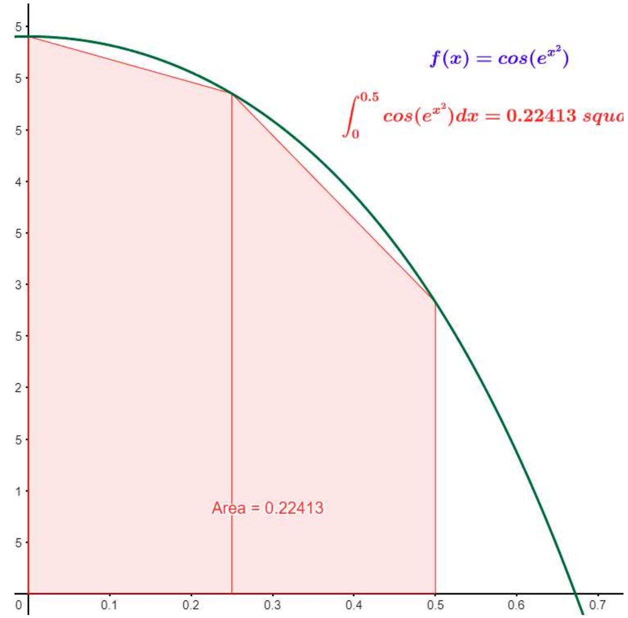
width of each trapezium

$$\int_a^b y\,dx \approx \frac{h}{2}\left(y_0 + 2(y_1 + \cdots + y_{n-1}) + y_n\right)$$

Area under curve

is approximately

[Trapezium rule for concave downwards, – GeoGebra]



Notes

Worked Example

$$I = \int_0^{\frac{\pi}{3}} \sec x \ dx$$

Use the trapezium rule with two strips to estimate I.

Worked Example

$$I = \int_0^{\frac{\pi}{2}} \sin x \; dx$$

a) Use the trapezium rule with four strips to estimate I

b) State, with a reason, whether your approximation is an underestimate or an overestimate

c) Find the percentage error of your estimate to the exact value of I

d) Give one way the trapezium rule can be used to give a more accurate approximation

11.10) Solving differential equations

Differential equations are equations involving a mix of variables and derivatives, e.g. y, x and $\frac{dy}{dx}$.

'<u>Solving</u>' these equations means to get y in terms of x (with no $\frac{dy}{dx}$).

Key Points

- Get y on to LHS by dividing (possibly factorising first).
- If after integrating you have ln on the RHS, make your constant of integration $\ln k$.
- Be sure to combine all your ln's together, e.g.:

$$2\ln|x+1| - \ln|x| \quad \rightarrow \quad \ln\left|\frac{(x+1)^2}{x}\right|$$

- Sub in boundary conditions to work out your constant – better to do sooner rather than later.
- Exam questions - partial fractions combined with differential equations.

Worked Example

Find the general solution to:

$$\frac{dy}{dx} = xy - y$$

Worked Example

Find the general solution to:

$$(1 - x^2)\frac{dy}{dx} = x\cot y$$

Worked Example

Find the particular solution to:

$$\frac{dy}{dx} = -\frac{3(y+2)}{(2x-1)(x-2)}$$

given that $x = 4$ when $y = 5$

Worked Example

Find the particular solution to:

$$\frac{dy}{dx} = -\frac{5}{y\sin^2 x}$$

given that $y = 4$ at $x = \frac{\pi}{4}$

Worked Example

Find the particular solution to:

$$\frac{dy}{dx} = xy\cos x$$

given that $y = 1$ at $x = \frac{\pi}{2}$

11.11) Modelling with differential equations

Worked Example

The rate of increase of a population P of microorganisms at time t, in hours, is given by

$$\frac{dP}{dt} = 6P, k > 0$$

Initially the population was of size 4.

a) Find a model for P in the form $P = Ae^{6t}$

b) Find, to the nearest hundred, the size of the population at time $t = 4$

c) Find the time at which the population will be 10000 times its starting value.

d) State one limitation of this model for large values of t

Worked Example

Water in a manufacturing plant is held in a large cylindrical tank of diameter 10m. Water flows out of the bottom of the tank through a tap at a rate proportional to the cube root of the volume.

(a) Show that t minutes after the tap is opened,

$\frac{dh}{dt} = -k\sqrt[3]{h}$ for some constant k.

(b) Show that the general solution of this differential equation may be written $h = (P - Qt)^{\frac{3}{2}}$, where P and Q are constants.

Initially the height of the water is 64m. 21 minutes later, the height is 27m.

(c) Find the values of the constants P and Q.

(d) Find the time in minutes when the water is at a depth of 8m.

Worked Example

A fluid reservoir initially containers 10000 litres of unpolluted fluid.
The reservoir is leaking at a constant rate of 200 litres per hour and it is suspected that contaminated fluid flows into the reservoir at a constant rate of 300 litres per day.
The contaminated fluid contains 4 grams of contaminant in every litre of fluid.
It is assumed that the contaminant instantly disperses throughout the reservoir upon entry.
Given that there are x grams of contaminant in the reservoir after t days,

(a) Show that the situation can be modelled by the differential equation

$$\frac{dx}{dt} = 1200 - \frac{2x}{100 + t}$$

(b) Hence find the number of grams of contaminant in the tank after 7 days.

(c) Explain how the model could be refined.

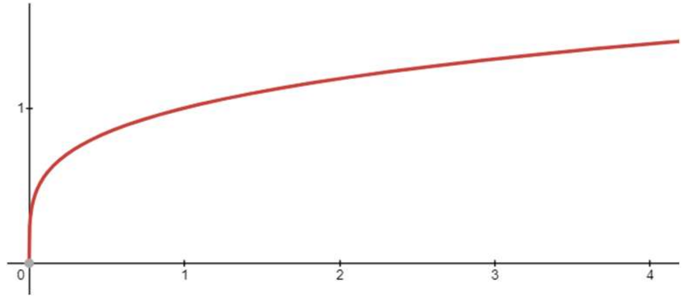
11.12) Integration as the limit of a sum

$$\lim_{\delta x \to 0} \sum_{x=a}^{b} f(x)\delta x = \int_a^b f(x)\,dx$$

Notes

Worked Example

The diagram shows a sketch of the curve with equation $y = \sqrt[4]{x}, x > 0$.



The area under the curve may be thought of as a series of thin strips of height y and width δx.
Calculate to 4 significant figures:

$$\lim_{\delta x \to 0} \sum_{2}^{3} \sqrt[4]{x}\, \delta x$$

Worked Example

Calculate to four significant figures:

$$\lim_{\delta x \to 0} \sum_{5}^{6} \cos x \, \delta x$$

Extract from Formulae book

Integration (+ constant)

$\mathrm{f}(x)$	$\int \mathrm{f}(x)\,\mathrm{d}x$
$\sec^2 kx$	$\frac{1}{k}\tan kx$
$\tan kx$	$\frac{1}{k}\ln\lvert\sec kx\rvert$
$\cot kx$	$\frac{1}{k}\ln\lvert\sin kx\rvert$
$\operatorname{cosec} kx$	$-\frac{1}{k}\ln\lvert\operatorname{cosec} kx + \cot kx\rvert,\quad \frac{1}{k}\ln\lvert\tan(\tfrac{1}{2}kx)\rvert$
$\sec kx$	$\frac{1}{k}\ln\lvert\sec kx + \tan kx\rvert,\quad \frac{1}{k}\ln\lvert\tan(\tfrac{1}{2}kx + \tfrac{1}{4}\pi)\rvert$

$$\int u\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x = uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x$$

Numerical Methods

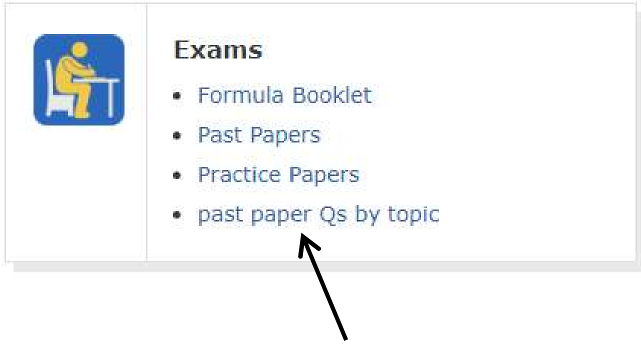
The trapezium rule: $\int_a^b y\,\mathrm{d}x \approx \tfrac{1}{2}h\{(y_0 + y_n) + 2(y_1 + y_2 + \ldots + y_{n-1})\}$, where $h = \dfrac{b-a}{n}$

4. Given that a is a positive constant and

$$\int_a^{2a} \frac{t+1}{t}\,\mathrm{d}t = \ln 7$$

show that $a = \ln k$, where k is a constant to be found.

(4)



Past paper practice by topic. Both new and old specification can be found via this link on hgsmaths.com

3(a)	Attempts $(x-2)^2+(y+5)^2=\ldots$	M1	1.1b
	Centre $(2, -5)$	A1	1.1b
		(2)	
(b)	Sets $k+2^2+5^2>0$	M1	2.2a
	$\Rightarrow k>-29$	A1ft	1.1b
		(2)	
			(4 marks)

Summary of Key Points

$f(x)$	How to deal with it	$\int f(x)\,dx$ (+constant)	Formula booklet?
$\boldsymbol{\sin x}$	Standard result	$-\cos x$	No
$\boldsymbol{\cos x}$	Standard result	$\sin x$	No
$\boldsymbol{\tan x}$	In formula booklet, but use $\int \frac{\sin x}{\cos x}\,dx$ which is of the form $\int \frac{kf'(x)}{f(x)}\,dx$	$\ln\lvert\sec x\rvert$	Yes
$\boldsymbol{\sin^2 x}$	For both $\sin^2 x$ and $\cos^2 x$ use identities for $\cos 2x$ $\cos 2x = 1 - 2\sin^2 x$ $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$	$\frac{1}{2}x - \frac{1}{4}\sin 2x$	No
$\boldsymbol{\cos^2 x}$	$\cos 2x = 2\cos^2 x - 1$ $\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$	$\frac{1}{2}x + \frac{1}{4}\sin 2x$	No
$\boldsymbol{\tan^2 x}$	$1 + \tan^2 x \equiv \sec^2 x$ $\tan^2 x \equiv \sec^2 x - 1$	$\tan x - x$	No
$\boldsymbol{cosec\ x}$	Would use substitution $u = cosec\ x + \cot x$, but too hard for exam.	$-\ln\lvert cosec\ x + \cot x\rvert$	Yes
$\boldsymbol{\sec x}$	Would use substitution $u = \sec x + \tan x$, but too hard for exam.	$\ln\lvert\sec x + \tan x\rvert$	Yes
$\boldsymbol{\cot x}$	$\int \frac{\cos x}{\sin x}\,dx$ which is of the form $\int \frac{f'(x)}{f(x)}\,dx$	$\ln\lvert sin\ x\rvert$	Yes

Summary of Key Points

$f(x)$	How to deal with it	$\int f(x)dx$ (+constant)	Formula booklet?
$\boldsymbol{cosec^2 x}$	By observation.	$-\boldsymbol{\cot x}$	No!
$\boldsymbol{sec^2 x}$	By observation.	$\tan x$	Yes (but memorise)
$\boldsymbol{cot^2 x}$	$1+\cot^2 x \equiv cosec^2 x$	$-\cot x - x$	No
$\boldsymbol{\sin 2x \cos 2x}$	For any product of sin and cos with same coefficient of x, use double angle. $\sin 2x \cos 2x \equiv \frac{1}{2}\sin 4x$	$-\frac{1}{8}\cos 4x$	No
$\boldsymbol{\frac{1}{x}}$		$\ln x$	No
$\boldsymbol{\ln x}$	Use IBP, where $u=\ln x, \frac{dv}{dx}=\ln x$	$x\ln x - x$	No
$\boldsymbol{\frac{x}{x+1}}$	Use algebraic division. $\frac{x}{x+1} \equiv 1-\frac{1}{x+1}$	$x-\ln\|x+1\|$	
$\boldsymbol{\frac{1}{x(x+1)}}$	Use partial fractions.	$\ln\|x\|-\ln\|x+1\|$	

Summary of Key Points

$f(x)$	How to deal with it	$\int f(x)\,dx$ (+constant)
$\dfrac{4x}{x^2+1}$	Reverse chain rule. Of form $\int \frac{kf'(x)}{f(x)}$	$2\ln\lvert x^2+1\rvert$
$\dfrac{x}{(x^2+1)^2}$	Power around denominator so NOT of form $\int \frac{kf'(x)}{f(x)}$. Rewrite as product. $x(x^2+1)^{-2}$ Reverse chain rule (i.e. "Consider $y = (x^2+1)^{-1}$" and differentiate)	$-\frac{1}{2}(x^2+1)^{-1}$
e^{2x+1} $\dfrac{1}{1-3x}$	For any function where 'inner function' is linear expression, divide by coefficient of x	$\frac{1}{2}e^{2x+1}$ $-\frac{1}{3}\ln\lvert 1-3x\rvert$
$x\sqrt{2x+1}$	IBP or use sensible substitution. $u = 2x+1$ or even better, $u^2 = 2x+1$.	$\frac{1}{15}(2x+1)^{\frac{3}{2}}(3x-1)$
$\sin^5 x\ \cos x$	Reverse chain rule.	$\frac{1}{6}\sin^6 x$