KING EDWARD VI
HANDSWORTH GRAMMAR
SCHOOL FOR BOYS

KING EDWARD VI
ACADEMY TRUST
BIRMINGHAM

Year 13
Pure Mathematics
P2 2 Functions

HGS Maths

Dr Frost Course

Name: ______________________________

Class: ______________________________

Contents

Prior knowledge check

Prior knowledge check

1 Make y the subject of each of the following:

a $5x = 9 - 7y$ **b** $p = \frac{2y + 8x}{5}$

c $5x - 8y = 4 + 9xy$ ← **GCSE Mathematics**

2 Write each expression in its simplest form.

a $(5x - 3)^2 - 4$ **b** $\frac{1}{2(3x - 5) - 4}$

c $\frac{\frac{x+4}{x+2} + 5}{\frac{x+4}{x+2} - 3}$ ← **GCSE Mathematics**

3 Sketch each of the following graphs. Label any points where the graph cuts the x- or y-axis.

a $y = e^x$ **b** $y = x(x + 4)(x - 5)$

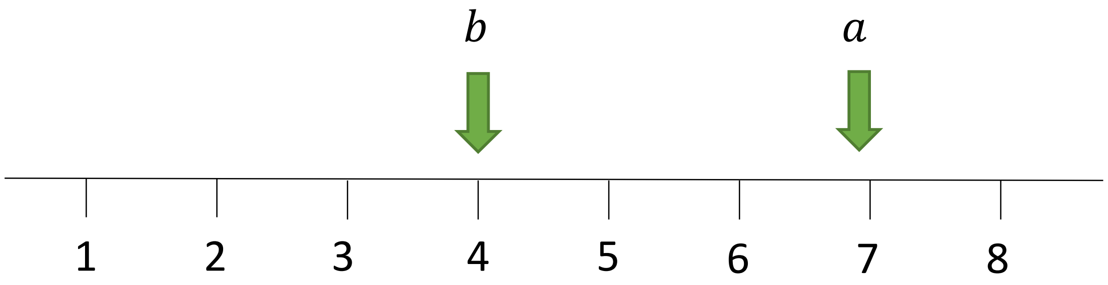
c $y = \sin x, 0 \leqslant x \leqslant 360°$ ← **Year 1**

4 $f(x) = x^2 - 3x$. Find the values of:

a $f(7)$ **b** $f(3)$ **c** $f(-3)$ ← **Year 1**

2.1) The modulus function

The modulus of a number a, written $|a|$, is its **non-negative** numerical value.
e.g. $|6| = 6$ and $|-7.1| = 7.1$



The modulus function is particularly useful in expressing a **difference**. We generally like to quote differences as positive values, but $b - a$ may be negative if b is smaller than a. By using $|b - a|$, we get round this problem!

More fundamentally, the modulus of a value gives us its '**magnitude**', i.e. size; from Mechanics, you should also be used to the notion the distances and speeds are quoted as positive values.
And in Pure Year 1 we saw the same notation used for vectors: $|\boldsymbol{a}|$ gives us the magnitude/length of the vector $\boldsymbol{a}$. It's the same function!

Notes

Worked example	Your turn
If $f(x) = \lvert 4x + 5 \rvert - 6$, find:	If $f(x) = \lvert 2x - 3 \rvert + 1$, find:
a) $f(5)$	a) $f(5)$
b) $f(-2)$	b) $f(-2)$
c) $f(1)$	c) $f(1)$

T.26 : 2A Qs 1-3, P.6: 2.1 Qs 1,2

Worked Example

570c: **Sketch graphs given in the form** $y = |ax + b|$

Plot the graph of $y = \left|-\frac{3}{2}x + 1\right|$ from $x = -2$ to $x = 4$

T.26 : 2A Qs 4, P.6: 2.1 Qs 3

Worked Example

Solve:

$$|3x - 2| = 7$$

Your Turn

Solve:

$$|2x - 3| = 5$$

$$x = -1, x = 4$$

Your Turn

Solve:

a) $|5x - 2| = 3 - \frac{1}{3}x$

b) $hence\ solve$: $|5x - 2| < 3 - \frac{1}{3}x$

T.26 : 2A Qs 11-13, P.6: 2.1 Qs 5-6

Worked Example

Solve:

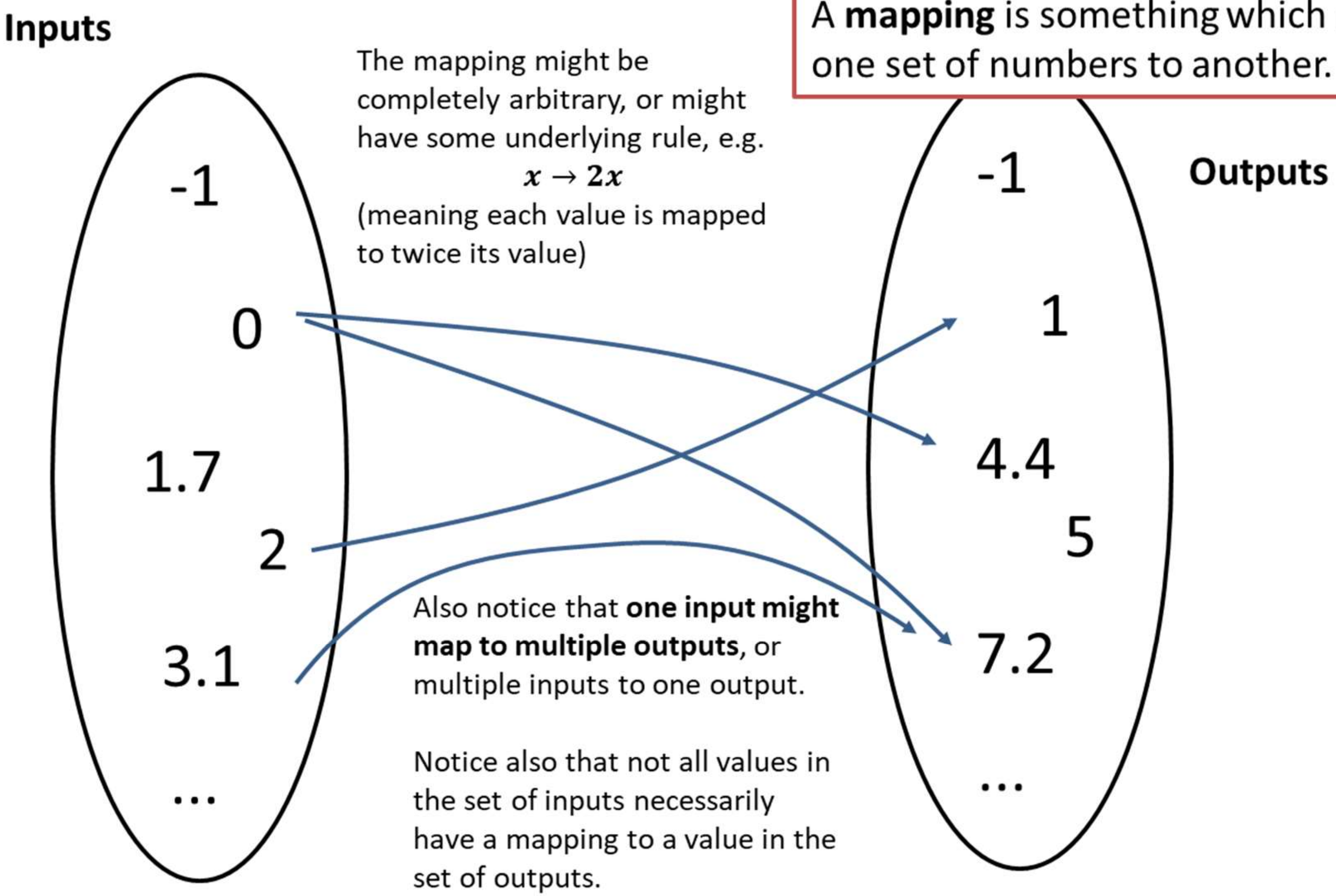
$$|x + 3| = 5x + 2$$

T.26 : 2A Qs 5+, P.6: 2.1 Qs 4,7+

2.2) Functions and mappings

What is a mapping?

A **mapping** is something which maps one set of numbers to another.



The mapping might be completely arbitrary, or might have some underlying rule, e.g.

$$x \to 2x$$

(meaning each value is mapped to twice its value)

Also notice that **one input might map to multiple outputs**, or multiple inputs to one output.

Notice also that not all values in the set of inputs necessarily have a mapping to a value in the set of outputs.

The **domain** is the set of possible inputs.

The **range** is the set of possible outputs.

Notes

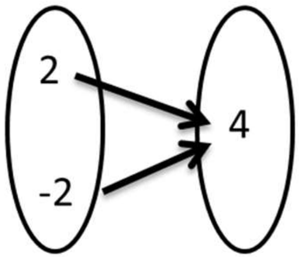
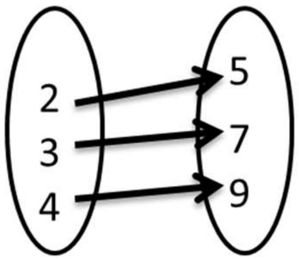
A function is: a mapping such that every element of the domain is mapped to exactly one element of the range.

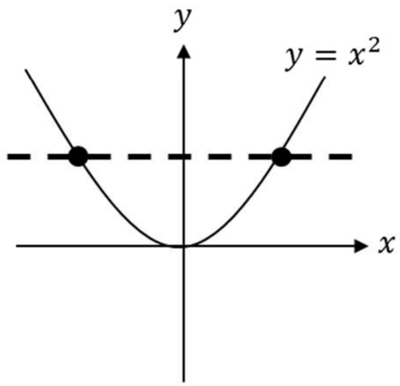
Notation: $f(x) = 2x + 1$ $\qquad f: x \to 2x + 1$

$f(x)$ refers to the **output** of the function.

One-to-one vs Many-to-one

While functions permit an input only to be mapped to one output, there's nothing stopping multiple different inputs mapping to the same output.

Type	Description	Example
Many-to-one function	Multiple inputs can map to the same output. 	$f(x) = x^2$ e.g. $f(2) = 4$ $f(-2) = 4$
One-to-one function	Each output has one input and vice versa. 	$f(x) = 2x + 1$



You can use the 'horizontal ray test' to see if a function is one-to-one or many-to-one.

Summary of Domain/Range

It is important that you can identify the range for **common graphs**, **using a suitable sketch**:

$f(x) = x^2, \qquad x \in \mathbb{R}$
Range: $\boldsymbol{f(x) \geq 0}$

$f(x) = \frac{1}{x}, \qquad x \in \mathbb{R}, x \neq 0$
Range: $\boldsymbol{f(x) \neq 0}$

$f(x) = \ln x, \qquad x \in \mathbb{R}, x > 0$
Range: $\boldsymbol{f(x) \in \mathbb{R}}$

$f(x) = e^x, \qquad x \in \mathbb{R},$
Range: $\boldsymbol{f(x) > 0}$

$f(x) = x^2 + 2x + 9, \qquad x \in \mathbb{R}$
Range: $\boldsymbol{f(x) \geq 8}$

Be careful in noting the domain – it may be 'restricted', which similarly restricts the range. Again, use a sketch!

$f(x) = x^2, \qquad x \in \mathbb{R}, -1 \leq x \leq 4$
Range: $\boldsymbol{0 \leq f(x) \leq 16}$

Notes

Exercise

State whether:

- the mapping is one-to-one, many-to-one, or one-to-many
- the mapping is a function

$f(x) = 2x - 3, \quad x \in \mathbb{R}$

$p(x) = x^3, \quad x \in \mathbb{R}$

$g(x) = x^2, \quad x \in \mathbb{R}$

$q(x) = \left|\frac{1}{x}\right|, \quad x \in \mathbb{R}$

$h(x) = \frac{1}{x}, \quad x \in \mathbb{R}$

$r(x) = \sqrt{x}, x \in \mathbb{R}, x \geq 0, \quad x \in \mathbb{R}$

$i(x) = \sqrt{x}, \quad x \in \mathbb{R}$

$s(x) = \pm\sqrt{x}, x \in \mathbb{R}, x \geq 0$

T.30 : 2B Qs 1-2, P.7: 2.2 Qs 1

Worked example

Write down the largest possible domain for:

$$f(x) = \frac{1}{x-3}$$

$$g(x) = \frac{2}{7x-21}$$

$$h(x) = \frac{3}{2x^2-x-3}$$

$$i(x) = \frac{4x+5}{x^2-64}$$

Your turn

Write down the largest possible domain for:

$$p(x) = \frac{6}{x+4}$$

$$q(x) = \frac{7}{5x+20}$$

$$r(x) = \frac{8}{3x^2+10x-8}$$

$$s(x) = \frac{9x-10}{x^2-16}$$

Worked example

Write down the largest possible domain for:

$$f(x) = \sqrt{x - 3}$$

$$g(x) = \sqrt{7x - 21}$$

$$h(x) = \sqrt{7x + 21}$$

$$i(x) = \sqrt{21 - 7x}$$

Your turn

Write down the largest possible domain for:

$$p(x) = \sqrt{x + 4}$$

$$q(x) = \sqrt{5x + 20}$$

$$r(x) = \sqrt{5x - 20}$$

$$s(x) = \sqrt{20 - 5x}$$

Worked example

Write down the largest possible domain for:

$$f(x) = \frac{\sqrt{x+3}}{x^2 - 2x}$$

$$g(x) = \frac{x^3 - 2x^2}{\sqrt{x^2 + 5x + 6}}$$

Your turn

Write down the largest possible domain for:

$$h(x) = \frac{\sqrt{x+4}}{x^4 - 25x^2}$$

Worked example

Find the range of the following functions:

$$f(x) = 2x - 3, \qquad x = \{1, 2, 3, 4\}$$

$$g(x) = 3 - 2x, \qquad x \in \mathbb{R}, x \leq 0$$

$$h(x) = 3 - 2x, \qquad x \in \mathbb{R}, 2 < x < 5$$

Your turn

Find the range of the following functions:

$$p(x) = 3x - 2, \qquad x = \{1, 2, 3, 4\}$$

$$q(x) = 2 - 3x, \qquad x \in \mathbb{R}, x > 0$$

$$r(x) = 2 - 3x, \qquad x \in \mathbb{R}, -3 < x \leq 4$$

Worked example

Find the range of the following functions:

$$f(x) = x^4, \qquad x = \{1, 2, 3, 4\}$$

$$g(x) = x^4, \qquad x \in \mathbb{R}, x \leq 0$$

$$h(x) = x^4, \qquad x \in \mathbb{R}, -2 \leq x < 5$$

Your turn

Find the range of the following functions:

$$p(x) = x^2, \qquad x = \{1, 2, 3, 4\}$$

$$q(x) = x^2, \qquad x \in \mathbb{R}, x > 0$$

$$r(x) = x^2, \qquad x \in \mathbb{R}, -3 < x \leq 4$$

Worked example

Find the range of the following functions:

$$f(x) = \frac{1}{x}, \qquad x = \{-1, -2, -3, -4\}$$

$$g(x) = \frac{1}{x-2}, \qquad x \in \mathbb{R}, x \leq 1$$

$$h(x) = \frac{1}{x+3}, \qquad x \in \mathbb{R}, -2 \leq x < 5$$

Your turn

Find the range of the following functions:

$$p(x) = \frac{1}{x}, \qquad x = \{1, 2, 3, 4\}$$

$$q(x) = \frac{1}{x+2}, \qquad x \in \mathbb{R}, x > -1$$

$$r(x) = \frac{1}{x-5}, \qquad x \in \mathbb{R}, -3 < x \leq 4$$

Worked example

Find the range of the following functions:

$$f(x) = \frac{1}{x}, \qquad x \in \mathbb{R}, x \neq 0$$

$$g(x) = \frac{1}{x} + 2, \qquad x \in \mathbb{R}, x \neq 0$$

Your turn

Find the range of the following functions:

$$h(x) = \frac{1}{x} - 3, \qquad x \in \mathbb{R}, x \neq 0$$

T.30 : 2B Qs 3-5, P.7: 2.2 Qs 2-3

Worked Example

564i: Determine the range of a rational function $f(x) = \frac{ax+b}{cx+d}$

$$h(x) = \frac{3x+6}{x-3}, \quad x \geq 8$$

Worked Example

Find the range of the following functions:

$g(x) = e^x - 4, \qquad x \in \mathbb{R}, x > 0$

$f(x) = \ln x + 5, \qquad x \in \mathbb{R}, x > 0$

$h(x) = -e^x - 3, \qquad x \in \mathbb{R}, x \leq 0$

$g(x) = \ln x - 4, \qquad x \in \mathbb{R}, x > 0$

Worked Example

564g: **Determine the range of a quadratic function by completing the square.**

$f(x) = x^2 + 4x - 7, \qquad x \in \mathbb{R}$

Find the range of $f(x)$

Worked Example

The function f is defined by $f(x) = x^2 - 8x + 27$ and has domain $x \geq a$. Given that $f(x)$ is a one-to-one function, find the smallest possible value of the constant a

Worked Example

The function $f(x)$ is defined by

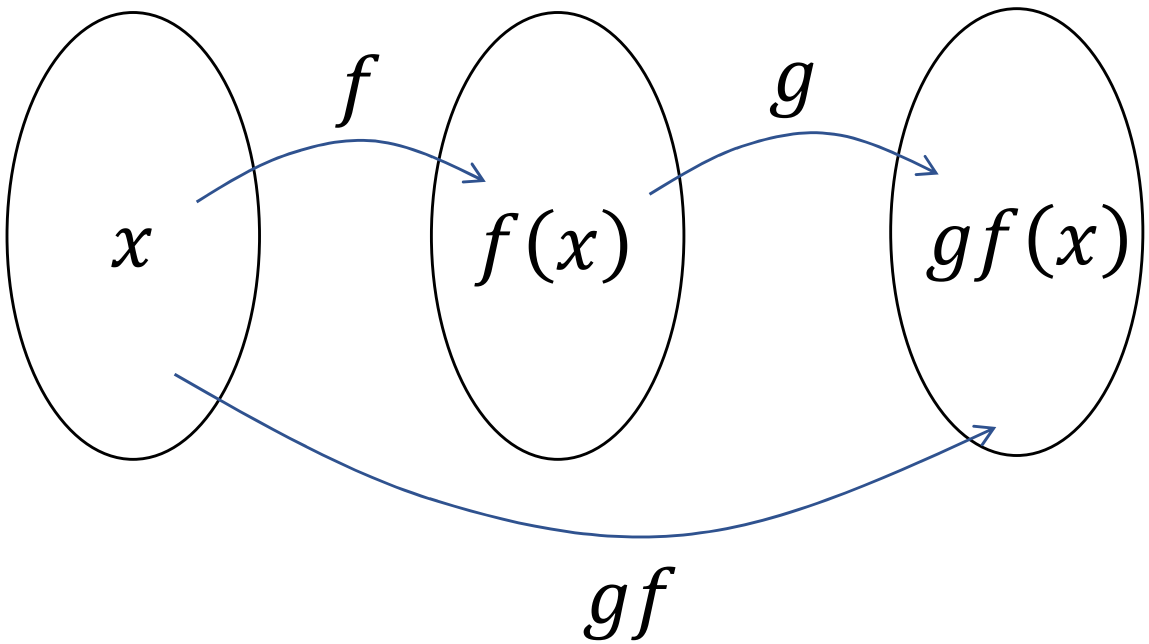
$$f: x \to \begin{cases} 2 - 5x, & x < 1 \\ x^2 - 3, & x \geq 1 \end{cases}$$

a) Sketch $y = f(x)$, and state the range of $f(x)$.

b) Solve $f(x) = 22$

T.30 : 2B Qs 6+, P.7: 2.2 Qs 4+

2.3) Composite functions



$gf(x)$ means $g(f(x))$, i.e. f is applied first, then g.

Sometimes we may apply multiple functions in succession to an input. These combined functions are known as a **composite function**.

Notes

Worked example

$f(x) = 3x - 2$, and $g(x) = x^2 - 4$

Find:

$fg(x)$

$gf(x)$

$f^2(x)$

$g^2(x)$

Your turn

$f(x) = 3x + 2$, and $g(x) = x^2 + 4$

Find:

$fg(x)$

$gf(x)$

$f^2(x)$

$g^2(x)$

Worked example

$f(x) = 3x - 2$, and $g(x) = x^2 - 4$

Find:

$fg(1)$

$gf(-2)$

$f^2(3)$

$g^2(-4)$

Your turn

$f(x) = 3x + 2$, and $g(x) = x^2 + 4$

Find:

$fg(4)$

$gf(-3)$

$f^2(2)$

$g^2(-1)$

T.34 : 2C Qs 1,2, P.8: 2.3 Qs 1-2

Worked example

$f(x) = 3x - 2$, and $g(x) = x^2 - 4$

Solve:

$$fg(a) = 13$$

$$gf(b) = 12$$

Your turn

$f(x) = 3x + 2$, and $g(x) = x^2 + 4$

Find:

$$fg(a) = 62$$

$$gf(b) = 293$$

Worked Example

The functions f and g are defined by

$$f: x \to |3x - 12|$$

$$g: x \to \frac{x+2}{3}$$

a) Find $fg(2)$

b) Solve $fg(x) = x$

T.34 : 2C Qs 3-5, P.8: 2.3 Qs 3-4

Worked Example

The function g is defined by

$$g: x \to 4 - 3x, \qquad x \in \mathbb{R}$$

Solve the equation

$$g^2(x) + [g(x)]^2 = 0$$

Worked Example

The functions f and g are defined by

$$f: x \to e^{3x} - 2, \qquad x \in \mathbb{R}$$
$$g: x \to 4\ln(x+1), \qquad x > -1$$

Find $fg(x)$, giving your answer in its simplest form.

Worked Example

The functions f and g are defined by

$$f: x \to 3^{2x} - 1, \qquad x \in \mathbb{R}$$

$$g: x \to 4\log_3(x + 5), \qquad x > -5$$

Find $fg(x)$, giving your answer in its simplest form.

Worked Example

$$f(x) = \frac{1}{x-1}, x \neq 1$$

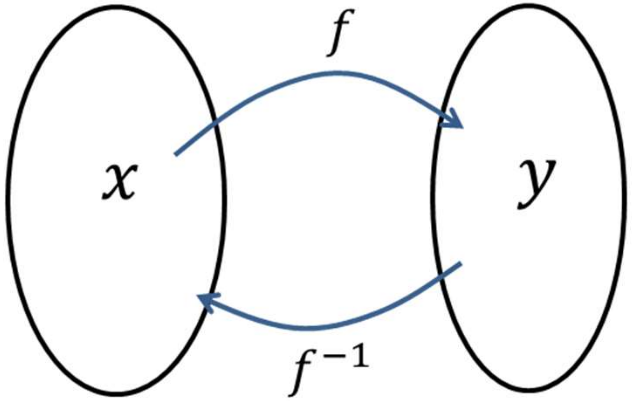
Find an expression for $f^2(x)$ and $f^3(x)$

Worked Example

A function f has domain $-3 \leq x \leq 12$ and is linear from $(-3, 9)$ to $(0, 6)$ and from $(0, 6)$ to $(12,10)$.
Find the value of $f^2(0)$

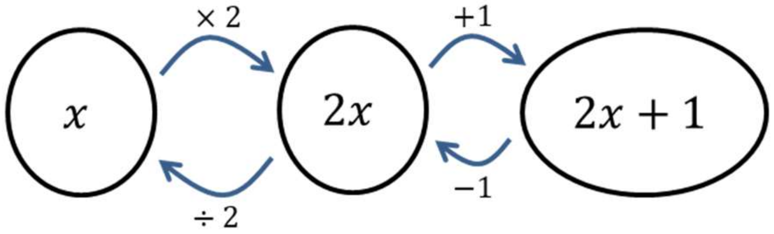
T.34 : 2C Qs 6+, P.8: 2.3 Qs 5+

2.4) Inverse functions



An inverse function f^{-1} **does the opposite of the original function**. For example, if $f(4) = 2$, then $f^{-1}(2) = 4$.

If $f(x) = 2x + 1$, we could do the opposite operations within the function in reverse order to get back to the original input:



Thus $f^{-1}(x) = \frac{x-1}{2}$

Notation: Just like f^2 means "apply f twice", f^{-1} means "apply f -1 times", i.e. once backwards!
This is why we write $\sin^{-1}(x)$ to mean "inverse sin".

This has appeared in exams before.

Explain why the function must be one-to-one for an inverse function to exist:
If the mapping was many-to-one, then the inverse mapping would be one-to-many. But this is not a function!

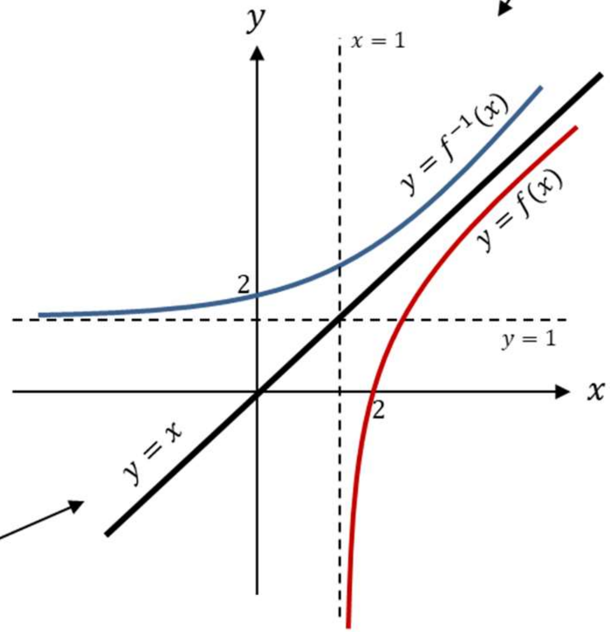
Graphing an Inverse Function

We saw that the inverse function effectively swaps the input x and output y. Thus the $\boldsymbol{x}$ **and** $\boldsymbol{y}$ **axis are swapped** when sketching the original function and its inverse.

And since the set of inputs and set of outputs is swapped…

The domain of $f(x)$ is the range of $f^{-1}(x)$ and vice versa.

Notice that x-intercepts become y-intercepts, and vertical asymptotes become horizontal ones.



$y = f(x)$ and $y = f^{-1}(x)$ have the line $y = x$ as a line of symmetry.

Domain of f: $\boldsymbol{x > 1}$

Range of f^{-1}: $\boldsymbol{f^{-1}(x) > 1}$

The domain of the function is the same as the range of the inverse, but remember that we write a domain in terms of x, but a range in terms of $f(x)$ or $f^{-1}(x)$.

Notes

Worked Example

Find the inverse functions:

$f(x) = 3x^2 - 5, \qquad x \geq 0$

$g(x) = 4x^2 + 6, \qquad x \geq 0$

T.38: 2D Qs 1-4, P.9 2.4 Qs 1

Worked Example

Find the inverse functions:

$f(x) = x^2 + 4x + 3, \qquad x \geq -2$

$g(x) = x^2 - 8x - 5, \qquad x \geq 5$

Worked Example

Find the inverse functions:

$f(x) = \frac{2}{x-5}, \qquad x \in \mathbb{R}, x \neq 5$

$g(x) = \frac{7}{x+2}, \qquad x \in \mathbb{R}, x \neq -2$

Worked Example

568a: Determine the inverse of exponential and logarithmic functions.

Given that $f(x) = 4 + \ln(x - 4)$, find $f^{-1}(x)$.

Worked Example

Find the inverse functions:

$$f(x) = \frac{x-2}{2x+1}, \qquad x \neq \frac{1}{2}$$

$$g(x) = \frac{2x+3}{4x-5}, \qquad x \neq \frac{5}{4}$$

Worked Example

$$f(x) = \sqrt{x - 3}\ \{x \in \mathbb{R}, x \geq 3\}$$

a) State the range of $f(x)$

b) Find the function $f^{-1}(x)$ and state its domain and range

c) Sketch $y = f(x)$, $y = f^{-1}(x)$ and $y = x$

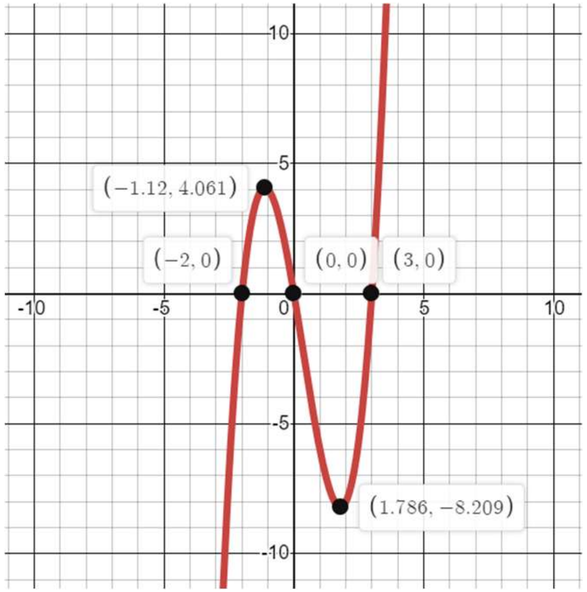
T.38: 2D Qs 5+ P.9 2.4 Qs 2+

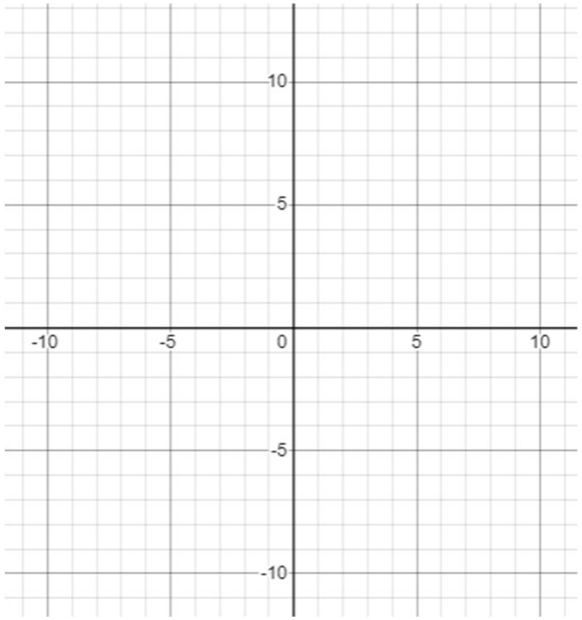
2.5) $y=|f(x)|$ and $y=f(|x|)$

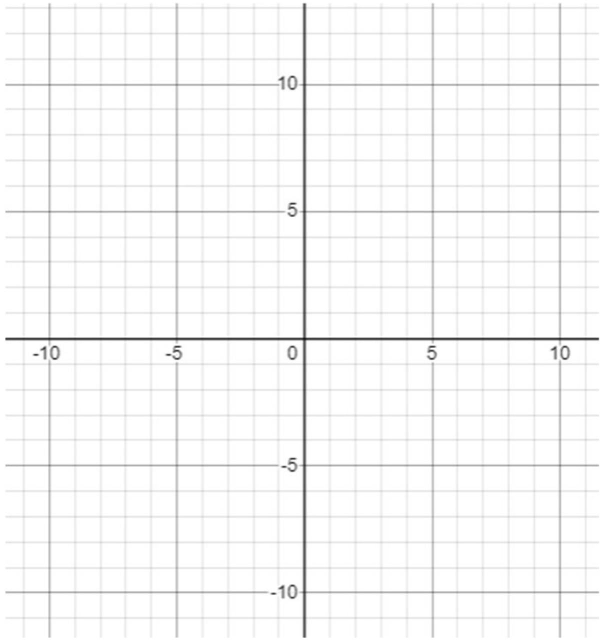
Notes

Worked Example

A sketch of $y = f(x)$ is shown.
Sketch $y = |f(x)|$ and $y = f(|x|)$ on separate axes.

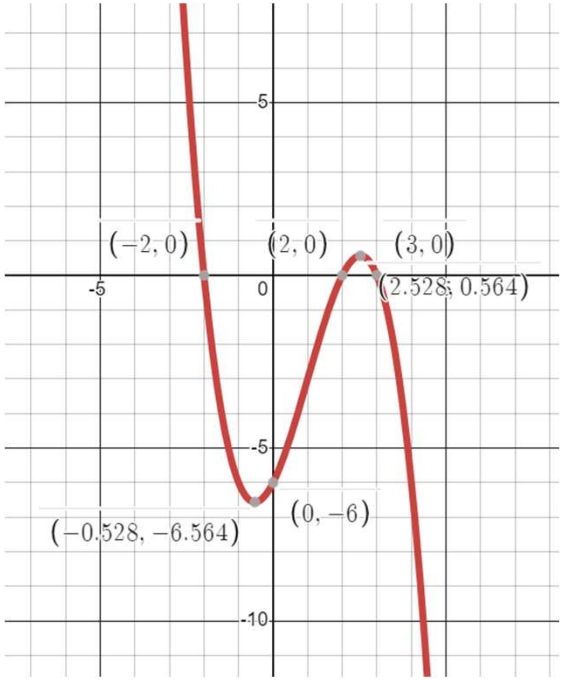


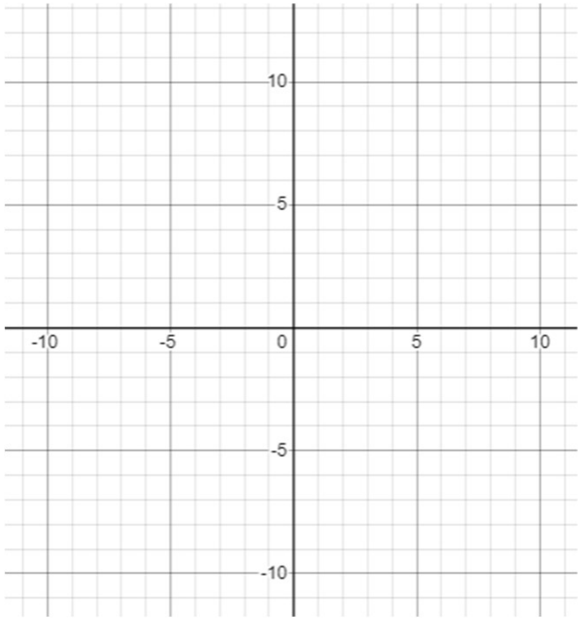


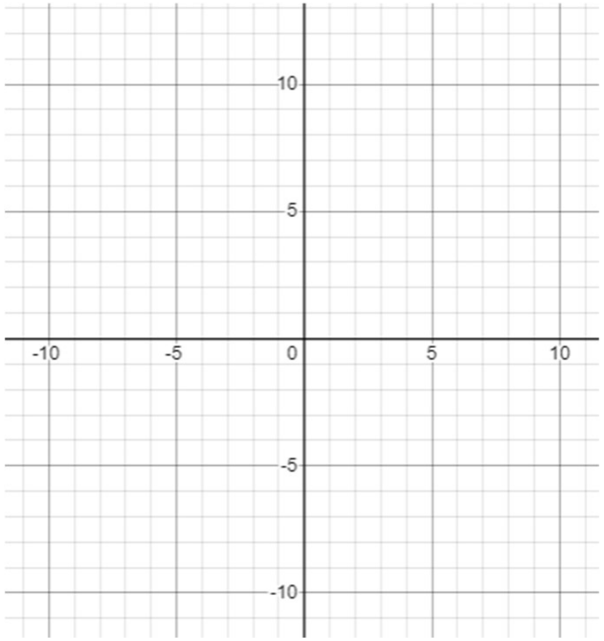


Your Turn

A sketch of $y = f(x)$ is shown.
Sketch $y = |f(x)|$ and $y = f(|x|)$ on separate axes.







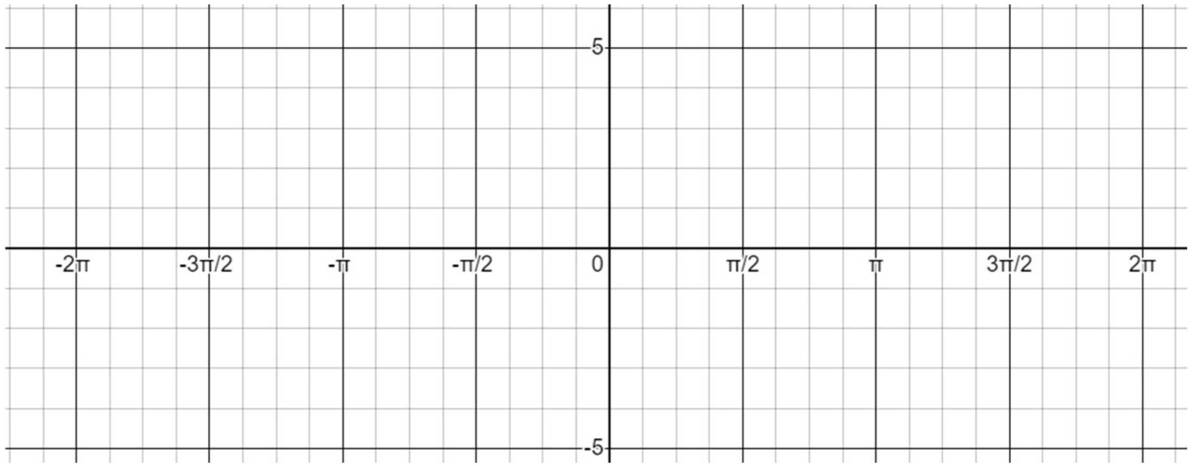
T.42: 2E Qs 1,3-10 P.10 2.5 Qs 3-7

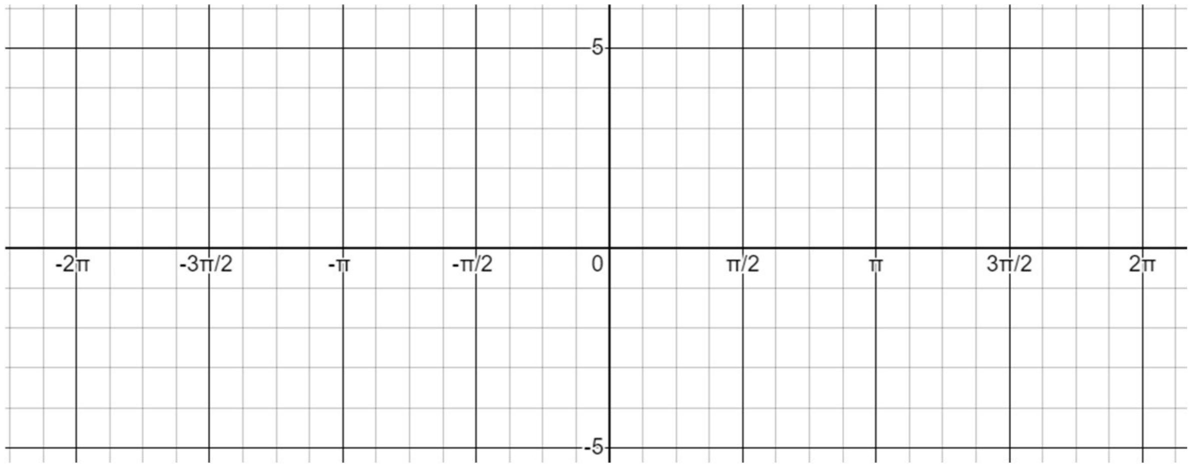
Worked Example

$$y = \cos x\,, \qquad -2\pi \le x \le 2\pi$$

Sketch:

a) $y = |\cos x|$ b) $y = \cos|x|$



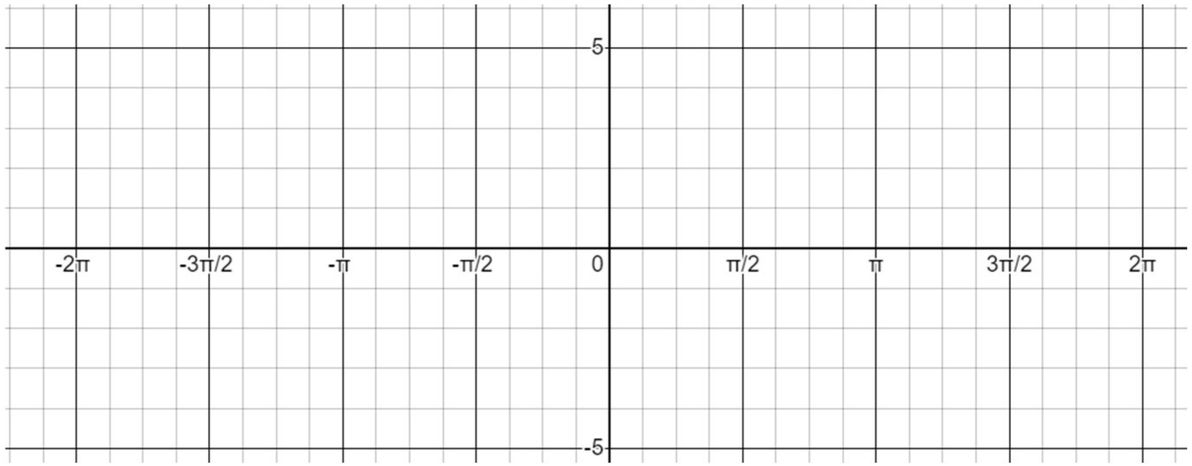


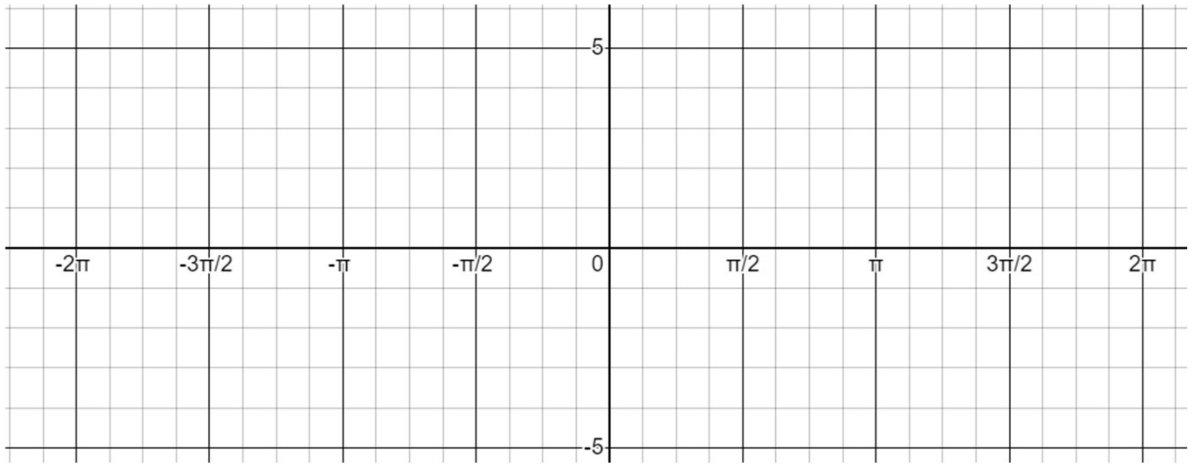
Your Turn

$$y = tan\, x\,, \qquad -2\pi \leq x \leq 2\pi$$

Sketch:

a) $y = |tan x|$ b) $y = \tan|x|$





T.42: 2E Qs 2 P.10 2.5 Qs 1-2

2.6) Combining transformations

	Affects which axis?	What we expect or opposite?
Change **inside** $f(\,)$	x	Opposite
Change **outside** $f(\,)$	y	What we expect

What if two x changes or two y changes?

$$y = 2f(x) + 1$$

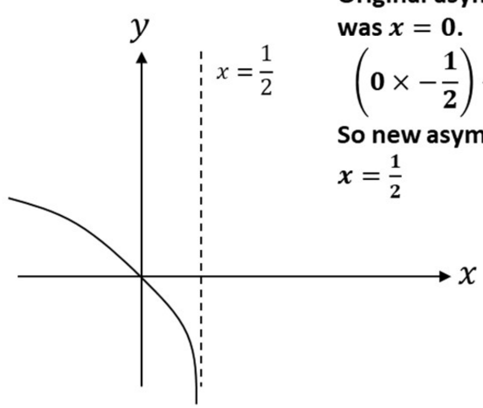
The y values are multiplied by 2, and <u>then</u> 1 is added.

$$y = f(2x + 1)$$

The easiest way is the think of the **inverse function** of $2x + 1$, i.e. $\frac{x-1}{2}$.
This gives us the changes to the x values, and in the correct order! In this case, we would -1 from the x values (translation 1 left) and then halve the x values (stretch on x-axis of scale factor $\frac{1}{2}$)

Sketch $y = \ln(1 - 2x)$

Inverse of $1 - 2x$ is $\frac{1-x}{2} = -\frac{1}{2}x + \frac{1}{2}$
So multiply x values by $-\frac{1}{2}$ and then add $\frac{1}{2}$.

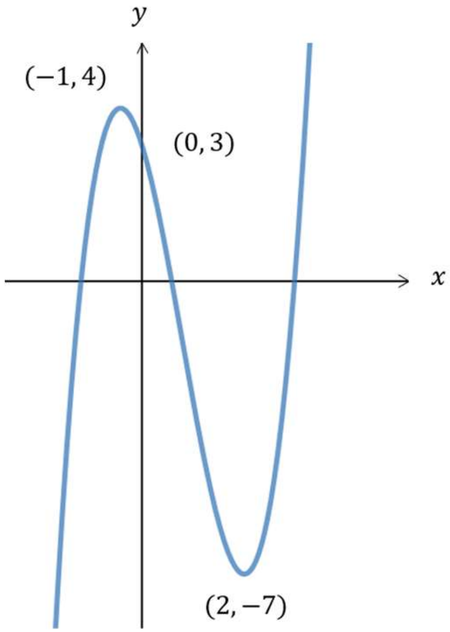


Original asymptote was $x = 0$.
$\left(0 \times -\frac{1}{2}\right) + \frac{1}{2} = \frac{1}{2}$
So new asymptote is $x = \frac{1}{2}$

Notes

Worked Example

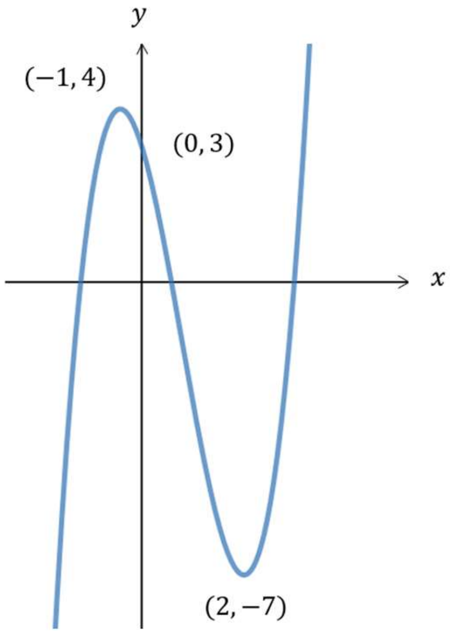
A sketch of the graph $y = f(x)$ is shown:



Sketch the graph of $y = -f(x) + 3$

Worked Example

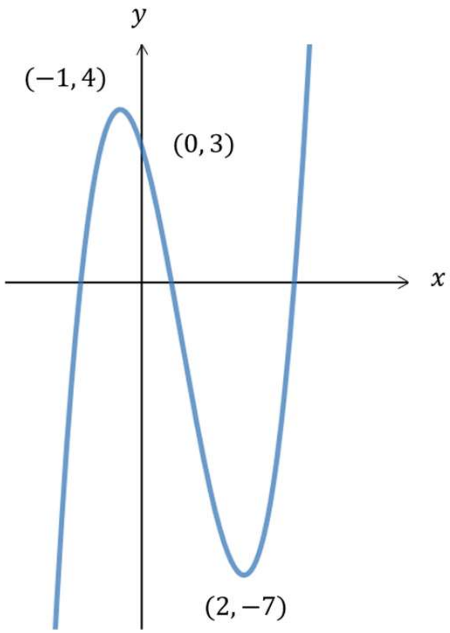
A sketch of the graph $y = f(x)$ is shown:



Sketch the graph of $y = f(-x) - 3$

Worked Example

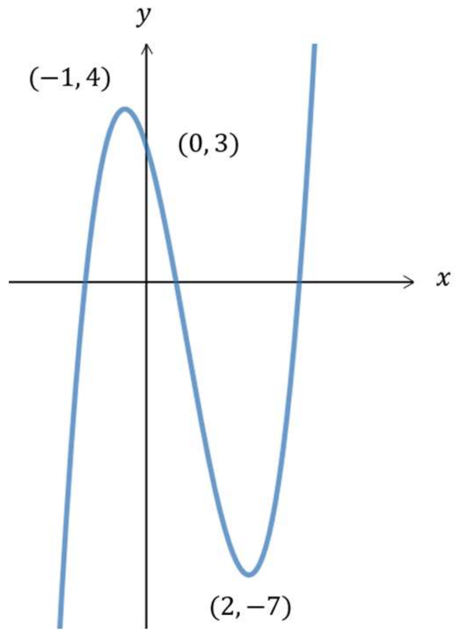
A sketch of the graph $y = f(x)$ is shown:



Sketch the graph of $y = -5f(x + 2) + 3$

Worked Example

A sketch of the graph $y = f(x)$ is shown:



Sketch the graph of $y = -f(|x|)$

T.47: 2F Qs all, P.12 2.6 Qs all

2.7) Solving modulus problems

Notes

Worked Example

$$f(x) = 2|x+1| - 3, x \in \mathbb{R}$$

(a) Sketch the graph of $y = f(x)$

(b) State the range of f.

(c) Solve the equation $f(x) = \frac{1}{3}x + 2$

Worked Example

$$f(x) = 6 - 2|x + 3|, x \in \mathbb{R}$$

(a) Sketch the graph of $y = f(x)$

(b) State the range of f.

(c) Solve the inequality $f(x) > 5$

Worked Example

$$f(x) = 6 + 3|x - 2|, x \in \mathbb{R}$$

State the range of values of k for which $f(x) = k$ has:

a) no solutions
b) exactly one solution
c) two distinct solutions

T.51: 2G Qs all, P.14 2.7 Qs all

Past Paper Questions

Figure 4 shows a sketch of the graph of $y = \mathrm{g}(x)$, where

$$\mathrm{g}(x) = \begin{cases} (x-2)^2 + 1 & x \leqslant 2 \\ 4x - 7 & x > 2 \end{cases}$$

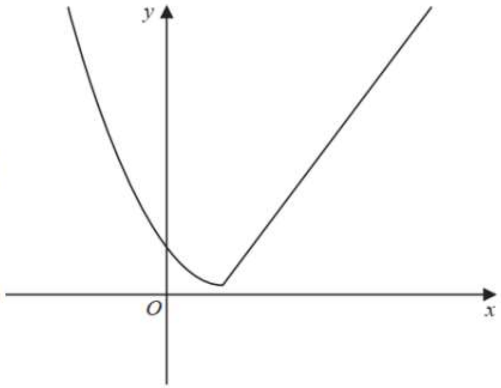


Figure 4

(a) Find the value of $\mathrm{gg}(0)$. **(2)**

(b) Find all values of x for which

$$\mathrm{g}(x) > 28$$

(4)

The function h is defined by

$$\mathrm{h}(x) = (x-2)^2 + 1 \qquad x \leqslant 2$$

(c) Explain why h has an inverse but g does not. **(1)**

(d) Solve the equation

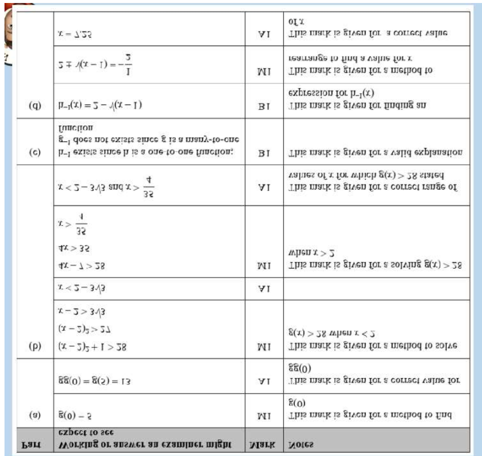
$$\mathrm{h}^{-1}(x) = -\frac{1}{2}$$

(3)

Exams

- Formula Booklet
- Past Papers
- Practice Papers
- past paper Qs by topic

Past paper practice by topic. Both new and old specification can be found via this link on hgsmaths.com



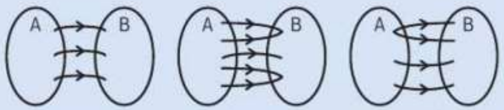
Summary of Key Points

1 A modulus function is, in general, a function of the type $y = |f(x)|$.

- When $f(x) \geqslant 0$, $|f(x)| = f(x)$
- When $f(x) < 0$, $|f(x)| = -f(x)$

2 To sketch the graph of $y = |ax + b|$, sketch $y = ax + b$ then reflect the section of the graph below the x-axis in the x-axis.

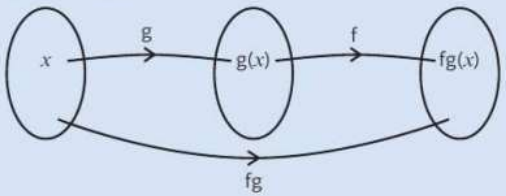
3 A mapping is a **function** if every input has a distinct output. Functions can either be **one-to-one** or **many-to-one**.



one-to-one function | many-to-one function | not a function

4 $fg(x)$ means apply g first, then apply f.

$fg(x) = f(g(x))$



5 Functions $f(x)$ and $f^{-1}(x)$ are inverses of each other. $ff^{-1}(x) = x$ and $f^{-1}f(x) = x$.

6 The graphs of $y = f(x)$ and $y = f^{-1}(x)$ are reflections of each another in the line $y = x$.

7 The domain of $f(x)$ is the range of $f^{-1}(x)$.

8 The range of $f(x)$ is the domain of $f^{-1}(x)$.

9 To sketch the graph of $y = |f(x)|$

- Sketch the graph of $y = f(x)$
- Reflect any parts where $f(x) < 0$ (parts below the x-axis) in the x-axis
- Delete the parts below the x-axis

10 To sketch the graph of $y = f(|x|)$

- Sketch the graph of $y = f(x)$ for $x \geqslant 0$
- Reflect this in the y-axis

11 $f(x + a)$ is a horizontal translation of $-a$.

12 $f(x) + a$ is a vertical translation of $+a$.

13 $f(ax)$ is a horizontal stretch of scale factor $\frac{1}{a}$

14 $af(x)$ is a vertical stretch of scale factor a.

15 $f(-x)$ reflects $f(x)$ in the y-axis.

16 $-f(x)$ reflects $f(x)$ in the x-axis.