KING EDWARD VI
HANDSWORTH GRAMMAR
SCHOOL FOR BOYS

KING EDWARD VI
ACADEMY TRUST
BIRMINGHAM

Year 13
Pure Mathematics
9 Differentiation
(part 2)

HGS Maths

Dr Frost Course

Name: ______________________________

Class: ______________________________

9.6) Differentiating trigonometric functions

Notes

Worked Example

Differentiate with respect to x:

$y = \cot x$

$y = \operatorname{cosec} x$

Worked Example

Differentiate with respect to x:

$y = \tan 2x$

$f(x) = \tan(-\frac{x}{3})$

Worked Example

Differentiate with respect to x:

$y = \cot 2x$

$f(x) = \cot(-\frac{x}{3})$

Worked Example

Differentiate with respect to x:

$y = \sec 2x$

$f(x) = \sec(-\frac{x}{3})$

Worked Example

Differentiate with respect to x:

$$y = \tan^4 2x$$

$$y = \cot^4 2x$$

T.248: Q 1-3 ,P.74: Q1-2

Worked Example

Differentiate with respect to x:

$$y = \frac{cosec\ 3x}{x^3}$$

T.248: Q 4-7 ,P.74: Q3-7

Worked Example

Given that $x = \cot y$, express $\frac{dy}{dx}$ in terms of x.

Worked Example

Differentiate with respect to x:

$y = \text{arc}cos\ x$

$y = \arctan x$

Worked Example

Differentiate with respect to x:

$y = \text{arc}cos\, x^4$

$y = \arctan x^3$

Worked Example

Given that $x = \operatorname{cosec} 3y$, find $\frac{dy}{dx}$ in terms of x

Worked Example

Given that $y = \arctan\left(\frac{1+x}{1-x}\right)$, find $\frac{dy}{dx}$

T.248: Q 8+ ,P.74: Q8

9.7) Parametric differentiation

Notes

Worked Example

Find the gradient at the point P where $t = 3$, on the curve given parametrically by

$$x = t^2 - t, \qquad y = \mathrm{t}^4 - 2, \qquad \mathrm{t} \in \mathbb{R}$$

T.251: Q 1 ,P.75: Q1

Worked Example

Find the equation of the tangent at the point where $t = \frac{\pi}{6}$, to the curve with parametric equations

$$x = \sqrt{5}\sin 2t\,, \qquad y = 8\cos^2 t\,, \qquad 0 \le t \le \pi$$

Worked Example

Find the equation of the normal at the point where $\theta = \frac{\pi}{3}$, to the curve with parametric equations

$$x = 2\cos\theta, \qquad y = 7\sin\theta$$

T.251: Q2+ ,P.75: Q2+

9.8) Implicit differentiation

Notes

Worked Example

Find:

$\frac{d}{dx}(y^4)$

$\frac{d}{dx}(3y^5)$

Worked Example

Find:

$$\frac{d}{dx}(\cos y)$$

$$\frac{d}{dx}(\tan 2y)$$

Worked Example

Find:

$\frac{d}{dx}(e^{y})$

$\frac{d}{dx}(e^{2y})$

Worked Example

Find:

$$\frac{d}{dx}(xy)$$

$$\frac{d}{dx}(x^2y)$$

Worked Example

Find:

$$\frac{d}{dx}(e^{xy})$$

$$\frac{d}{dx}(e^{x^2y})$$

Worked Example

Find:

$$\frac{d}{dx}(\cos(x+y))$$

$$\frac{d}{dx}(\tan(x^2-4y))$$

Worked Example

Find $\frac{dy}{dx}$ where:

$$x^4 - x + y^2 - 3y = 5$$

T.255: Q 1-3 ,P.76: Q1-3

Worked Example

Find $\frac{dy}{dx}$ at the point $(1, 1)$, given that:

$$6x^2y - \frac{4x}{y^2} = 2$$

Worked Example

A curve is described by:

$$x^3 + 4y^2 = -12xy$$

Find the gradient of the curve at the points where $x = 8$

Worked Example

$$x^2 + y^2 + 20x + 4y - 8xy = -75$$

Find the values of y for which $\frac{dy}{dx} = 0$

T.255: Q 4-6,P.76: Q4-7

Worked Example

A curve has equation

$$x^2 + 4xy + y^2 - x = 35$$

Find the equation of the tangent to the curve at the point $(2, 3)$.
Give your answer in the form $ax + by + c = 0$, where a, b and c are integers

Worked Example

The curve $ye^{-4x} = 4x - y^2$.
Find the equation of the normal(s) to the curve at the point where $x = 0$.
Give your answer in the form $ax + by + c = 0$

T.256: Q 7+6,P.76: Q8

9.9) Using second derivatives

Notes

Worked Example

Find the interval on which the function is concave:

$f(x) = x^3 - 2x + 5$

$g(x) = 2x^3 - 5x^2 - 6$

Worked Example

Find the interval on which the function is convex:

$f(x) = x^3 - 2x + 5$

$g(x) = 2x^3 - 5x^2 - 6$

Worked Example

Show that the function is convex for all real values of x:

$f(x) = e^{3x} + x^2$ $g(x) = e^{4x} + x^4$

T.259: Q 1-2, P.77: Q1-2

Worked Example

Determine if there is a point of inflection on the curve with equation $y = (x - 3)^4$

Worked Example

A curve C has equation

$$y = \frac{1}{4}x^2 \ln x - 3x + 7, x > 0$$

Find where C is convex

T.259: Q 3+, P.77: Q3+

9.10) Rates of change

Notes

Worked Example

Given that the area of a circle A cm^2 is related to its radius r cm by the formula $A = \pi r^2$, and that the rate of change of its radius in cm s^{-1} is given by $\frac{dr}{dt} = 5$, find $\frac{dA}{dt}$ when $r = 3$.

T.263: Q 1-5, P.77: Q1-4

Worked Example

Atmospheric pressure decreases as altitude increases. The rate at which atmospheric pressure decreases is proportional to the current air pressure. Write down a differential equation for the rate of change of the atmospheric pressure.

Worked Example

A metal bar is heated from a starting temperature. It is noted that the rate of gain of temperature of the metal bar is proportional to the difference between the temperature of the metal bar and its surroundings. Write a differential equation to represent this situation.

Worked Example
Determine the rate of change of the area A of a circle when the radius $r = 7$cm, given that the radius is changing at a rate of $4\ cm\ s^{-1}$.

Worked Example

A cube is expanding uniformly as it is heated. At time t seconds, the length of each edge is x cm and the volume of the cube is $V\ cm^3$.
Given that the volume increases at a constant rate of $0.024\ cm^3\ s^{-1}$, find the rate of increase of the total surface area of the cube in $cm^2\ s^{-1}$ when $x = 4$

Worked Example
A right circular cylindrical metal rod is expanding as it is heated. After t seconds the radius of the rod is x cm and the length of the rod is $5x$ cm. The cross-sectional area of the rod is increasing at the constant rate of $0.016\ cm^2\ s^{-1}$. Find the rate of increase of the volume of the rod when $x = 4$

T.263: Q 6+, P.78: Q5+

Summary of Key Points

1 For small angles, measured in radians:

- $\sin x \approx x$
- $\cos x \approx 1 - \frac{1}{2}x^2$

2
- If $y = \sin kx$, then $\frac{dy}{dx} = k\cos kx$
- If $y = \cos kx$, then $\frac{dy}{dx} = -k\sin kx$

3
- If $y = e^{kx}$, then $\frac{dy}{dx} = ke^{kx}$
- If $y = \ln x$, then $\frac{dy}{dx} = \frac{1}{x}$

4 If $y = a^{kx}$, where k is a real constant and $a > 0$, then $\frac{dy}{dx} = a^{kx}k\ln a$

5 The **chain rule** is: $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

where y is a function of u and u is another function of x.

6 The chain rule enables you to differentiate a function of a function. In general,

- if $y = (f(x))^n$ then $\frac{dy}{dx} = n(f(x))^{n-1}f'(x)$
- if $y = f(g(x))$ then $\frac{dy}{dx} = f'(g(x))g'(x)$

7 $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$

8 The **product rule**:

- If $y = uv$ then $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$, where u and v are functions of x.
- If $f(x) = g(x)h(x)$ then $f'(x) = g(x)h'(x) + h(x)g'(x)$

9 The **quotient rule**:

- If $y = \frac{u}{v}$, then $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$ where u and v are functions of x.
- If $f(x) = \frac{g(x)}{h(x)}$, then $f'(x) = \frac{h(x)g'(x) - g(x)h'(x)}{(h(x))^2}$

10
- If $y = \tan kx$, then $\frac{dy}{dx} = k\sec^2 kx$
- If $y = \operatorname{cosec} kx$, then $\frac{dy}{dx} = -k\operatorname{cosec} kx\cot kx$
- If $y = \sec kx$, then $\frac{dy}{dx} = k\sec kx\tan kx$
- If $y = \cot kx$, then $\frac{dy}{dx} = -k\operatorname{cosec}^2 kx$

11
- If $y = \arcsin x$, then $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$
- If $y = \arccos x$, then $\frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}$
- If $y = \arctan x$, then $\frac{dy}{dx} = \frac{1}{1+x^2}$

12 If x and y are given as functions of a parameter, t: $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$

13
- $\frac{d}{dx}(f(y)) = f'(y)\frac{dy}{dx}$
- $\frac{d}{dx}(y^n) = ny^{n-1}\frac{dy}{dx}$
- $\frac{d}{dx}(xy) = x\frac{dy}{dx} + y$

14
- The function $f(x)$ is **concave** on a given interval if and only if $f''(x) \leqslant 0$ for every value of x in that interval.
- The function $f(x)$ is **convex** on a given interval if and only if $f''(x) \geqslant 0$ for every value of x in that interval.

15 A **point of inflection** is a point at which $f''(x)$ changes sign.

16 You can use the chain rule to connect rates of change in situations involving more than two variables.

T.265: mixed ex ,P79: BSG Qs