KING EDWARD VI HANDSWORTH GRAMMAR SCHOOL FOR BOYS

KING EDWARD VI ACADEMY TRUST BIRMINGHAM

Year 13
Pure Mathematics
9 Differentiation
(part 1)

HGS Maths

Dr Frost Course

Name: ________________________________

Class: ________________________________

Contents

Prior knowledge check

Prior knowledge check

1 Differentiate:

a $3x^2 - 5x$ **b** $\frac{2}{x} - \sqrt{x}$

c $4x^2(1 - x^2)$

← Year 1, Chapter 12

2 Find the equation of the tangent to the curve with equation $y = 8 - x^2$ at the point $(3, -1)$.

← Year 1, Chapter 12

3 The curve C is defined by the parametric equations

$$x = 3t^2 - 5t, \qquad y = t^3 + 2, \qquad t \in \mathbb{R}$$

Find the coordinates of any points where C intersects the coordinate axes.

← Section 8.4

4 Solve $2 \operatorname{cosec} x - 3 \sec x = 0$ in the interval $0 \leqslant x \leqslant 2\pi$, giving your answers correct to 3 significant figures.

← Section 6.3

9.1) Differentiating $\sin x$ and $\cos x$

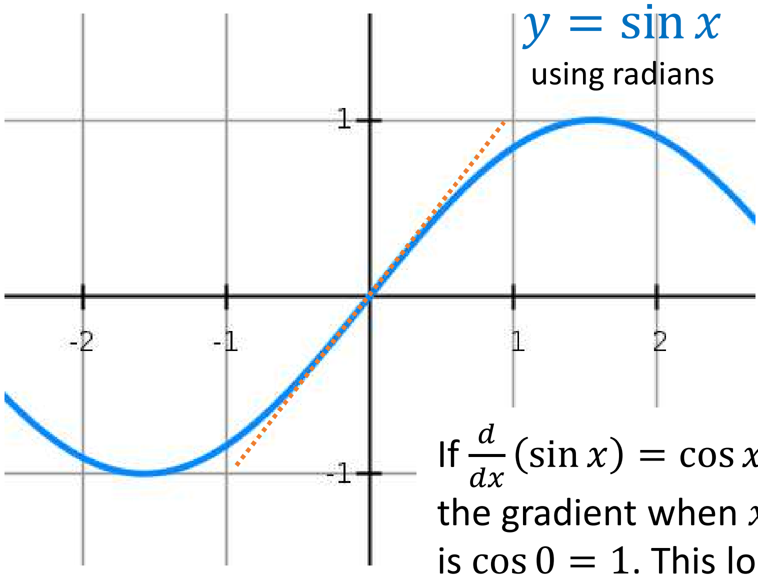
Why does this result only hold in radians?

Explanation 1:

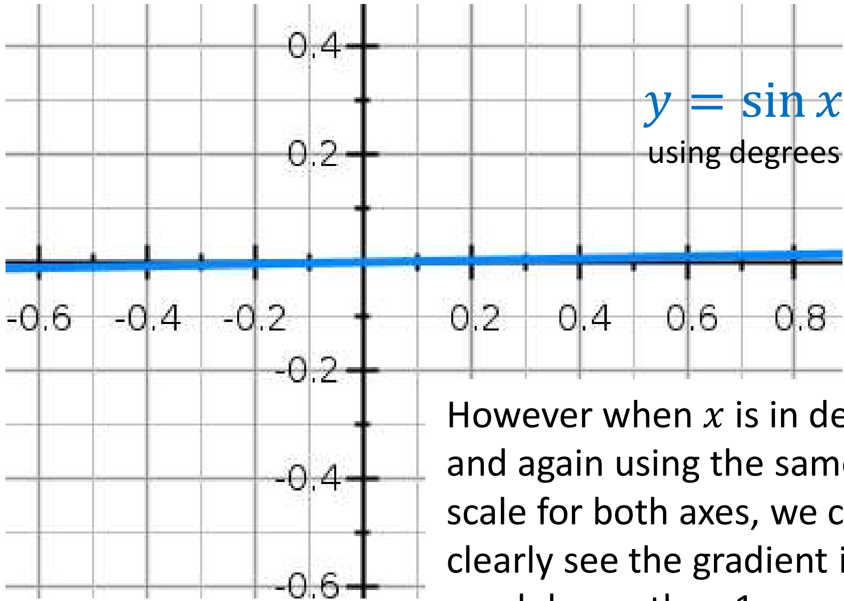
The approximations $\sin x \approx x$ and $\cos x \approx 1 - \frac{1}{2}x^2$ (as $x \to 0$) only holds if x is in radians (we saw why in the chapter on radians). The proof that $\frac{d}{dx}(\sin x) = \cos x$ made use of these approximations.

Explanation 2:

We can see by observation if we look at the graph of $\sin x$ in radians and in degrees.



If $\frac{d}{dx}(\sin x) = \cos x$ then the gradient when $x = 0$ is $\cos 0 = 1$. This looks right from the graph.



However when x is in degrees, and again using the same scale for both axes, we can clearly see the gradient is much lower than 1.

Notes

Worked Example

Prove, from first principles, that the derivative of $\sin x$ is $\cos x$.

You may assume that as $h \to 0, \frac{\sin h}{h} \to 1$ and $\frac{\cos h - 1}{h} \to 0$

Worked Example

Find:

$$\frac{d}{dx}(\sin 5x)$$

$$\frac{d}{dx}(\cos \frac{1}{2}x)$$

$$\frac{d}{dx}(6 \sin 7x - 4 \cos 3x)$$

T.234: Q 1-3 ,P.70 Q 1

Worked Example

A curve has equation $y = \frac{1}{4}x - \cos 3x$.

Find the stationary points on the curve in the interval $0 \leq x \leq \pi$

Worked Example

A curve has equation $y = \sin 5x + 3x$. Find the stationary points on the curve in the interval $0 \le x \le \frac{3}{5}\pi$

T.234: Q 4-6 ,P.70 Q 2,3

Worked Example

A curve has equation $y = \sin 4x - \cos 3x$. Find the equation of the tangent to the curve at the point $(\pi, 1)$

Worked Example

A curve has equation $y = 3x^2 - \sin x$. Show that the equation of the normal to the curve at the point with x-coordinate π is

$$x + (6\pi + 1)y - \pi(18\pi^2 - 3\pi - 1) = 0$$

T.234: 7+,P.70 Q 4+

9.2) Differentiating exponentials and logarithms

Notes

Proof

Prove that the derivative of a^{kx} is $a^{kx}k\ln a$

Worked Example

Find:

$\frac{d}{dx}(5e^{2x})$

$\frac{d}{dx}\left(4e^{\frac{1}{3}x}\right)$

$\frac{d}{dx}(3e^{-0.5x})$

Worked Example

Find:

$\frac{d}{dx}(2^x)$

$\frac{d}{dx}(3^x)$

$\frac{d}{dx}(4^x)$

Worked Example

Find:

$\frac{d}{dx}\left(5(2^x)\right)$

$\frac{d}{dx}\left(4(3^x)\right)$

$\frac{d}{dx}(3(4^x))$

Worked Example

Find:

$\frac{d}{dx}(2^{5x})$

$\frac{d}{dx}(3^{4x})$

$\frac{d}{dx}(4^{3x})$

Worked Example

Find:

$\frac{d}{dx}(\ln 2x)$

$\frac{d}{dx}(\ln 3x)$

$\frac{d}{dx}(\ln 4x)$

Worked Example

Find:

$\frac{d}{dx}(2\ln x)$

$\frac{d}{dx}(3\ln x)$

$\frac{d}{dx}(4\ln x)$

Worked Example

Find:

$\frac{d}{dx}(\ln x^2)$

$\frac{d}{dx}(\ln x^3)$

$\frac{d}{dx}(\ln x^4)$

Worked Example

Find:

$$\frac{d}{dx}\left(\frac{3 + 2e^{5x}}{7e^{4x}}\right)$$

$$\frac{d}{dx}\left(\frac{7 - 5e^{-4x}}{3e^{2x}}\right)$$

Worked Example

Find:

$$\frac{d}{dx}(e^x - 3)^2$$

T.236: Q1-3, P.70 Q 1-3

Worked Example

The population P of a species after t days can be modelled using $P = 640(2^{-3t})$

a) Determine how many days have elapsed before there are 20 individuals left
b) Determine the rate of change of the population after 3 days.

Worked Example

A curve has the equation

$$y = e^{4x} - \ln x$$

Show that the equation of the tangent to the curve at the point with x-coordinate 1 is:

$$y = (4e^4 - 1)x - 3e^4 + 1$$

Worked Example

A curve has the equation

$$y = 5(2^{3x})$$

Find the equation of the normal to the curve at the point with x-coordinate 1 in the form $y = ax + b$, where a and b are constants to be found in exact form

T.236: Q4+,P.70 Q 4+

9.3) The chain rule

Notes

Worked Example

597a: **Apply the chain rule to differentiate functions of the form $f(x) = a(bx+c)^n$ for rational n**

Given that $y = 9\sqrt[3]{9x+10}$

find $\frac{dy}{dx}$

597a: **Apply the chain rule to differentiate functions of the form $f(x) = a(bx+c)^n$ for rational n**

Given that $y = \frac{1}{\sqrt[3]{8x+4}}$

find $\frac{dy}{dx}$

Worked Example

597c: Apply the chain rule to differentiate functions of the form $\ln(ax+b)$, e^{ax+b}, $\sin(ax+b)$ or $\cos(ax+b)$ where the inner function is linear.

Given that $y = 7\ln(5x+9)$

find $\frac{dy}{dx}$

Worked Example

Differentiate with respect to x:

$y = e^{x^2+x}$

$f(x) = e^{x^3-2x+1}$

Worked Example

Differentiate with respect to x:

$$y = \ln(\sin x)$$

T.239: Q1,2,P.71 Q 1-3

Worked Example

597c: Apply the chain rule to differentiate functions of the form $\ln(ax+b)$, e^{ax+b}, $\sin(ax+b)$ or $\cos(ax+b)$ where the inner function is linear.

Given that $y = 9\sin(7x - 9)$

find $\frac{dy}{dx}$

Worked Example

Differentiate with respect to x:

$y = \sin^4 x$ $\quad\quad$ $f(x) = \cos^3 x$

Worked Example

Differentiate with respect to x:

$y = \sin^4 3x$ $\qquad$ $f(x) = \cos^3 2x$

Worked Example

Differentiate with respect to x:

$$y = e^{e^{-x}}$$

Worked Example

Given that $y = \sqrt{2x^5 - 2}$, find $\frac{dy}{dx}$ at $(3,22)$

Worked Example

A curve C has equation

$$y = \frac{3}{(2-5x)^4}, x \neq \frac{2}{5}$$

Find an equation for the normal to C at the point with x-coordinate 1 in the form $ax + by + c = 0$, where a, b and c are integers

T.239: Q3-5,P.71 Q 4-8

Your Turn

A curve C has equation

$$y = \frac{4}{(3-2x)^5}, x \neq \frac{3}{2}$$

Find an equation for the normal to C at the point with x-coordinate 2 in the form

$ax + by + c = 0$, where a, b and c are integers

$x + 40y + 158 = 0$

Worked Example

Find the gradient of:

$x = (1 + 2y)^3$ when $y = 1$

$x = (3y^2 - 2)^4$ at $(1, -1)$

Worked Example

Find $\frac{dy}{dx}$ when:

$x = 2y^2 + y$

$x = 3y^4 - y^2 - 1$

T.239: Q6+

9.4) The product rule

Notes

Worked Example

597f: Use the product rule with simple functions.

Given that

$$y = 4x^8e^x$$

find $\frac{dy}{dx}$.

Given that

$$y = 9\ln(x)e^x$$

find $\frac{dy}{dx}$.

Worked Example

597f: Use the product rule with simple functions.

Given that

$$y = 8x^6 \sin(x)$$

find $\frac{dy}{dx}$.

Worked Example

Differentiate with respect to x:

$y = e^{3x} \sin^4 2x$

$f(x) = e^{2x} \cos^3 4x$

Worked Example

Determine the coordinates of the turning point:

$y = xe^{3x}$

$f(x) = xe^{4x}$

T.242: Q1-3,P.72 Q 1-5

Worked Example

Differentiate with respect to x:

$$y = x^2\sqrt{5x - 2}$$

Worked Example

Find the equation of the tangent to the curve with equation $y = x^2 \sin(x^2)$ at the point $\left(\frac{\sqrt{\pi}}{2}, \frac{\pi\sqrt{2}}{8}\right)$ in the form $ax + by + c = 0$ where a, b and c are exact constants.

Worked Example

Differentiate with respect to x:

$$\frac{dy}{dx} = \frac{1}{5x(x+1)^{\frac{1}{2}}}$$

T.242: Q4+,P.72 Q 6+

9.5) The quotient rule

Notes

Worked Example

597h: Use the quotient rule with simple functions.

Given that

$$y = \frac{2\sin(x)}{\ln(x)}$$

find $\frac{dy}{dx}$.

Worked Example

Differentiate with respect to x:

$y = \frac{x^2}{\ln 5x}$

$y = \frac{\ln 4x}{x^4}$

Worked Example

Differentiate with respect to x:

$$y = \frac{\cos 3x}{x^4}$$

$$f(x) = \frac{x^4}{\cos 3x}$$

Worked Example

597i: Use the quotient rule and the chain rule within a single expression.

Given that

$$y = \frac{(4x^8 + 9x^9)}{e^{8x^2+2x}}$$

find $\dfrac{dy}{dx}$.

T.244: Q1-4,P.73 Q 1-3

Worked Example

Find the stationary point of

$$y = \frac{\cos x}{e^{3x}}, 0 < x < \pi$$

Worked Example

Find the equation of the tangent to the curve $y = \frac{e^{\frac{1}{4}x}}{x}$ at the point $(4, \frac{1}{4}e)$

T.245: Q5+,P.73 Q 4+