KING EDWARD VI
HANDSWORTH GRAMMAR
SCHOOL FOR BOYS

KING EDWARD VI
ACADEMY TRUST
BIRMINGHAM

Year 13
Pure Mathematics
P2 4 Binomial Expansion

HGS Maths

Dr Frost Course

Name: ________________________________

Class: ________________________________

Contents

Pure Year 1 Recap

Remember that for small integer n you could use a row of Pascal Triangle for the Binomial coefficients, descending powers of the first term and ascending powers of the second.
If the first term is 1, we can ignore the powers of 1.

$$\begin{aligned}
(1+x)^5 &= 1+5x+10x^2+10x^3+5x^4+x^5\\
(1+2x)^4 &= 1+4(2x)+6(2x)^2+4(2x)^3+(2x)^4\\
&= 1+8x+24x^2+32x^3+16x^5\\
(1-3x)^3 &= 1+3(-3x)+3(-3x)^2+(-3x)^3\\
&= 1-9x+27x^2-27x^3
\end{aligned}$$

Do you remember the simple way to find your Binomial coefficients?

Hopefully you can see the pattern by this point.

$$\binom{n}{1} = \boldsymbol{n} \qquad \binom{n}{2} = \frac{\boldsymbol{n(n-1)}}{\mathbf{2!}} \qquad \binom{n}{3} = \frac{\boldsymbol{n(n-1)(n-2)}}{\mathbf{3!}} \qquad \binom{n}{4} = \frac{\boldsymbol{n(n-1)(n-2)(n-3)}}{\mathbf{4!}}$$

$$\binom{10}{3} = \frac{\mathbf{10\times 9\times 8}}{\mathbf{6}} = \mathbf{120} \qquad \binom{-1}{2} = \frac{\mathbf{-1\times -2}}{\mathbf{2}} = \mathbf{1} \qquad \binom{-2}{3} = \frac{\mathbf{-2\times -3\times -4}}{\mathbf{6}} = \mathbf{-4}$$

$$\binom{0.5}{2} = \frac{\mathbf{0.5\times -0.5}}{\mathbf{2}} = -\frac{\mathbf{1}}{\mathbf{8}}$$

Note: You can work out a 'choose' value, in the same way, when the top number is negative or fractional, but your calculator can not do this directly.

Worked Example

Find the binomial expansion of:

$$(1 - 2x)^5$$

4.1) Expanding $(1+x)^n$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots + {}^nC_r\, x^n$$

Notes

Worked Example

Find the first four terms in the binomial expansion of:

$$\frac{1}{(1-x)^3}$$

Worked Example

Find the first four terms in the binomial expansion of:

$$\sqrt{1 + 6x}$$

Worked Example – K583a

Find the binomial expansion of

$$(1-4y)^{-3}$$

in ascending powers of y up to and including the term in y^2.

Give your answer in its simplest form.

Worked Example

State when the binomial expansion is valid:

$$\frac{1}{1+2x}$$

$$\frac{1}{(1-3x)^4}$$

$$(1+5x)^{\frac{3}{2}}$$

$$\frac{1}{\sqrt{1+\frac{x}{7}}}$$

Worked Example

State when the binomial expansion is valid:

$$\frac{2 - x}{\sqrt{1 + 3x}}$$

$$\frac{5 + x}{(1 - 2x)^4}$$

$$(1 + 5x)^{\frac{3}{2}}\sqrt{1 - \frac{x}{4}}$$

T.96: 4A Qs 1-3, P.29: 4.1 Qs 1,2

Worked Example

By substituting $x = 0.07$ into the binomial expansion for $\sqrt{1-4x}$, find a decimal approximation to $\sqrt{2}$

Worked Example

By substituting $x = 0.04$ into the binomial expansion for $\sqrt{1-4x}$, find a decimal approximation to $\sqrt{21}$ to 5 decimal places

Worked Example

Find the series expansion, in ascending powers of x, up to and including the x^3 term, of $\sqrt{1+7x}$. Find the percentage error in using $x = 0.01$ in this series expansion to estimate $\sqrt{107}$

Worked Example

Find the x^2 term in the series expansion of:

$$\frac{5-x}{\sqrt{1-3x}}$$

Worked Example

Find the first three terms in the series expansion of:

$$\sqrt{\frac{1+2x}{1-3x}}$$

Worked Example

Find the series expansion, in ascending powers of x, up to and including the x^2 term for:

$$\frac{6}{1-3x}-\frac{4}{1-2x}$$

Worked Example

In the expansion of $(1 + kx)^{-3}$ the coefficient of x^2 is 4 and $k > 0$. Find k

T.96: 4A Q 6,9, P.29: 4.1 Q 5,9

4.2) Expanding $(a + bx)^n$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots + {}^nC_r\, x^n$$

Notes

Quick Fire 1st step

What would be the first step in finding the Binomial expansion of each of these?

$(a+bx)^n$	$a^n\left(1+\frac{b}{a}x\right)^n$	$\left\lvert\frac{b}{a}x\right\rvert < 1$	$\lvert x\rvert < \cdots$
$(2+x)^{-3}$			
$(9+2x)^{\frac{1}{2}}$			
$(8-x)^{\frac{1}{3}}$			
$(5-2x)^{-3}$			
$(16+3x)^{-\frac{1}{2}}$			

Worked Example

Find first four terms in the binomial expansion of $\sqrt{2+x}$

State the values of x for which the expansion is valid.

Worked Example K583b

Find the first **four** terms in ascending powers of x in the binomial expansion of

$$(2-5x)^{-2}$$

Give your answer in its simplest form.

T.99: 4B Qs1-4, P.30: 4.2 Q1-3

Worked Example

Find first three terms in ascending powers of x of the series expansion of $\frac{3x+4}{\sqrt{2-5x}}$

State the values of x for which the expansion is valid.

Worked Example

Find the series expansion, in ascending powers of x, up to and including the x^2 term for:

$$\frac{6}{2-3x}-\frac{4}{5+2x}$$

T.99: 4B Qs 5,6,8, P.30: 4.2 Qs5,7

Worked Example

Use the binomial expansion of $\sqrt{8+9x}$ up to the x^2 term to estimate $\sqrt{11}$, giving your answer as a single fraction

Worked Example

Find the percentage error in approximating $\sqrt{53}$ using $x = \frac{1}{9}$ in the series expansion of $\sqrt{6 - x}$ up to and including the x^2 term.

T.99: 4B Q 7, P.30: 4.2 Qs6,8

4.3) Using partial fractions

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots + {}^nC_r\, x^n$$

Notes

Worked Example

Find the cubic approximation of

$$\frac{4+5x}{(1-x)(2+x)}$$

and state the range of values of x for which the expansion is valid

T.102: 4C Qs 1,2, P.31: 4.3 Qs 1,2

Worked Example

Find the quadratic approximation of

$$\frac{2x^2 - 5x - 10}{x^2 - x - 2}$$

T.102: 4C Qs 3-6, P.31: 4.3 Qs 3-6

Worked Example

Find the quadratic approximation of

$$\frac{40x^2 - 37x + 9}{(4x-1)^2(x+2)}$$

T.102: 4C Qs 7+, P.31: 4.3 Qs 7+

Extract from Formulae book

Binomial series

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots + b^n \quad (n \in \mathbb{N})$$

where $\binom{n}{r} = {}^nC_r = \dfrac{n!}{r!(n-r)!}$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1 \times 2}x^2 + \ldots + \frac{n(n-1)\ldots(n-r+1)}{1 \times 2 \times \ldots \times r}x^r + \ldots \quad (|x| < 1, n \in \mathbb{R})$$

A2 2018 Paper 1

Binomial Expansion

11. (a) Use binomial expansions to show that $\sqrt{\dfrac{1+4x}{1-x}} \approx 1 + \dfrac{5}{2}x - \dfrac{5}{8}x^2$

(6)

A student substitutes $x = \dfrac{1}{2}$ into both sides of the approximation shown in part (a) in an attempt to find an approximation to $\sqrt{6}$

(b) Give a reason why the student **should not** use $x = \dfrac{1}{2}$

(1)

(c) Substitute $x = \dfrac{1}{11}$ into

$$\sqrt{\frac{1+4x}{1-x}} = 1 + \frac{5}{2}x - \frac{5}{8}x^2$$

to obtain an approximation to $\sqrt{6}$. Give your answer as a fraction in its simplest form.

(3)

Exams

- Formula Booklet
- Past Papers
- Practice Papers
- past paper Qs by topic

Past paper practice by topic. Both new and old specification can be found via this link on hgsmaths.com

Question	Scheme	Marks	AOs
11 (a)	$\sqrt{\frac{1+4x}{1-x}} = (1+4x)^{0.5}\times(1-x)^{-0.5}$	B1	3.1a
	$(1+4x)^{0.5} = 1+0.5\times(4x)+\frac{0.5\times-0.5}{2}\times(4x)^2$	M1	1.1b
	$(1-x)^{-0.5} = 1+(-0.5)(-x)+\frac{(-0.5)\times(-1.5)}{2}(-x)^2$	M1	1.1b
	$(1+4x)^{0.5}=1+2x-2x^2$ and $(1-x)^{-0.5}=1+0.5x+0.375x^2$ oe	A1	1.1b
	$(1+4x)^{0.5}\times(1-x)^{-0.5} = (1+2x-2x^2 \ldots)\times\left(1+\frac{1}{2}x+\frac{3}{8}x^2\ldots\right)$ $=1+\frac{1}{2}x+\frac{3}{8}x^2+2x+x^2-2x^2+\ldots$ $=A+Bx+Cx^2$	dM1	2.1
	$=1+\frac{5}{2}x-\frac{5}{8}x^2 \ldots$ *	A1*	1.1b
		(6)	
(b)	Expression is valid $\lvert x\rvert<\frac{1}{4}$. Should not use $x=\frac{1}{2}$ as $\frac{1}{2}>\frac{1}{4}$	B1	2.3
		(1)	
(c)	Substitute $x=\frac{1}{11}$ into $\sqrt{\frac{1+4x}{1-x}}=1+\frac{5}{2}x-\frac{5}{8}x^2$	M1	1.1b
	$\sqrt{\frac{3}{2}}=\frac{1183}{968}$	A1	1.1b
	$\frac{1183}{484}$ or $\frac{2904}{1183}$ (as $\sqrt{6}$ oe)	A1	2.1
		(3)	
(10 marks)			

Summary of Key Points

Summary of key points

1 This form of the binomial expansion can be applied to negative or fractional values of n to obtain an infinite series:

$$(1+x)^n = 1 + nx + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-2)x^3}{3!} + \ldots + \frac{n(n-1)\ldots(n-r+1)x^r}{r!} + \ldots$$

The expansion is valid when $|x| < 1$, $n \in \mathbb{R}$.

2 The expansion of $(1+bx)^n$, where n is negative or a fraction, is valid for $|bx| < 1$, or $|x| < \frac{1}{|b|}$.

3 The expansion of $(a+bx)^n$, where n is negative or a fraction, is valid for $\left|\frac{b}{a}x\right| < 1$ or $|x| < \left|\frac{a}{b}\right|$.

T.104: mixed ex, P.33: BSG