KING EDWARD VI HANDSWORTH GRAMMAR SCHOOL FOR BOYS

KING EDWARD VI ACADEMY TRUST BIRMINGHAM

Year 13
Applied Mathematics
M2 8 Further Kinematics

HGS Maths

Dr Frost Course

Name: ______________________________

Class: ______________________________

Contents

Prior knowledge check

Prior knowledge check

1 For the vectors $\mathbf{s} = \begin{pmatrix} 5 \\ -12 \end{pmatrix}$ and $\mathbf{t} = \begin{pmatrix} 4 \\ -7 \end{pmatrix}$, find:

a $3\mathbf{s} + \mathbf{t}$ **b** $2\mathbf{s} - 5\mathbf{t}$

c the unit vector in the direction of **s**.

Give your answers in the form $a\mathbf{i} + b\mathbf{j}$.

←Pure Year 1, Chapter 11

2 A particle moves in a straight line with acceleration 5 m s^{-2}. The initial velocity of the particle is 3 m s^{-1}. When $t = 4$ seconds, find:

a the velocity of the particle

b the displacement from the starting point.

←Year 1, Chapter 9

3 **a** Differentiate: **i** $3e^{2x}$ **ii** $2\sin 3x$

b Integrate: **i** $4e^{3x+1}$ **ii** $5\cos 2\pi x$

←Pure Year 2, Chapters 9, 11

8.1) Vectors in kinematics

✐ Position vector $\boldsymbol{r}$ of particle:

$$\boldsymbol{r} = \boldsymbol{r_0} + \boldsymbol{v}t$$

where r_0 is initial position and $\boldsymbol{v}$ is velocity.

T.162 8A: Q1

Notes

Worked Example

A particle starts from the position vector $(7\boldsymbol{i} - 2\boldsymbol{j})$ m and moves with constant velocity $(-3\boldsymbol{i} + \boldsymbol{j})$ ms^{-1}.

(a) Find the position vector of the particle 2 seconds later.

(b) Find the time at which the particle is due north of the origin.

Worked Example

A particle P has velocity $(-\boldsymbol{i} + 5\boldsymbol{j})$ ms^{-1}. The particle moves with constant acceleration $\boldsymbol{a} = (4\boldsymbol{i} + 7\boldsymbol{j})$ ms^{-2}. Find:

(a) the speed of the particle at time $t = 6$ seconds.

(b) the bearing on which it is travelling at time $t = 6$ seconds.

T.162 8A: QS 2-10, P.68 8.1: Qs 1-6

Worked Example

An ice skater is skating on a large flat ice rink. At time $t = 0$ the skater is at a fixed point O and is travelling with velocity $(-4\boldsymbol{i} - 9\boldsymbol{j})\ ms^{-1}$.
At time $t = 5$ s the skater is travelling with velocity $(-34\boldsymbol{i} + 29\boldsymbol{j})\ ms^{-1}$.
Relative to O, the skater has position vector $\boldsymbol{s}$ at time t seconds.
Modelling the ice skater as a particle with constant acceleration, find:

(a) The acceleration of the ice skater
(b) An expression for $\boldsymbol{s}$ in terms of t
(c) The time at which the skater is directly south-west of O.

A second skater travels so that she has position vector
$\boldsymbol{r} = (-132\boldsymbol{i} + (6 - 22t)\boldsymbol{j})$ m relative to O at time t.
(d) Show that the two skaters will meet.

Worked Example

A ship S is moving with constant velocity $(2\boldsymbol{i} + 4\boldsymbol{j})\ kmh^{-1}$.
At time $t = 0$, the position vector of S is $(-3\boldsymbol{i} + 5\boldsymbol{j})\ km$.
A ship T is moving with constant velocity $(6\boldsymbol{i} + n\boldsymbol{j})\ kmh^{-1}$
At time $t = 0$, the position vector of T is $(-15\boldsymbol{i} + 2\boldsymbol{j})\ km$.
The two ships meet at point P.
Find the value of n and the distance OP

T.162 8A: Qs 11+, P.68 8.1: Qs 7+

8.2) Vector methods with projectiles

Notes

Worked Example

A ball is struck by a racket from a point A which has position vector $40j$ m relative to a fixed origin O. Immediately after being struck, the ball has velocity $(7i + 10j)$ ms^{-1}, where i and j are unit vectors horizontally and vertically respectively. After being struck, the ball travels freely under gravity until it strikes the ground at point B.

(a) Find the speed of the ball 3 seconds after being struck.

(b) Find an expression for the position vector, r, of the ball relative to O at time t seconds.

(c) Hence determine the distance OB.

Worked Example

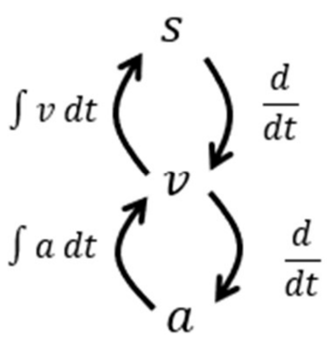
The point O is a fixed point on a horizontal plane.
A ball is projected from O with velocity $(4\boldsymbol{i} + 8\boldsymbol{j})\ ms^{-1}$.
The ball passes through a point A at time t seconds after projection. The point B is on the horizontal plane vertically below A. It is given that $OB = 4AB$. Find:

a) The value of t

b) The speed of the ball at the instant it passes through A

T.166 8B: Qs all, P.69 8.2: Qs all

8.3) Variable acceleration in one dimension



In Mechanics Yr1 we saw that velocity was the rate of change of displacement, and thus $v = \frac{ds}{dt}$. Similarly acceleration is the rate of change of velocity, and thus $a = \frac{dv}{dt}$

Let's stick to one-dimension for the moment, but you may need to **differentiate more complex functions of t that use Pure Year 2 techniques**.

Notes

Worked Example

A particle is moving in a straight line with acceleration at time t seconds given by

$$a = \cos 5\pi t \text{ ms}^{-2}, \qquad t \geq 0$$

The velocity of the particle at time $t = 0$ is $\frac{1}{5\pi} ms^{-1}$. Find:

(a) an expression for the velocity at time t seconds

(b) the maximum speed

(c) the distance travelled in the first 6 seconds.

Worked Example

A particle of mass 12kg is moving on the positive x-axis. At time t seconds the displacement, s, of the particle from the origin is given by

$$s = 3t^{\frac{5}{2}} + \frac{e^{-3t}}{4} \text{ m}, \qquad t \geq 0$$

(a) Find the velocity of the particle when $t = 2.5$.

Given that the particle is acted on by a single force of variable magnitude F N which acts in the direction of the positive x-axis,

(b) Find the value of F when $t = 4$

T.168 8C: Qs all, P.71 8.3: Qs all

8.4) Differentiating vectors

Suppose that $\boldsymbol{v} = \begin{pmatrix} t^2 \\ \sin t \end{pmatrix}$. What would be the acceleration?

We can simply differentiate the $\boldsymbol{i}$ and $\boldsymbol{j}$ components independently:

$$\boldsymbol{a} = \frac{d\boldsymbol{v}}{dt} = \begin{pmatrix} 2t \\ \cos t \end{pmatrix}$$

✎ If $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j}$ then $\boldsymbol{v} = \frac{d\boldsymbol{r}}{dt} = \dot{\boldsymbol{r}} = \dot{x}\boldsymbol{i} + \dot{y}\boldsymbol{j}$

and $\boldsymbol{a} = \frac{d\boldsymbol{v}}{dt} = \frac{d^2\boldsymbol{r}}{dt} = \ddot{\boldsymbol{r}} = \ddot{x}\boldsymbol{i} + \ddot{y}\boldsymbol{j}$

Notational note: Dot notation is a short-hand for differentiation <u>with respect to time</u>: $\dot{x} = \frac{dx}{dt}$
Its use is common in Physics.

Notes

Worked Example

A particle P of mass 1.6kg is acted on by a single force $\mathbf{F}$ N. Relative to a fixed origin O, the position vector of P at time t seconds is $\boldsymbol{r}$ metres, where

$$\boldsymbol{r} = 5t^3\boldsymbol{i} + 20t^{-\frac{1}{5}}\boldsymbol{j}, \qquad t \geq 0$$

Find:

(a) the speed of P when $t = 2$

(b) the acceleration of P as a vector when $t = 4$

(c) $\mathbf{F}$ when $t = 4$.

T.171 8D: Qs all, P.72 8.4: Qs all

8.5) Integrating vectors

Notes

Worked Example

A particle P is moving in a plane. At time t seconds, its velocity $\boldsymbol{v}$ ms^{-1} is given by

$$\boldsymbol{v} = 2t\boldsymbol{i} + \frac{1}{3}t^2\boldsymbol{j}, \qquad t \geq 0$$

When $t = 0$, the position vector of P with respect to a fixed O is $(5\boldsymbol{i} - 4\boldsymbol{j})$ m.
Find the position vector of P at time t seconds.

Worked Example

A particle P is moving in a plane so that, at time t seconds, its acceleration is $(3\boldsymbol{i} - 4t\boldsymbol{j})\ ms^{-2}$.
When $t = 2$, the velocity of P is $-3\boldsymbol{j}\ ms^{-1}$ and the position vector of P is $(20\boldsymbol{i} + 3\boldsymbol{j})$ m with respect to a fixed origin O. Find:

(a) the angle between the direction of motion of P and $\boldsymbol{j}$ when $t = 3$

(b) the distance of P from O when $t = 0$.

T.175 8E: Qs 1-3, P.73 8.5: Qs 1-3

Worked Example

The velocity of a particle P at time t seconds is
$\left((6t^2 - 4)\boldsymbol{i} + 10\boldsymbol{j}\right)\ ms^{-1}$.
When $t = 0$, the position vector of P with respect to a fixed origin O is $(5\boldsymbol{i} - 3\boldsymbol{j})\ m$.
A second particle Q moves with constant velocity
$(3\boldsymbol{i} + 5\boldsymbol{j})\ ms^{-1}$.
When $t = 0$, the position vector of Q with respect to the fixed origin O is $2\boldsymbol{j}\ m$.
Prove that P and Q collide.

T.175 8E: Qs 4+, P.73 8.5: Qs 4+

Mechanics

Kinematics

For motion in a straight line with constant acceleration:

$$v = u + at$$

$$s = ut + \tfrac{1}{2}at^2$$

$$s = vt - \tfrac{1}{2}at^2$$

$$v^2 = u^2 + 2as$$

$$s = \tfrac{1}{2}(u + v)t$$

8. [*In this question* **i** *and* **j** *are horizontal unit vectors due east and due north respectively and position vectors are given relative to the fixed point O.*]

A particle P moves with constant acceleration.
At time $t = 0$, the particle is at O and is moving with velocity $(2\mathbf{i} - 3\mathbf{j})\,\text{m s}^{-1}$
At time $t = 2$ seconds, P is at the point A with position vector $(7\mathbf{i} - 10\mathbf{j})\,\text{m}$.

(a) Show that the magnitude of the acceleration of P is $2.5\,\text{m s}^{-2}$

(4)

At the instant when P leaves the point A, the acceleration of P changes so that P now moves with constant acceleration $(4\mathbf{i} + 8.8\mathbf{j})\,\text{m s}^{-2}$

At the instant when P reaches the point B, the direction of motion of P is north east.

(b) Find the time it takes for P to travel from A to B.

(4)

Exams

- Formula Booklet
- Past Papers
- Practice Papers
- past paper Qs by topic

Past paper practice by topic. Both new and old specification can be found via this link on hgsmaths.com

Question	Scheme	Marks	AOs
8(a)	Use of $\mathbf{r} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$: $(7\mathbf{i} - 10\mathbf{j}) = 2(2\mathbf{i} - 3\mathbf{j}) + \frac{1}{2}\mathbf{a}2^2$	M1	3.1b
	$\mathbf{a} = (1.5\mathbf{i} - 2\mathbf{j})$	A1	1.1b
	$\lvert\mathbf{a}\rvert = \sqrt{1.5^2 + (-2)^2}$	M1	1.1b
	$= 2.5\ \text{m s}^{-2}$ * GIVEN ANSWER	A1*	2.1
		(4)	
(b)	Use of $\mathbf{v} = \mathbf{u} + \mathbf{a}t$: $(2\mathbf{i} - 3\mathbf{j}) + 2(1.5\mathbf{i} - 2\mathbf{j})$	M1	3.1b
	$= (5\mathbf{i} - 7\mathbf{j})$	A1	1.1b
	$\mathbf{v} = (5\mathbf{i} - 7\mathbf{j}) + t(4\mathbf{i} + 8.8\mathbf{j}) = (5 + 4t)\mathbf{i} + (8.8t - 7)\mathbf{j}$ and $(5 + 4t) = (8.8t - 7)$	M1	3.1b
	$t = 2.5$ (s)	A1	1.1b
		(4)	
(8 marks)			

Summary of Key Points

Summary of key points

1 If a particle starts from the point with position vector $\mathbf{r}_0$ and moves with constant velocity $\mathbf{v}$, then its displacement from its initial position at time t is $\mathbf{v}t$ and its position vector $\mathbf{r}$ is given by $\mathbf{r} = \mathbf{r}_0 + \mathbf{v}t$.

2 For an object moving in a plane with constant acceleration:

- $\mathbf{v} = \mathbf{u} + \mathbf{a}t$
- $\mathbf{r} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$

where

- $\mathbf{u}$ is the initial velocity
- $\mathbf{a}$ is the acceleration
- $\mathbf{v}$ is the velocity at time t
- $\mathbf{r}$ is the displacement at time t.

3 If $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$, then $\mathbf{v} = \frac{d\mathbf{r}}{dt} = \dot{\mathbf{r}} = \dot{x}\mathbf{i} + \dot{y}\mathbf{j}$

and $\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2} = \ddot{\mathbf{r}} = \ddot{x}\mathbf{i} + \ddot{y}\mathbf{j}$

4 $\mathbf{v} = \int \mathbf{a}\,dt$ and $\mathbf{r} = \int \mathbf{v}\,dt$

T.177 mixed ex, P.75 BSG