Write your name here

Surname	Other names

Pearson Edexcel GCE

Centre Number | Candidate Number

A level Further Mathematics
Core Pure Mathematics
Practice Paper 3

You must have:
Mathematical Formulae and Statistical Tables (Pink)

Total Marks

Instructions

- Use black ink or ball-point pen.
- If pencil is used for diagrams/sketches/graphs it must be dark (HB or B).
- Fill in the boxes at the top of this page with your name, centre number and candidate number.
- Answer all the questions and ensure that your answers to parts of questions are clearly labelled.
- Answer the questions in the spaces provided – there may be more space than you need.
- You should show sufficient working to make your methods clear. Answers without working may not gain full credit.
- Inexact answers should be given to three significant figures unless otherwise stated.

Information

- A booklet 'Mathematical Formulae and Statistical Tables' is provided.
- There are **7** questions in this question paper. The total mark for this paper is **70**.
- The marks for each question are shown in brackets – use this as a guide as to how much time to spend on each question.
- Calculators must not be used for questions marked with a * sign.

Advice

- Read each question carefully before you start to answer it.
- Try to answer every question.
- Check your answers if you have time at the end.
- If you change your mind about an answer, cross it out and put your new answer and any working underneath.

1. (a) Express $\dfrac{2}{4r^2-1}$ in partial fractions.

(2)

(b) Hence use the method of differences to show that

$$\sum_{r=1}^{n} \frac{1}{4r^2-1} = \frac{n}{2n+1}$$

(3)

(Total 5 marks)

[Mark scheme for Question 1]

[Examiner Comments]

2.

$$z_1 = 3\text{i} \text{ and } z_2 = \frac{6}{1+\text{i}\sqrt{3}}$$

(a) Express z_2 in the form $a + \text{i}b$, where a and b are real numbers.

(2)

(b) Find the modulus and the argument of z_2, giving the argument in radians in terms of π.

(4)

(c) Show the three points representing z_1, z_2 and $(z_1 + z_2)$ respectively, on a single Argand diagram.

(2)

(Total 8 marks)

[Mark scheme for Question 2]

[Examiner Comments]

---

3. The curve C_1 has equation $y = 3\sinh 2x$, and the curve C_2 has equation $y = 13 - 3e^{2x}$.

(a) Sketch the graph of the curves C_1 and C_2 on one set of axes, giving the equation of any asymptote and the coordinates of points where the curves cross the axes.

(4)

(b) Solve the equation $3\sinh 2x = 13 - 3e^{2x}$, giving your answer in the form $\frac{1}{2}\ln k$, where k is an integer.

(5)

(Total 9 marks)

Mark scheme for Question 3

Examiner Comments

---

4. (a) Use the standard results for $\sum_{r=1}^{n} r$ and $\sum_{r=1}^{n} r^2$ to show that

$$\sum_{r=1}^{n} (3r^2 + 8r + 3) = \frac{1}{2}n(2n+5)(n+3)$$

for all positive integers n.

(5)

Given that

$$\sum_{r=1}^{12} (3r^2 + 8r + 3 + k(2^{r-1})) = 3520$$

(b) find the exact value of the constant k.

(4)

(Total 9 marks)

Mark scheme for Question 4

Examiner Comments

5.

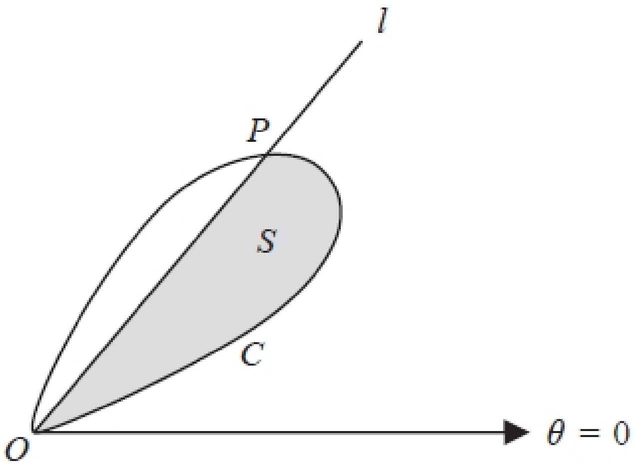


Figure 1

Figure 1 shows a curve C with polar equation $r = a\sin 2\theta$, $0 \leq \theta \leq \frac{\pi}{2}$, and a half-line l.

The half-line l meets C at the pole O and at the point P. The tangent to C at P is parallel to the initial line. The polar coordinates of P are (R, ϕ).

(a) Show that $\cos\phi = \frac{1}{\sqrt{3}}$

(6)

(b) Find the exact value of R.

(2)

The region S, shown shaded in Figure 1, is bounded by C and l.

(c) Use calculus to show that the exact area of S is

$$\frac{1}{36}a^2\left(9\arccos\left(\frac{1}{\sqrt{3}}\right)+\sqrt{2}\right)$$

(7)

(Total 15 marks)

[Mark scheme for Question 5]

[Examiner Comments]

6. $$f(n) = 2^n + 6^n$$

(a) Show that $f(k + 1) = 6f(k) - 4(2^k)$.

(3)

(b) Hence, or otherwise, prove by induction that, for $n \in \mathbb{Z}^+$, $f(n)$ is divisible by 8.

(4)

(Total 7 marks)

Mark scheme for Question 6

Examiner Comments

---

7. At the start of the year 2000, a survey began of the number of foxes and rabbits on an island.

At time t years after the survey began, the number of foxes, f, and the number of rabbits, r, on the island are modelled by the differential equations

$$\frac{df}{dt} = 0.2f + 0.1r$$

$$\frac{dr}{dt} = -0.2f + 0.4r$$

(a) Show that $\frac{d^2 f}{dt^2} - 0.6\frac{df}{dt} + 0.1f = 0$

(3)

(b) Find a general solution for the number of foxes on the island at time t years.

(4)

(c) Hence find a general solution for the number of rabbits on the island at time t years.

(3)

At the start of the year 2000 there were 6 foxes and 20 rabbits on the island.

(d) (i) According to this model, in which year are the rabbits predicted to die out?

(ii) According to this model, how many foxes will be on the island when the rabbits die out?

(iii) Use your answers to parts (i) and (ii) to comment on the model.

(7)

(Total 17 marks)

Mark scheme for Question 7

Examiner Comments

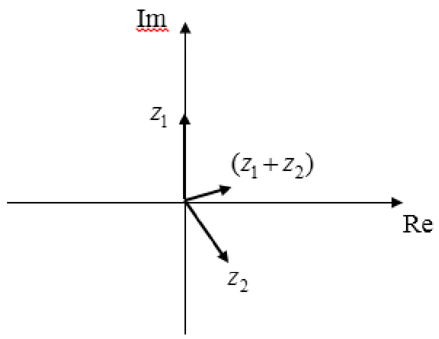
TOTAL FOR PAPER: 70 MARKS

Further Core Pure Mathematics – Practice Paper 03 – Mark scheme –

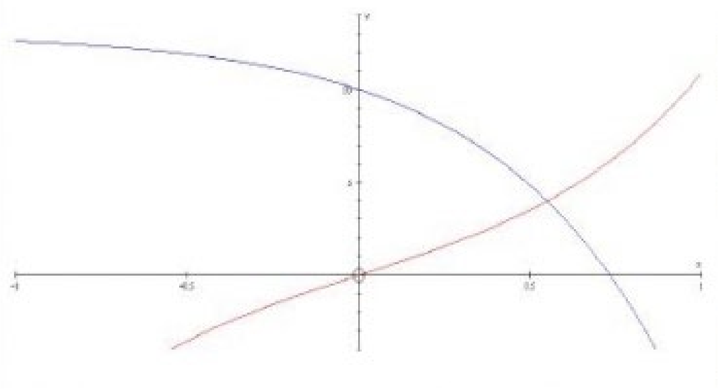
Mark scheme for Question 1 (Examiner comment) (Return to Question 1)

Question	Scheme	Marks
1(a)	$\frac{2}{4r^2-1}=\frac{A}{2r+1}+\frac{B}{2r-1}$	
	$2=A(2r-1)+B(2r+1) \Rightarrow A=-1, \quad B=1$	
	$\frac{2}{4r^2-1}=\frac{1}{2r-1}-\frac{1}{2r+1}$	M1A1
		(2)
(b)	$(2)\sum_{r=1}^{n}\frac{1}{4r^2-1}=\sum_{r=1}^{n}\left(\frac{1}{2r-1}-\frac{1}{2r+1}\right)$	
	$=1-\frac{1}{3}+\frac{1}{3}-\frac{1}{5}+\frac{1}{5}-\frac{1}{7}+\ldots+\frac{1}{2n-1}-\frac{1}{2n+1}=1-\frac{1}{2n+1}$	M1A1 ft
	$=\frac{2n+1-1}{2n+1}$	
	$\sum_{r=1}^{n}\frac{1}{4r^2-1}=\frac{n}{2n+1}$ *	
	$(2)\sum_{r=1}^{n}\frac{1}{4r^2-1}=\sum_{r=1}^{n}\left(\frac{1}{2r-1}-\frac{1}{2r+1}\right)$	A1
		(3)
		(5 marks)

Mark scheme for Question 2 (Examiner comment) (Return to Question 2)

Question	Scheme	Marks
2(a)	$z_2 = \dfrac{6(1-i\sqrt{3})}{(1+i\sqrt{3})(1-i\sqrt{3})} = \dfrac{6(1-i\sqrt{3})}{4}$	M1
	$z_2 = \dfrac{6(1-i\sqrt{3})}{4}\left(= \dfrac{3}{2} - i\dfrac{3}{2}\sqrt{3}\right)$	A1
		(2)
(b)	$\lvert z_2 \rvert = \sqrt{\left(\dfrac{3}{2}\right)^2 + \left(\dfrac{3\sqrt{3}}{2}\right)^2} = \sqrt{\dfrac{9}{4} + \dfrac{27}{4}}$	M1
	The modulus of z_2 is 3	A1
	$\tan\theta = (\pm)\sqrt{3}$ and attempts to find θ	M1
	and the argument is $-\dfrac{\pi}{3}$	A1
		(4)
(c)		M1A1
		(2)
		(8 marks)

Mark scheme for Question 3 (Examiner comment) (Return to Question 3)

Question	Scheme	Marks
3(a)		
	Graph of $y = 3\sinh 2x$	**B1**
	Shape of $-e^{2x}$ graph	**B1**
	Asymptote: $y = 13$	**B1**
	Value 10 on y axis and value 0.7 or $\frac{1}{2}\ln\left(\frac{13}{3}\right)$ on x axis	**B1**
		(4)
(b)	Use definition $\frac{3}{2}(e^{2x} - e^{-2x}) = 13 - 3e^{2x} \rightarrow 9e^{4x} - 26e^{2x} - 3 = 0$ to form quadratic	**M1A1**
	$\therefore e^{2x} = -\frac{1}{9}$ or 3	**DM1** **A1**
	$\therefore x = \frac{1}{2}\ln(3)$	**B1**
		(5)
		(9 marks)

Mark scheme for Question 4 (Examiner comment) (Return to Question 4)

Question	Scheme	Marks
4(a)	$\sum_{r=1}^{n}\left(3r^2+8r+3\right)$	
	$= \frac{3}{6}n(n+1)(2n+1)+\frac{8}{2}n(n+1)+3n$	**M1A1**
	$= \frac{1}{2}n(n+1)(2n+1)+4n(n+1)+3n$	**B1**
	$= \frac{1}{2}n\left((2n+1)(n+1)+8(n+1)+6\right)$	
	$= \frac{1}{2}n\left(2n^2+3n+1+8n+8+6\right)$	**M1**
	$= \frac{1}{2}n\left(2n^2+11n+15\right)$	
	$= \frac{1}{2}n(2n+5)(n+3)$ (*)	**A1*cso**
		(5)
(b)	$\sum_{r=1}^{12}\left(3r^2+8r+3+k(2^{r-1})\right) = 3520$	
	$\sum_{r=1}^{12}\left(3r^2+8r+3\right) = \frac{1}{2}(12)(29)(15)\ \{= 2610\}$	**M1**
	$\sum_{r=1}^{12}(2^{r-1}) = \frac{1(1-2^{12})}{1-2}\ \{= 4095\}$	**M1**
	So, $2610 + 4095k = 3520 \Rightarrow 4095k = 910$	**A1**
	giving, $k = \frac{2}{9}$	**A1**
		(4)
		(9 marks)

Question 4 notes:

(i)(a)

M1: Attempts determinant, equates to zero and attempts to solve for a in order to establish the restriction for a

A1: Provides the correct condition for a if **M** has an inverse

Notes for Question 4 *continued*
(i)(b) **B1:** A correct matrix of minors or cofactors **M1:** For a complete method for the inverse **A1ft:** Two correct rows following through their determinant **A1ft:** Fully correct inverse following through their determinant
(ii) **B1:** Shows the statement is true for $n = 1$ **M1:** Assumes the statement is true for $n = k$ **M1:** Attempts to multiply the correct matrices **A1:** Correct matrix in terms of k **A1:** Correct matrix in terms of $k + 1$ **A1:** Correct complete conclusion

Mark scheme for Question 5 (Examiner comment) (Return to Question 5)

Question	Scheme	Marks
5(a)	$(y =)\ r\sin\theta = a\sin 2\theta\sin\theta$	M1
	$\left(\frac{dy}{d\theta} =\right) a(2\cos 2\theta\sin\theta + \sin 2\theta\cos\theta)$	M1 depA1
	$\left(\frac{dy}{d\theta} =\right) 2a\sin\theta\,(\cos 2\theta\ + \cos^2\theta)$	M1
	At P $\frac{dy}{d\theta} = 0 \Rightarrow \sin\theta$ (n/a) or $2\cos^2\theta - 1 + \cos^2\theta = 0$, $\sin\theta = 0$ not needed	M1
	$3\cos^2\theta = 1$	
	$\cos^2\theta = \frac{1}{\sqrt{3}}$ *	A1cso
		(6)
(b)	$r = a\sin 2\theta = 2a\sin\theta\cos\theta$	
	$r = 2a\sqrt{\left(1-\frac{1}{3}\right)}\sqrt{\frac{1}{3}} = 2a\frac{\sqrt{2}}{3}$	M1A1
		(2)
(c)	Area = $\int_0^{\phi}\frac{1}{2}r^2 d\theta = \frac{1}{2}a^2\int_0^{\phi}\sin^2 2\theta\, d\theta$	M1
	$= \frac{1}{2}a^2\int_0^{\phi}\frac{1}{2}(1-\cos 4\theta)\,d\theta$	M1
	$= \frac{1}{4}a^2\left[\theta - \frac{1}{4}\sin 4\theta\right]_0^{\phi}$	M1A1
	$= \frac{1}{4}a^2\left[\phi - \frac{1}{4}\left(4\sin\phi\cos\phi\left(2\cos^2\phi - 1\right)\right)\right]$	M1dep on 2nd M mark
	$= \frac{1}{4}a^2\left[\arccos\left(\frac{1}{\sqrt{3}}\right) - \left(\sqrt{\frac{2}{3}}\times\sqrt{\frac{1}{3}}\times\left(\frac{2}{3}-1\right)\right)\right]$	M1 dep(all Ms)
	$\frac{1}{36}a^2\left[9\arccos\left(\frac{1}{\sqrt{3}}\right) + \sqrt{2}\right]$ *	A1
		(7)
		(15 marks)

Mark scheme for Question 6 (Examiner comment) (Return to Question 6)

Question	Scheme		Marks
6(a)	(a) LHS = $f(k+1)=2^{k+1}+6^{k+1}$	OR RHS = $=6f(k)-4(2^k)=6(2^k+6^k)-4(2^k)$	**M1**
	$=2(2^k)+6(6^k)$	$=2(2^k)+6(6^k)$	**M1**
	$=6(2^k+6^k)-4(2^k)=6f(k)-4(2^k)$	$=2^{k+1}+6^{k+1}=f(k+1)$ (*)	**A1**
	OR $f(k+1)-6f(k)=2^{k+1}+6^{k+1}-6(2^k+6^k)$		
	Alternative		
	OR $f(k+1)-6f(k)=2^{k+1}+6^{k+1}-6(2^k+6^k)$		**M1**
	$=(2-6)(2^k)=-4.2^k$,	and so $f(k+1)=6f(k)-4(2^k)$	**A1A1**
			(3)
(b)	$n=1$: $f(1)=2^1+6^1=8$, **which is divisible** by 8		**B1**
	Either Assume f(*k*) divisible by 8 and try to use $f(k+1)=6f(k)-4(2^k)$	**Or** Assume f(*k*) divisible by 8 and try to use $f(k+1)-f(k)$ or $f(k+1)+f(k)$ including factorising $6^k=2^k3^k$	**M1**
	Show $4(2^k)=4\times 2(2^{k-1})=8(2^{k-1})$ or $8(\frac{1}{2}2^k)$ Or valid statement	$=2^3 2^{k-3}(1+5.3^k)$ or $=2^3 2^{k-3}(3+7.3^k)$ o.e.	**A1**
	Deduction that result is implied for $n=k+1$ and so is true for positive integers by induction (may include $n=1$ true here)	Deduction that result is implied for $n=k+1$ and so is true for positive integers by induction (must include explanation of why $n=2$ is also true here)	**A1cso**
			(4)
			(7 marks)

Mark scheme for Question 7 (Examiner comment) (Return to Question 7)

Question	Scheme	Marks	AOs
7(a)	$r=10\frac{df}{dt}-2f \Rightarrow \frac{dr}{dt}=10\frac{d^2f}{dt^2}-2\frac{df}{dt}$	M1	2.1
	$10\frac{d^2f}{dt^2}-2\frac{df}{dt}=-0.2f+0.4\left(10\frac{df}{dt}-2f\right)$	M1	2.1
	$\frac{d^2f}{dt^2}-0.6\frac{df}{dt}+0.1f=0$ *	A1*	1.1b
		(3)	
(b)	$m^2-0.6m+0.1=0 \Rightarrow m=\frac{0.6\pm\sqrt{0.6^2-4\times0.1}}{2}$	M1	3.4
	$m=0.3\pm0.1\text{i}$	A1	1.1b
	$f=e^{\alpha t}\left(A\cos\beta t+B\sin\beta t\right)$	M1	3.4
	$f=e^{0.3t}\left(A\cos 0.1t+B\sin 0.1t\right)$	A1	1.1b
		(4)	
(c)	$\frac{df}{dt}=0.3e^{0.3t}\left(A\cos 0.1t+B\sin 0.1t\right)+0.1e^{0.3t}\left(B\cos 0.1t-A\sin 0.1t\right)$	M1	3.4
	$r=10\frac{df}{dt}-2f$ $=e^{0.3t}\left(\left(3A+B\right)\cos 0.1t+\left(3B-A\right)\sin 0.1t\right)$ $-2e^{0.3t}\left(A\cos 0.1t+B\sin 0.1t\right)$	M1	3.4
	$r=e^{0.3t}\left(\left(A+B\right)\cos 0.1t+\left(B-A\right)\sin 0.1t\right)$	A1	1.1b
		(3)	
(d)(i)	$t=0,\ f=6,\ \Rightarrow\ A=6$	M1	3.1b
	$t=0,\ r=20,\ \Rightarrow\ B=14$	M1	3.3
	$r=e^{0.3t}(20\cos 0.1t+8\sin 0.1t)=0$	M1	3.1b
	$\tan 0.1t=-2.5$	A1	1.1b
	2019	A1	3.2a
		(5)	
(d)(ii)	3750 foxes	B1	3.4
		(1)	

Question	Scheme	Marks	AOs
7(d)(iii)	e.g. the model predicts a large number of foxes are on the island when the rabbits have died out and this may not be sensible.	**B1**	3.5a
		(1)	
			(17 marks)

Notes:

(a)
M1: Attempts to differentiate the first equation with respect to *t*
M1: Proceeds to the printed answer by substituting into the second equation
A1*: Achieves the printed answer with no errors

(b)
M1: Uses the model to form and solve the auxiliary equation
A1: Correct values for *m*
M1: Uses the model to form the CF
A1: Correct CF

(c)
M1: Differentiates the expression for the number of foxes
M1: Uses this result to find an expression for the number of rabbits
A1: Correct equation

(d)(i)
M1: Realises the need to use the initial conditions in the model for the number of foxes
M1: Realises the need to use the initial conditions in the model for the number of rabbits to find both unknown constants
M1: Obtains an expression for *r* in terms of *t* and sets = 0
A1: Rearranges and obtains a correct value for tan
A1: Identifies the correct year

(d)(ii)
B1: Correct number of foxes

(d)(iii)
B1: Makes a suitable comment on the outcome of the model

Further Core Pure Mathematics – Practice Paper 03 – Examiner report –

Examiner comment for Question 1 (Mark scheme) (Return to Question 1)

1. The vast majority of students confidently separated the given function into partial fractions and demonstrated good understanding of the method of differences. Some failed to appreciate the need to divide by 2 for the required summation. Good explanation and presentation of the method was generally in evidence.

Examiner comment for Question 2 (Mark scheme) (Return to Question 2)

2. Almost all candidates made a confident attempt at part (a). However, in part (b) a significant number of candidates neglected to calculate the modulus, indicating that they did not read the question properly. When the modulus was used it was usually found correctly. Almost all candidates recognised the use of the tan function to calculate the argument. However, many candidates did not make use of an argand diagram or their knowledge of complex numbers to use the correct sign for the argument, leaving their answer as $\frac{\pi}{3}$. The Argand diagram in part (c) was frequently completed successfully, although not all candidates could place z_2 in the correct quadrant, even if they had correctly calculated part (a).

Examiner comment for Question 3 (Mark scheme) (Return to Question 3)

3. This was a well-answered question in general with 34% of the candidates gaining full marks and just over 9% fewer than 4 marks.
In part (a) of the 2 graphs required to be sketched, $y = 3\sinh 2x$ was drawn best. Occasionally a cosh graph was given and very occasionally the curvature was incorrect but this mark was usually gained. The exponential curve was less well drawn; in transforming e^x, candidates seemed to miss the reflection and showed exponential curves decreasing from second to first or fourth quadrants and flattening off. The mark most often not gained was for the asymptote; the *equation* of the asymptote was asked for and so just marking 13 on the y-axis was not sufficient, and some candidates lost the mark for including other asymptotes. Sometimes the values of the two intercepts were correctly found in the script but wrongly attached to the graph..

Part (b) was very well answered. The majority of candidates knew what to do and while some made basic algebraic errors with signs or constants most successfully arrived at the correct quadratic in e^{2x}. Most then solved correctly for x, rejecting or ignoring the root $-\frac{1}{9}$, although a small number of candidates gave 2 answers, thus losing the final mark.

Examiner comment for Question 4 (Mark scheme) (Return to Question 4)

4. Almost all candidates gained full marks in part (a), helped by knowing what they were trying to achieve. In some cases, this involved more than one attempt. A good proportion initially extracted $n/2$ as a common factor so that the six terms in the brackets were easy to simplify. Those who chose 1/6 or $n/6$ as their factor made a lot of work for themselves, which sometimes led to errors. Part (b) proved to be a good discriminator. Whilst many candidates successfully used $n = 12$ in the result from part (a), recognition of the geometric progression frequently challenged candidates, many of whom ignored the summation. It was quite rare to see use of the geometric sum formula, with a fair number listing and then summing the twelve terms, for which they could score full marks, but wasted time, and sometimes caused

errors. A few candidates apparently realised they should use a sum formula but chose that of an arithmetic progression.

Examiner comment for Question 5 **(Mark scheme)** **(Return to Question 5)**

5. This was unsurprisingly a challenging question for many candidates, although the majority scored well in part (a) and in the early stages of part (c). There was a good number of clear accurate solutions which demonstrated a thorough understanding of this topic but many candidates made what are quite elementary errors for Further Mathematics students. There was much poor use of trigonometric identities, even such basic ones as $\sin 2\theta = 2\sin\theta\cos\theta$ and an inability to determine $\cos\theta$ from $\sin\theta = \frac{\sqrt{2}}{\sqrt{3}}$ or $\tan\theta = \sqrt{2}$ using an appropriate method. Some couldn't get from $\sin\theta = ...$ or $\tan\theta = ...$ to $\cos\theta = ...$ without using the inverse trigonometric functions on their calculators. Relatively few established why the positive root was correct. This difficulty was also evident in part (b) where calculators were often used to give decimal answers. It was prevalent in part (c), where the majority of candidates knew the correct area formula, substituted for y correctly and were able to change $\sin^2 2\theta$ into an expression in $\cos 4\theta$. Most however used their calculators to reach the given answer rather than find an exact value of $\sin 4\theta$ using identities after their (mainly successful) integration.

Many candidates used the double angle formula for $\sin 2\theta$ to obtain expressions in $\sin\theta$ or $\cos\theta$ before differentiating. A wide variety of different methods to reach the solution were seen, depending on when the identities were used. A minority of candidates obtained incorrect derivatives of y, as a result of incorrect differentiation of $\cos\theta$ and/or $\sin\theta$ and/or $\sin 2\theta$ and/or $\sin^2\theta$ and/or $\cos^3\theta$. Perhaps inevitably there were sign errors in many responses. Integration and differentiation notation is still challenging for a significant number of candidates.

Examiner comment for Question 6 **(Mark scheme)** **(Return to Question 6)**

6. Well explained logical explanations to both parts were rare and indicated a very good candidate.

In part (a) those who began with the RHS and attempted to reach the LHS had most success. Elegant proofs were in the minority, with many candidates forced to start with the LHS and the RHS separately, and then attempting to meet in the middle. Those who adopt this approach should be aware of the need to reach a conclusion. Considering $f(k + 1) - 6f(k)$ was a productive starting point for some, yielding neat efficient proofs.

Part (b) was probably the toughest question on the paper. Far too many candidates ignored the "hence" and just considered $f(k + 1) - f(k)$, because that is what they usually do . Of these, very few could deal convincingly with the 6^k term that resulted, and just as few returned to making $f(k + 1)$ the subject before drawing a conclusion. Of those taking the approach suggested in the question, relatively few could demonstrate convincingly that $f(k + 1)$ was divisible by 8. Many immediately reverted to the original definitions and went nowhere. One very good candidate proved that 4×2^k was divisible by 8 using induction, then used that result in the induction method to answer the question.

There was some, fortunately not too common, highly creative work with powers, but the major problems were the confused and muddled style of candidates' answers. It was rare to see A and B divisible by 8 => A ± B divisible by 8. Also an accurate, concise, end statement making reference to the result being true for all positive integers (or equivalent) was extremely rare. Yet it was clear that teachers had tried hard to instil the essentials of the argument into their pupils, with many setting out “Basis, Assumption, Induction, Conclusion” or similar. Indeed 21% of the candidates achieved full marks on this question.

Examiner comment for Question 7 **(Mark scheme)** **(Return to Question 7)**

7. No Examiner's Report available for this question. (Taken from SAMs)