Mark Scheme (Standardisation)
November 2021

Pearson Edexcel GCE in A Level Further Mathematics
Paper 9FM0/3A

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General Marking Guidance

- All candidates must receive the same treatment. Examiners must mark the first candidate in exactly the same way as they mark the last.
- Mark schemes should be applied positively. Candidates must be rewarded for what they have shown they can do rather than penalised for omissions.
- Examiners should mark according to the mark scheme not according to their perception of where the grade boundaries may lie.
- There is no ceiling on achievement. All marks on the mark scheme should be used appropriately.
- All the marks on the mark scheme are designed to be awarded. Examiners should always award full marks if deserved, i.e. if the answer matches the mark scheme. Examiners should also be prepared to award zero marks if the candidate's response is not worthy of credit according to the mark scheme.
- Where some judgement is required, mark schemes will provide the principles by which marks will be awarded and exemplification may be limited.
- When examiners are in doubt regarding the application of the mark scheme to a candidate's response, the team leader must be consulted.
- Crossed out work should be marked UNLESS the candidate has replaced it with an alternative response.

General Instructions for Marking

1. The total number of marks for the paper is 75.

2. The Edexcel Mathematics mark schemes use the following types of marks:

- **M** marks: method marks are awarded for 'knowing a method and attempting to apply it', unless otherwise indicated.
- **A** marks: Accuracy marks can only be awarded if the relevant method (M) marks have been earned.
- **B** marks are unconditional accuracy marks (independent of M marks)
- Marks should not be subdivided.

3. Abbreviations

These are some of the traditional marking abbreviations that will appear in the mark schemes.

- bod – benefit of doubt
- ft – follow through
- the symbol $\sqrt{}$ will be used for correct ft
- cao – correct answer only
- cso - correct solution only. There must be no errors in this part of the question to obtain this mark
- isw – ignore subsequent working
- awrt – answers which round to
- SC: special case
- oe – or equivalent (and appropriate)
- dep – dependent
- indep – independent
- dp decimal places
- sf significant figures
- $\ast$ The answer is printed on the paper
- $\sqcap$ The second mark is dependent on gaining the first mark

4. All A marks are 'correct answer only' (cao.), unless shown, for example, as A1 ft to indicate that previous wrong working is to be followed through. After a misread however, the subsequent A marks affected are treated as A ft, but manifestly absurd answers should never be awarded A marks.

Question	Scheme	Marks	AOs
1(a)	Uses $b^2 = a^2\left(1-e^2\right)$ to find a value of *e* look for $20 = 36\left(1-e^2\right)$	**M1**	1.1b
	$e = \frac{2}{3} \Rightarrow$ foci are $\left(\pm 6 \times \text{"their } e\text{"}, 0\right)$	**dM1**	1.1b
	Foci are $\left(\pm 4, 0\right)$	**A1**	1.1b
		(3)	
	Alternative Sets up an equation such as $2\sqrt{p^2+b^2} = 2a$ where *p* is the *x* coordinate of the foci $2\sqrt{p^2+20} = 12$	**M1**	1.1b
	Solves to find the value of *p*	**dM1**	1.1b
	Foci are $\left(\pm 4, 0\right)$	**A1**	1.1b
		(3)	
(b)	Directrices are $x = (\pm)\frac{6}{\text{their } e}$	**M1**	1.1b
	$x = \pm 9$ only	**A1**	1.1b
		(2)	
			(5 marks)

Notes:

(a)

M1: Uses $b^2 = a^2(1-e^2)$ to obtain a value of *e* (allow if $-\frac{2}{3}$ also given)

dM1: Uses $a = 6$ and their value of *e* with $0 < e < 1$, to find at least one focus using $\left((\pm)ae, 0\right)$

A1: Correct foci – both required, including *y* coordinates.

Alternative

M1: Sets up an equation using total distance from foci to point on ellipse = 2a

dM1: Solves to find a value for the *x* coordinate of the foci

A1: Correct foci – both required, including *y* coordinates.

(b)

M1: Uses $x = (\pm)\frac{a}{e}$ with $a = 6$ and their *e* to attempt directrices.

A1: Correct directrices, both required and no other lines

Question	Scheme	Marks	AOs
2(i)	$\dfrac{\sin x-\cos x+1}{\sin x+\cos x-1}=\dfrac{\dfrac{2t}{1+t^2}-\dfrac{1-t^2}{1+t^2}+1}{\dfrac{2t}{1+t^2}+\dfrac{1-t^2}{1+t^2}-1}=\ldots$	**M1**	1.1b
	$=\dfrac{2t-(1-t^2)+1+t^2}{2t+1-t^2-(1+t^2)}$ or $=\dfrac{2t-1+t^2+1+t^2}{2t+1-t^2-1-t^2)}$ numerator $=\dfrac{2t-(1-t^2)+1+t^2}{1+t^2}$ denominator = $=\dfrac{2t+1-t^2-(1+t^2)}{1+t^2}$ and divides	**M1**	2.1
	$=\dfrac{2t^2+2t}{2t-2t^2}\left(=\dfrac{1+t}{1-t}\right)$	**A1**	1.1b
	$=\dfrac{t+1}{1-t}\times\dfrac{1+t}{1+t}=\dfrac{t^2+2t+1}{1-t^2}=\dfrac{1+t^2}{1-t^2}+\dfrac{2t}{1-t^2}$ Alt: $\sec x+\tan x=\dfrac{1+t^2}{1-t^2}+\dfrac{2t}{1-t^2}=\dfrac{1+2t+t^2}{1-t^2}$	**M1**	3.1a
	$=\dfrac{1}{\cos x}+\tan x=\sec x+\tan x$ * Alt: $=\dfrac{(t+1)^2}{(1-t)(1+t)}=\dfrac{1+t}{1-t}=LHS$ hence result proved.*	**A1***	2.1
		(5)	
(ii)	$\displaystyle\int_{(0)}^{\left(\frac{\pi}{2}\right)}\frac{5}{4+2\cos\theta}\,\mathrm{d}\theta=\int_{(0)}^{(1)}\frac{5}{4+2\frac{1-t^2}{1+t^2}}\times\frac{2}{1+t^2}\,\mathrm{d}t$ **Alternatively** $\dfrac{\mathrm{d}t}{\mathrm{d}\theta}=\dfrac{1}{2}\sec^2\left(\dfrac{\theta}{2}\right)=\dfrac{1}{\cos\theta+1}$ leading to $\displaystyle\int_{(0)}^{\left(\frac{\pi}{2}\right)}\frac{5}{4+2\cos\theta}\,\mathrm{d}\theta=\int_{(0)}^{(1)}\frac{5\cos\theta+5}{4+2\cos\theta}\,\mathrm{d}t=\int_{(0)}^{(1)}\frac{5\left(\frac{1-t^2}{1+t^2}\right)+5}{4+2\left(\frac{1-t^2}{1+t^2}\right)}\,\mathrm{d}t$	**M1**	2.1
	$\displaystyle=\int_{(0)}^{(1)}\frac{10}{4(1+t^2)+2(1-t^2)}\,\mathrm{d}t=\int_{(0)}^{(1)}\frac{5}{3+t^2}\,\mathrm{d}t$ o.e.	**A1**	1.1b
	$=\left[5\times\dfrac{1}{\sqrt{3}}\arctan\left(\dfrac{t}{\sqrt{3}}\right)\right]_{(0)}^{(1)}$	**M1**	1.1b

	$=\frac{5}{\sqrt{3}}\left(\arctan\left(\frac{1}{\sqrt{3}}\right)-0\right)$ or $\frac{5}{\sqrt{3}}\left(\arctan\left(\frac{\tan\left(\frac{\pi}{4}\right)}{\sqrt{3}}\right)-\arctan\left(\frac{\tan(0)}{\sqrt{3}}\right)\right)$	**M1**	2.2a
	$=\frac{5\pi\sqrt{3}}{18}$ oe in a surd form e.g. $\frac{5\pi}{6\sqrt{3}}$	**A1**	1.1b
		(5)	
			(10 marks)

Notes:

(i)

M1: Applies the t-formulae to the left-hand side of expression. Allow slips in signs of the terms.

M1: Multiplies numerator and denominator through by $1+t^2$ (allow if they forget to multiply the 1's). Alternative works separately on the numerator and denominator to combine terms and then divides.

A1: Correct $\frac{\text{quadratic}}{\text{quadratic}}$ with terms gathered, award where first seen, need not have cancelled 2t for this mark.

M1: Cancels 2t, multiplies numerator and denominator by $1+t$ and splits to sum of two terms. If working from both sides, this mark is for substituting the t-formulae into the right-hand side and combining to single fraction.

A1*: Correct completion to given result. No errors in proof. If working from both sides, a suitable conclusion is needed, e.g "hence proven".

(ii)

M1: Applies the substitution including the use of $\mathrm{d}\theta=\frac{2}{1+t^2}\mathrm{d}t$ (Limits not needed for first three marks).

A1: Simplifies correctly to a recognisable integrable form.

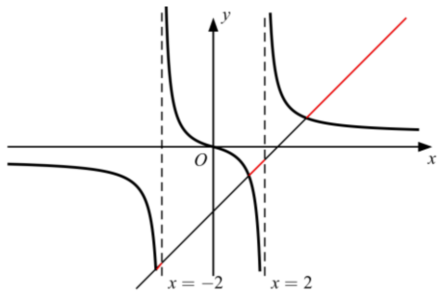
M1: Integrates to the form $K\arctan\left(\frac{t}{a}\right)$ where a^2 is their constant term.

M1: Deduces correct limits and applies them the correct way round OR deduces integral in terms of θ from their integration and applies original limits the correct way.

A1: $\frac{5\pi\sqrt{3}}{18}$ or equivalent in surd form.

Note use of calculator does not lead to the exact value required in the question = 1.51149947

This can score M1 A1 M0 M1 A0

Question	Scheme	Marks	AOs
3	For $x<0$ need $2x-5>\dfrac{x}{-x-2}$ and for $x\geqslant 0$ need $2x-5>\dfrac{x}{x-2}$ and goes on to find the critical values for each.	**M1**	3.1a
	For $x\geqslant 0$: $2x-5=\dfrac{x}{x-2}\Rightarrow 2x^2-10x+10=0\Rightarrow x=...$	**M1**	1.1b
	$x=\dfrac{5\pm\sqrt{5}}{2}$ (oe) awrt 3.62 and awrt 1.38	**A1**	1.1b
	For $x<0$: $2x-5=\dfrac{x}{-x-2}\Rightarrow -2x^2+10=0\Rightarrow x=...$	**M1**	1.1b
	$x=-\sqrt{5}$ only ($\sqrt{5}$ must be rejected at some stage)	**A1**	2.3
	 Uses graph or other means to identify correct regions. Asymptotes must have been considered, but may miss the region near $x=-2$ So e.g. "$-\sqrt{5}<x<-2$" or "$\dfrac{5-\sqrt{5}}{2}<x<2$" or "$x>\dfrac{5+\sqrt{5}}{2}$"	**M1**	3.1a
	Inequality holds when $-\sqrt{5}<x<-2$ or $\dfrac{5-\sqrt{5}}{2}<x<2$ or $x>\dfrac{5+\sqrt{5}}{2}$ Accept equivalent notation, e.g $\left(-\sqrt{5},-2\right)\cup\left(\dfrac{5-\sqrt{5}}{2},2\right)\cup\left(\dfrac{5+\sqrt{5}}{2},\infty\right)$	**A1ft** **A1**	2.2a 2.5
		(8)	
			(8 marks)

Notes:

M1: Considers the two cases of $x<0$ and $x\geqslant 0$ to find critical values. Don't be concerned which side the $x=0$ case is considered part of. Allow if "=" used when considering C.V.s. This mark is for the overall strategy, so both cases must be considered, or equivalent complete longer methods.

M1: Correct method for intersection of line and curve for x positive.

A1: Line and curve intersect at $x=\dfrac{5\pm\sqrt{5}}{2}$

M1: Correct method for intersection of line and curve for x negative.

A1: Line and curve intersect at $x=-\sqrt{5}$ Must have rejected the positive value for this mark (though may be done later)

M1: Uses the graph (or other method) to identify at least one correct region, which must include consideration of the vertical asymptotes. Implied by two correct intervals being given for their critical values. Allow if $y=2x-5$ is added to the sketch and at least two (not necessarily correct) intervals produced as long as the points $x=\pm 2$ are excluded.

A1ft: At least one correct interval identified following through their solutions (as long as it is sensible).

A1: Fully correct solution, all three intervals given – accept alternative notations, may be just listed (no need for unions shown).

Multiplying both sides by $(x-2)^2$ or $(|x|-2)^2$ can score a maximum of M0 M1 A1 M0 A1 M1 A1ft A0

M0 M1: for multiplying through by $(x-2)^2$

$(2x-5)(x-2)^2 > x(x-2)$

$(x-2)\left[(2x-5)(x-2)-x\right] > 0$ leading to a value for x

$(x-2)(x^2-5x+5) > 0$

A1: Line and curve intersect at $x=\dfrac{5\pm\sqrt{5}}{2}$

M0A0: Not finding the point of intersection for negative x

M1 A1ft: for either "$x>\dfrac{5+\sqrt{5}}{2}$" or "$\dfrac{5-\sqrt{5}}{2}<x<2$"

A0:

If they multiply through by $(-x-2)^2$ the other marks can be scored

Question	Scheme	Marks	AOs
4(a)	$\mathbf{n}=\begin{pmatrix}1\\-2\\25\end{pmatrix}$ or any non-zero scalar multiple thereof	**B1**	1.2
		(1)	
(b)	$\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & -2 & 25\\ 3 & -2 & -1\end{vmatrix}=\begin{pmatrix}(-2)(-1)-(-2)(25)\\ -((1)(-1)-(3)(25))\\ (1)(-2)-(3)(-2)\end{pmatrix}=...$	**M1**	1.1b
	$=\begin{pmatrix}52\\76\\4\end{pmatrix}=4\begin{pmatrix}13\\19\\1\end{pmatrix}$ (or correct multiple for their normal vector used)	**A1**	2.1
		(2)	
(c)	Landing direction is perpendicular to $\mathbf{n}\times\mathbf{v_A}$ and $\mathbf{n}$ so required direction is given by $\begin{pmatrix}13\\19\\1\end{pmatrix}\times\begin{pmatrix}1\\-2\\25\end{pmatrix}=...$ **Alternatively** recognises the recognises the landing direction is the line of intersection of the plane containing $\mathbf{n}$ and $\mathbf{v_A}$ and the plane representing the field. Finds the equation of the plane containing $\mathbf{n}$ and $\mathbf{v_A}$ $\begin{pmatrix}x\\y\\z\end{pmatrix}.\begin{pmatrix}13\\19\\1\end{pmatrix}=\begin{pmatrix}3\\2\\-1\end{pmatrix}.\begin{pmatrix}13\\19\\1\end{pmatrix}$ $x-2y+25z=0$ and $13x+19y+z=0$	**M1**	3.1b
	$=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 13 & 19 & 1\\ 1 & -2 & 25\end{vmatrix}=\begin{pmatrix}(19)(25)-(-2)(1)\\ -((13)(25)-(1)(1))\\ (13)(-2)-(1)(19)\end{pmatrix}=...$ **Alternative** Selects a value for either *x*, *y* or *z* and solves simultaneously e.g $z=-5$ leading to $x-2y=125$ and $13x+19y=5 \Rightarrow x=...,\ y=..$	**M1**	3.4
	$=\begin{pmatrix}477\\-324\\-45\end{pmatrix}$ or any positive multiple thereof, e.g $\begin{pmatrix}53\\-36\\-5\end{pmatrix}$ or $\begin{pmatrix}1908\\-1296\\-180\end{pmatrix}$	**A1**	1.1b
		(3)	
(d)	Any acceptable reason e.g - Paths would not be linear - May have some lateral movement	**B1**	3.5b

	- Could be affected by cross winds - Field might not be flat		
		(1)	
			(7 marks)

Notes:

(a)
B1: Correct normal vector (any non-zero scalar multiple thereof is fine)

(b)
M1: Uses their **n** and $\mathbf{v}_A$ in cross product formula. Allow slips in coordinates as long as the intent is clear.
A1: Correct work leading to a multiple of the required vector – may be a different multiple to the one shown if their **n** was different.

(c)
M1: Uses a correct strategy to find the direction, ie realises $\mathbf{v}_A$ must be perpendicular to both the vector from (b) and **n.** Allow for vectors used either way round.
Other methods may be possible – e.g finds the plane containing **n** and $\mathbf{v}_A$ and solves with the plane representing the field.
M1: Uses their answer to (b) with the normal vector to find a vector in the direction required. Allow for vectors either way round.
Alternative find the line of intersection of the plane containing **n** and $\mathbf{v}_A$ and the field.
A1: A correct direction vector, as shown or any positive multiple. For this mark the direction should be correct – so order of vectors must have been correct, or adapted to correct direction if initially incorrect.

(d)
B1: Any correct limitation given. See scheme for examples.

Question	Scheme	Marks	AOs
5(a)	Equations of asymptotes of H are $y=\pm\frac{1}{2}x$ oe e.g $y=\pm\frac{3}{6}x$ or $\frac{x}{6}=\pm\frac{y}{3}$	**B1**	1.1b
		(1)	
(b)	For parabola $2y\frac{\mathrm{d}y}{\mathrm{d}x}=32\Rightarrow\frac{\mathrm{d}y}{\mathrm{d}x}=\ldots$ or $y=\sqrt{32x}\Rightarrow\frac{\mathrm{d}y}{\mathrm{d}x}=\ldots x^{-\frac{1}{2}}$ or $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}}=\frac{2a}{2at}=\ldots$	**M1**	2.1
	Finds the gradient of the normal using $m_N=\frac{-1}{\text{their }\frac{\mathrm{d}y}{\mathrm{d}x}}$ $m_N=-\frac{y}{16}$ or $-t$ or $-\frac{\sqrt{x}}{2\sqrt{2}}$ so $m_N=(\pm)\frac{1}{2}\Rightarrow y=(\pm)8, x=2$	**M1**	3.1a
	Finds the equation of either l_1 or l_2 $y-\text{"8"}=\text{"their }m_N\text{"}(x-\text{"2"})$ or $y-\text{"}-8\text{"}=\text{"their }m_N\text{"}(x-\text{"2"})$	**M1**	1.1b
	l_1 is $y+8=\frac{1}{2}(x-2)$ and l_2 is $y-8=-\frac{1}{2}(x-2)$ oe $y=\frac{1}{2}x-9$ and $y=-\frac{1}{2}x+9$	**A1**	1.1b
		(4)	
(c)	Meet $H\Rightarrow\frac{x^2}{36}-\frac{\left(\pm\left(\frac{1}{2}x-9\right)\right)^2}{9}=1\Rightarrow\frac{x^2}{36}-\frac{\frac{1}{4}x^2-9x+81}{9}=1\Rightarrow x=\ldots$ or $\frac{(18\pm2y)^2}{36}-\frac{y^2}{9}=1\Rightarrow\frac{81\pm18y+y^2}{9}-\frac{y^2}{9}=1\Rightarrow y=\ldots$	**M1**	2.1
	One correct point of intersection $(10,\pm4)$	**A1**	2.2a
	Area OPQ is $\frac{1}{2}\times10\times(4-(-4))=\ldots$ $-\frac{1}{2}\begin{vmatrix}10 & 4 & 0\\10 & -4 & 0\end{vmatrix}=-\frac{1}{2}\left\lvert-40-40\right\rvert$ $-\frac{1}{2}\begin{vmatrix}0 & 0 & 1\\10 & 4 & 1\\10 & -4 & 1\end{vmatrix}=-\frac{1}{2}\left\lvert0-0+\left[10\times-4-10\times4\right]\right\rvert$	**dM1**	1.1b
	$=40$	**A1**	1.1b
		(4)	

(9 marks)
Notes:
(a) **B1:** Correct equations for the asymptotes of H seen or implied, any form and need not be simplified. **(b) Note M1 M1 A1 A1 on ePen** **M1:** A correct method to find the gradient of the parabola. **M1:** Finds the gradient of the normal and sets their normal gradient equal to their asymptote gradient to obtain at least one point on C where normal is parallel to an asymptote **M1:** Finds the equation of either l_1 or l_2 **A1:** Correct equation for each normal, $y+8=\frac{1}{2}(x-2)$ and $y-8=-\frac{1}{2}(x-2)$. Ignore labelling. **(c)** **M1:** Substitutes for x or y into the equation of the hyperbola and solves for their variable. **A1:** Achieves one correct coordinate $x=10$ and $y=\pm 4$ **dM1:** Dependent on previous method**.** Correct method for the area of their triangle e.g, $\frac{1}{2}\times\text{their }10\times\text{twice their }4$ or equivalent determinant methods. **A1:** Area is 40. Correct answer only.

Question	Scheme	Marks	AOs
6(a)	$y=(1+\ln x)^2 \Rightarrow \frac{dy}{dx}=k(1+\ln x)\times\frac{1}{x}$ or $y=1+2\ln x+(\ln x)^2 \Rightarrow \frac{dy}{dx}=\frac{A}{x}+B\ln x\times\frac{1}{x}$	**M1**	1.1b
	$y=(1+\ln x)^2 \Rightarrow \frac{dy}{dx}=2(1+\ln x)\times\frac{1}{x}$ or $y=1+2\ln x+(\ln x)^2 \Rightarrow \frac{dy}{dx}=\frac{2}{x}+2\ln x\times\frac{1}{x}$	**A1**	1.1b
	$\frac{d^2y}{dx^2}=\frac{k\left(\frac{1}{x}\right)\times x-k(1+\ln x)\times 1}{x^2}$ or $\frac{k}{x}x^{-1}+k(1+\ln x)(-x^{-2})$ or $\frac{d^2y}{dx^2}=-\frac{A}{x^2}+\frac{\frac{B}{x}\times x-2\ln x\times 1}{x^2}$	**M1**	1.1b
	$\frac{d^2y}{dx^2}=\frac{\left(\frac{2}{x}\right)\times x-2(1+\ln x)}{x^2}$ or $\frac{2}{x}x^{-1}+2(1+\ln x)(-x^{-2})$ Leading to $\frac{d^2y}{dx^2}=-\frac{2\ln x}{x^2}$ * achieved from correct work.	**A1***	2.1
		(4)	
(b)	$\frac{d^3y}{dx^3}=\frac{\pm\frac{C}{x}\times x^2\pm Dx\ln x}{x^4}$ or $\frac{d^3y}{dx^3}=(-2\ln x)(-Cx^{-3})+\left(-\frac{D}{x}\right)(x^{-2})$	**M1**	1.1b
	$\frac{d^3y}{dx^3}=-\frac{\frac{2}{x}\times x^2-4x\ln x}{x^4}$ or $\frac{-\frac{2}{x}\times x^2-(-2\ln x)(2x)}{x^4}$ or $-\frac{2}{x^3}+\frac{4\ln x}{x^3}$	**A1**	1.1b
		(2)	
(c)	$y(1)=1, y'(1)=2, y''(1)=0, y'''(1)=-2$	**M1**	1.1b
	$[y]=1+2(x-1)+\frac{0}{2!}(x-1)^2+\frac{-2}{3!}(x-1)^3+\ldots$	**M1**	2.5
	$[y]=1+2(x-1)-\frac{1}{3}(x-1)^3+\ldots$ or $[y]=-1+2x-\frac{1}{3}(x-1)^3+\ldots$	**A1**	1.1b
		(3)	
(d)	$\frac{2x-1-(1+\ln x)^2}{(x-1)^3}=\frac{2x-1-1-2(x-1)+\frac{1}{3}(x-1)^3+\ldots}{(x-1)^3}=\frac{\frac{1}{3}(x-1)^3+\ldots}{(x-1)^3}$	**M1**	1.1b

	Simplifies and realises that terms cancel to leave a constant term $\dfrac{\frac{1}{3}\cancel{(x-1)^3}+\ldots}{\cancel{(x-1)^3}}=\dfrac{1}{3}$	**M1**	3.1a
	Hence $\lim\limits_{x\to 1}\dfrac{2x-1-(1+\ln x)^2}{(x-1)^3}=\dfrac{1}{3}$ as all remaining terms will become zero in the limit as they are multiples of $(x-1)^k$, which tends to 0.	**A1**	2.4
		(3)	
			(12 marks)

Notes:

(a)

M1: Attempts the first derivative, including use of the chain rule. May expand first. E.g. accept forms as shown.

A1: Correct first derivative, need not be simplified.

M1: Attempts second derivative using quotient rule or product rule – examples as shown, or equivalents accepted.

A1*: Correct result achieved from correct work.

(b)

M1: Applies quotient rule or product rule to achieve third derivative. If formula is quoted it must be correct, if not accept derivatives of the form shown as there may be confusion with the minus sign.

A1: Correct third derivative, any form.

(c)

M1: Find value of derivatives at $x = 1$.

M1: Applies Taylor series expansion

A1: Correct series, may be unsimplified, isw once correct series seen. Must be using a correct third derivative.

(d)

M1: Applies the series to the limit and cancels terms in numerator to leave term in $(x-1)^3$ and above only (may not see +… for this mark)

M1: Simplifies and realises that the $x-1^{\ 3}$ cancels and achieves a constant A

A1: Correct limit deduced with reasoning given why the remaining terms disappear.

Question	Scheme	Marks	AOs
7 **Way 1**	Position of A is given by $\overrightarrow{OA}=\begin{pmatrix}12+9\lambda\\16+6\lambda\\-8+2\lambda\end{pmatrix}$	**B1**	3.1a
	So have $\dfrac{12+9\lambda}{\sqrt{(12+9\lambda)^2+(16+6\lambda)^2+(-8+2\lambda)^2}}=\dfrac{3}{7}$	**M1**	1.1b
	$\Rightarrow 49\left(3\left(4+3\lambda\right)\right)^2=9\left((12+9\lambda)^2+(16+6\lambda)^2+(-8+2\lambda)^2\right)$ $\Rightarrow 2880\lambda^2+7200\lambda+2880=0$ or $2\lambda^2+5\lambda+2=0$	**M1** **A1**	3.1a 2.1
	$\Rightarrow\left(2\lambda+1\right)\left(\lambda+2\right)=0\Rightarrow\lambda=\ldots$	**M1**	1.1b
	Substitutes a value of λ to find a position for A e.g. $\overrightarrow{OA}=\begin{pmatrix}12+9\left(-\frac{1}{2}\right)\\16+6\left(-\frac{1}{2}\right)\\-8+2\left(-\frac{1}{2}\right)\end{pmatrix}=\ldots$	**M1**	1.1b
	Coordinates of A are $\left(\dfrac{15}{2},13,-9\right)$ only	**A1**	2.3
		(7)	
7 **Way 2**	Direction of $\overrightarrow{OA}$ is given by $\mathbf{d}=\begin{pmatrix}\frac{3}{7}k\\\beta k\\\gamma k\end{pmatrix}$ or use of $\left(\dfrac{3}{7}\right)^2+\beta^2+\gamma^2=1$	**B1**	3.1a
	$\begin{pmatrix}\frac{3}{7}k-12\\\beta k-16\\\gamma k+8\end{pmatrix}\times\begin{pmatrix}9\\6\\2\end{pmatrix}=\mathbf{0}\Rightarrow\begin{cases}6\left(\frac{3}{7}k-12\right)-9\left(\beta k-16\right)=0\\2\left(\frac{3}{7}k-12\right)-9\left(\gamma k+8\right)=0\\2\left(\beta k-16\right)-6\left(\gamma k+8\right)=0\end{cases}$	**M1**	2.1
	$\Rightarrow\beta k=\dfrac{2}{7}k+8$ and $\gamma k=\dfrac{1}{3}\left(\dfrac{2}{7}k-32\right)$ $\Rightarrow\dfrac{9k^2}{49}+\left(\dfrac{2}{7}k+8\right)^2+\dfrac{1}{9}\left(\dfrac{2}{7}k-32\right)^2=k^2\Rightarrow 2k^2-7k-490=0$	**M1** **A1**	3.1a 1.1b
	$\Rightarrow\left(2k-35\right)\left(k+14\right)=0\Rightarrow k=\ldots$	**M1**	1.1b

	$k > 0$ as direction cosine for first ordinate is positive, so need $k = \frac{35}{2}$ hence $\overrightarrow{OA} = \begin{pmatrix} \frac{3}{7} \times \frac{35}{2} \\ \frac{2}{7} \times \frac{35}{2} + 8 \\ \frac{1}{3}\left(\frac{2}{7} \times \frac{35}{2} - 32\right) \end{pmatrix} = \ldots$	**M1**	2.3
	Coordinates of A are $\left(\frac{15}{2}, 13, -9\right)$ only	**A1**	1.1b
		(7)	
(7 marks)			

Way 1
B1: Starts a correct procedure by parametrising the line correctly.
M1: Uses the direction cosine of $\frac{3}{7}$ to form an equation in λ
M1: Realises need to square, to form quadratic in λ and gathers terms.
A1: Correct quadratic – three terms only or rearranged to complete square and solve, but need not have common factors all cancelled.
M1: Solves their three term quadratic, any valid method.
M1: Substitutes aa value for λ into the equation of the line to find a position for A.
A1: Correct coordinates only

Way 2
B1: Starts correct procedure by using the direction cosines to parametrise $\overrightarrow{OA}$ or attempting to use the Pythagorean property of the direction cosines.
M1: Uses their $\overrightarrow{OA}$ as multiple of direction cosines in the line equation to produce simultaneous equations.
M1: Solves the system (no need to see check for consistency of third equation) to find βk and γk, or just β and γ in terms of k and proceeds to form a quadratic in k using the Pythagorean property of the direction cosines.
A1: A correct quadratic in k reduced to three terms etc.
M1: Solves their three term quadratic, any valid method.
M1: Substitutes aa value for λ into the equation of the line to find a position for A.
A1: Correct coordinates only

Question	Scheme	Marks	AOs
8(a)	At 6 hours $t = 0.25$ so "h" is 0.25	**B1**	3.1b
	At $t = 0$ $\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{3+\cosh 0}{3\times 3^2 \cosh 0} - \frac{1}{3}(3)\tanh 0 = ...\left(= \frac{4}{27}\right)$	**M1**	3.4
	So $x_1 \approx 3 + "0.25" \times "\frac{4}{27}" = ...$	**M1**	1.1b
	After 6 hours concentration of the pollutant is approximately awrt 3.04 ppm (3 s.f.) or $\frac{82}{27}$ ppm	**A1**	3.2a
		(4)	
(b)	$\frac{\mathrm{d}u}{\mathrm{d}t} = 3x^2 \times \frac{\mathrm{d}x}{\mathrm{d}t}$ or $\frac{1}{3x^2}\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}t}$ or $\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{\mathrm{d}u}{\mathrm{d}x} \times \frac{\mathrm{d}x}{\mathrm{d}t} = 3x^2\left(\frac{3+\cosh t}{3x^2 \cosh t} - \frac{1}{3}x\tanh t\right)$	**B1**	2.2a
	So $\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{3+\cosh t}{3x^2\cosh t} - \frac{1}{3}x\tanh t \to \frac{1}{3x^2}\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{3+\cosh t}{3x^2\cosh t} - \frac{1}{3}x\tanh t$ $\to \frac{\mathrm{d}u}{\mathrm{d}t} = \frac{3}{\cosh t} + 1 - u\tanh t$ $\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{3+\cosh t}{3x^2\cosh t} - \frac{1}{3}x\tanh t \to \frac{1}{3u^{\frac{2}{3}}}\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{3+\cosh t}{3u^{\frac{2}{3}}\cosh t} - \frac{1}{3}u^{\frac{1}{3}}\tanh t$ $\to \frac{\mathrm{d}u}{\mathrm{d}t} = \frac{3}{\cosh t} + 1 - u\tanh t$	**M1**	2.1
	$\frac{\mathrm{d}u}{\mathrm{d}t} + u\tanh t = 1 + \frac{3}{\cosh t}$ *	**A1***	1.1b
		(3)	
(c)	$\text{I.F.} = \exp\left(\int \tanh t \,\mathrm{d}t\right) = \exp(\ln\cosh t) = \cosh t$	**B1**	2.2a
	$\Rightarrow u'\cosh t' = \int '\cosh t'\left(1 + \frac{3}{\cosh t}\right)\mathrm{d}t = \left\{\int \cosh t + 3 \,\mathrm{d}t\right\}$	**M1**	1.1b
	$\Rightarrow u\cosh t = \sinh t + 3t(+c)$	**M1**	1.1b
	$u\cosh t = \sinh t + 3t + c$ or $u = \tanh t + \frac{3t}{\cosh t} + \frac{c}{\cosh t}$ oe	**A1**	1.1b
		(4)	
(d)	$t = 0 \Rightarrow x = 3, u = 27 \Rightarrow c = 27\cosh 0 - \sinh 0 - 3(0) = 27$	**M1**	3.4
	$\Rightarrow x = \left(\tanh t + \frac{3t + "27"}{\cosh t}\right)^{\frac{1}{3}}$	**M1**	3.4

	$x=\left(\tanh t+\dfrac{3t+27}{\cosh t}\right)^{\frac{1}{3}}$ (oe)	**A1**	3.2a
		(3)	
(e)	$x(0.25)=\left(\tanh 0.25+\dfrac{(0.75+27)}{\cosh 0.25}\right)^{\frac{1}{3}}=...(=3.0055...)$	**M1**	3.4
	% error is $\dfrac{3.0055...-3.037...}{3.0055...}\times 100=...$	**M1**	1.1b
	Estimate in (a) is an overestimate by 1.05% (3 s.f.)	**A1**	3.2a
		(3)	
			(17 marks)

Notes:

(a)

B1: Identifies a correct step length for the situation – 6 hours is a quarter of a day, so h = 0.25

M1: Uses $"y_0"=x(0)=3$ and $t=0$ to find $"\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)_0"=\left(\dfrac{\mathrm{d}x}{\mathrm{d}t}\right)_0$. Accept with whichever notation used, as long as it is clear they are attempting the correct things.

M1: Applies the approximation formula with their "h" and their $"\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)_0"$

A1: For awrt 3.04 ppm. Accept $\dfrac{82}{27}$ ppm

(b)

B1: A correct equation relating $\dfrac{\mathrm{d}u}{\mathrm{d}t}$ and $\dfrac{\mathrm{d}x}{\mathrm{d}t}$ from the chain rule.

M1: Makes a complete substitution for x and $\dfrac{\mathrm{d}x}{\mathrm{d}t}$ in equation (I) or a complete substitution for u and $\dfrac{\mathrm{d}u}{\mathrm{d}t}$ in equation (II)

A1*: Simplifies correctly to achieve the given result.

(c)

B1: Correct integrating factor found or spotted. Allow for $\mathrm{e}^{\ln\cosh t}$

M1: Applies IF to achieve $u"\cosh t"=\displaystyle\int"\cosh t"\left(1+\frac{3}{\cosh t}\right)\mathrm{d}t$

M1: A reasonable attempt to integrate the RHS. Need not include constant of integration. If I.F. correct allow for $\pm\sinh t+3t(+c)$

A1: Correct general solution, either implicit or explicit form including the context of integration (award when first seen and isw)

(d)

M1: Uses the initial conditions in an appropriate equation to find the constant of integration. Either t = 0 and u = 27 in the answer to (c), or t = 0 and x = 3 if substitution for x occurs first.

M1: Reverses the substitution and rearranges to find equation for x, with evaluated constant included. **A1:** Correct equation, any equivalent form, but must be $x = \dots$
(e) **M1:** Uses their model solution to find the value at $t = 0.25$ **M1:** Applies $\frac{\text{actual value} - \text{estimate}}{\text{actual value}} \times 100$ with their values. **A1:** States part (a) is overestimate by 1.05%