KING EDWARD VI
HANDSWORTH GRAMMAR
SCHOOL FOR BOYS

KING EDWARD VI
ACADEMY TRUST
BIRMINGHAM

Year 12

Unit P5: Algebra and Functions 2

HGS Maths

Dr Frost Course

Name: _____

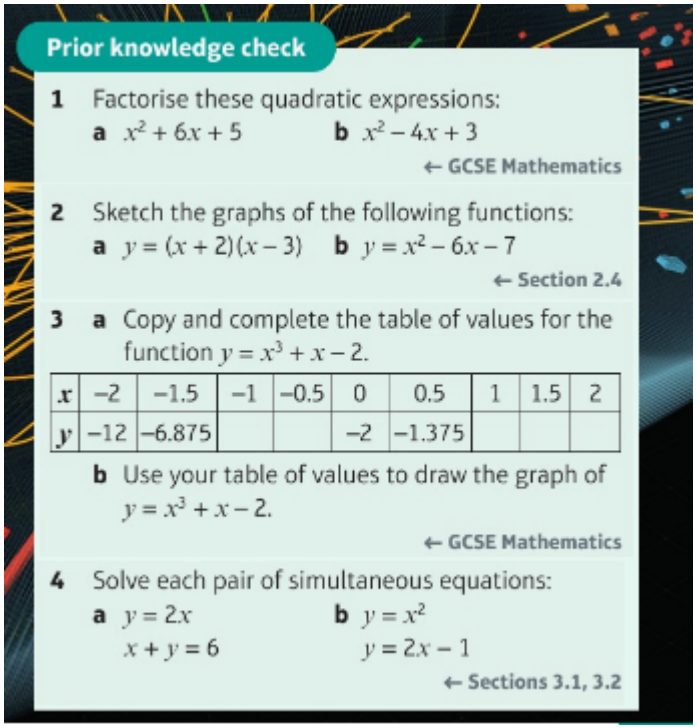
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**red italics are pre-requisite work from summer task*

Prior knowledge check



Prior knowledge check

1 Factorise these quadratic expressions:

a $x^2 + 6x + 5$ **b** $x^2 - 4x + 3$

← GCSE Mathematics

2 Sketch the graphs of the following functions:

a $y = (x + 2)(x - 3)$ **b** $y = x^2 - 6x - 7$

← Section 2.4

3 **a** Copy and complete the table of values for the function $y = x^3 + x - 2$.

x	−2	−1.5	−1	−0.5	0	0.5	1	1.5	2
y	−12	−6.875			−2	−1.375			

b Use your table of values to draw the graph of $y = x^3 + x - 2$.

← GCSE Mathematics

4 Solve each pair of simultaneous equations:

a $y = 2x$
$x + y = 6$

b $y = x^2$
$y = 2x - 1$

← Sections 3.1, 3.2

4.1) Cubic graphs

In Chapter 2 we briefly saw that a **polynomial** expression is of the form:

$$a + bx + cx^2 + dx^3 + ex^3 + \cdots$$

where $a, b, c, d, e, \ldots$ are constants (which could be 0).

The **order** of a polynomial is its highest power.

Order	Name
0	Constant (e.g. "4")
1	Linear (e.g. "$2x - 1$")
2	Quadratic (e.g. "$x^2 + 3$")
3	Cubic
4	Quartic
5	Quintic

These are covered in Chapter 5.

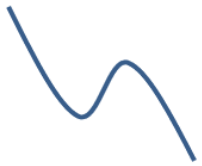
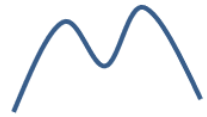
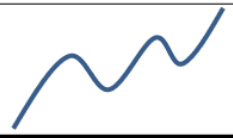
Chapter 2 explored the graphs for these.

We will cover these now.

While these are technically beyond the A Level syllabus, we will look at how to sketch polynomials in general.

Summary

e.g. If $y = 2x^2 + 3$, try a large positive value like $x = 1000$. We can see we'd get a large positive y value. Thus as $x \to \infty$, $y \to \infty$

Equation	If $a > 0$	Resulting Shape	If $a < 0$	Resulting Shape
$y = ax^2 + bx + c$	As $x \to \infty, y \to \infty$ As $x \to -\infty, y \to \infty$		As $x \to \infty, y \to -\infty$ As $x \to -\infty, y \to -\infty$	
$y = ax^3 + bx^2 + cx + d$	As $x \to \infty, y \to \infty$ As $x \to -\infty, y \to -\infty$		As $x \to \infty, y \to -\infty$ As $x \to -\infty, y \to \infty$	
$y = ax^4 + bx^3 + cx^2 + dx + e$	As $x \to \infty, y \to \infty$ As $x \to -\infty, y \to \infty$		As $x \to \infty, y \to -\infty$ As $x \to -\infty, y \to -\infty$	
$y = ax^5 + bx^4 + \cdots$	As $x \to \infty, y \to \infty$ As $x \to -\infty, y \to -\infty$		As $x \to \infty, y \to -\infty$ As $x \to -\infty, y \to \infty$	

If $a > 0$, what therefore can we say about the shape if:

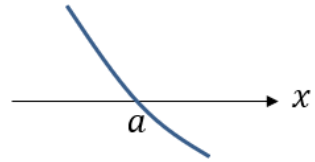
- **The order is odd**: It goes uphill (from left to right)
- **The order is even**: The tails go upwards.

(And we have the opposite if $a < 0$)

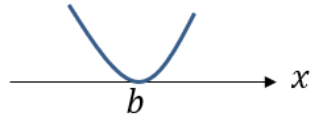
Important notes

If we sketched $y = (x - a)(x - b)^2(x - c)^3$ what happens on the x-axis at:

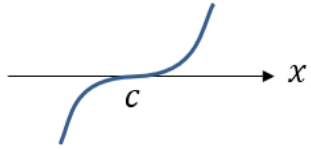
$x = a$: The line **crosses** the axis.



$x = b$: The line **touches** the axis.



$x = c$: **Point of inflection** on the axis.



Notes

Worked Example

Sketch the graph of:

$$y = x^3 - 4x^2 - 5x$$

Sketch the graph of:

$$y = (x + 4)^3$$

Your Turn

Sketch the graph of:

$$y = x^3 - 2x^2 - 3x$$

Sketch the graph of:

$$y = (x - 2)^3$$

Worked Example

Sketch the graph of:

$$y = -(x + 4)^3$$

Sketch the graph of:

$$y = (4 - x)^3$$

T.62 4A: Qs 1-4, P.26 4.1 Qs 1-3

Your Turn

Sketch the graph of:

$$y = -(x - 2)^3$$

Sketch the graph of:

$$y = (2 - x)^3$$

Worked Example

A curve is a positive cubic, touches the x-axis at 3 and crosses the x-axis at -2. Write a possible equation for the curve.

T.62 4A: Qs 5+, P.26 4.1 Qs 4+

4.2) Quartic graphs

Worked Example

Sketch the graph of:

$$y = (x + 3)(x + 4)(x - 3)(x - 4)$$

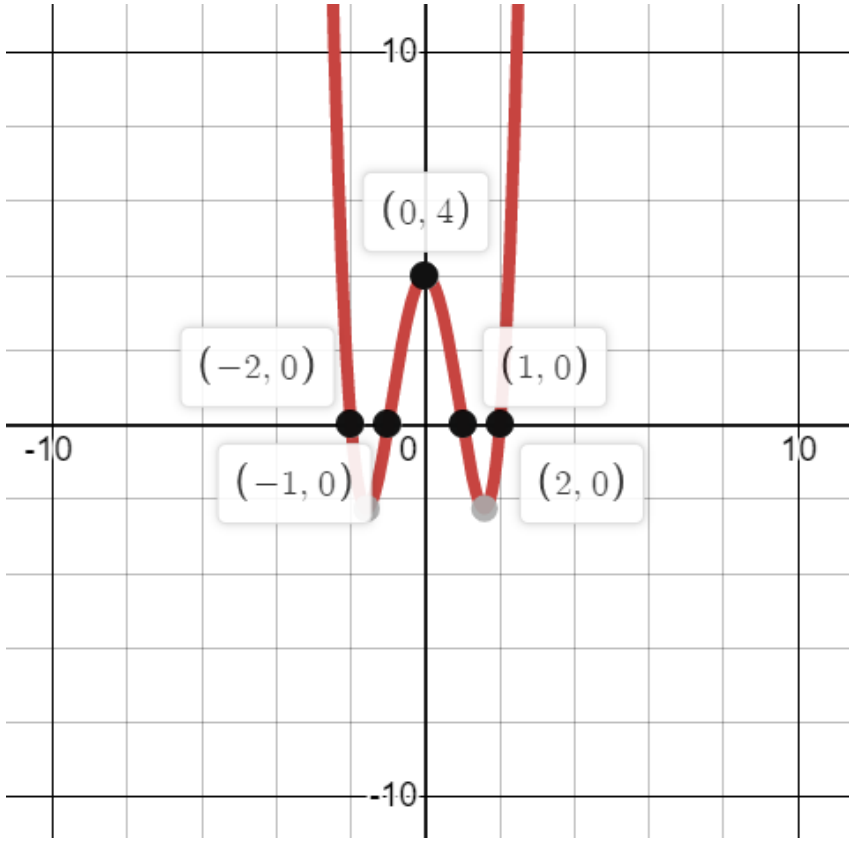
Sketch the graph of:

$$y = x(x - 3)^2(2 - x)$$

Your Turn

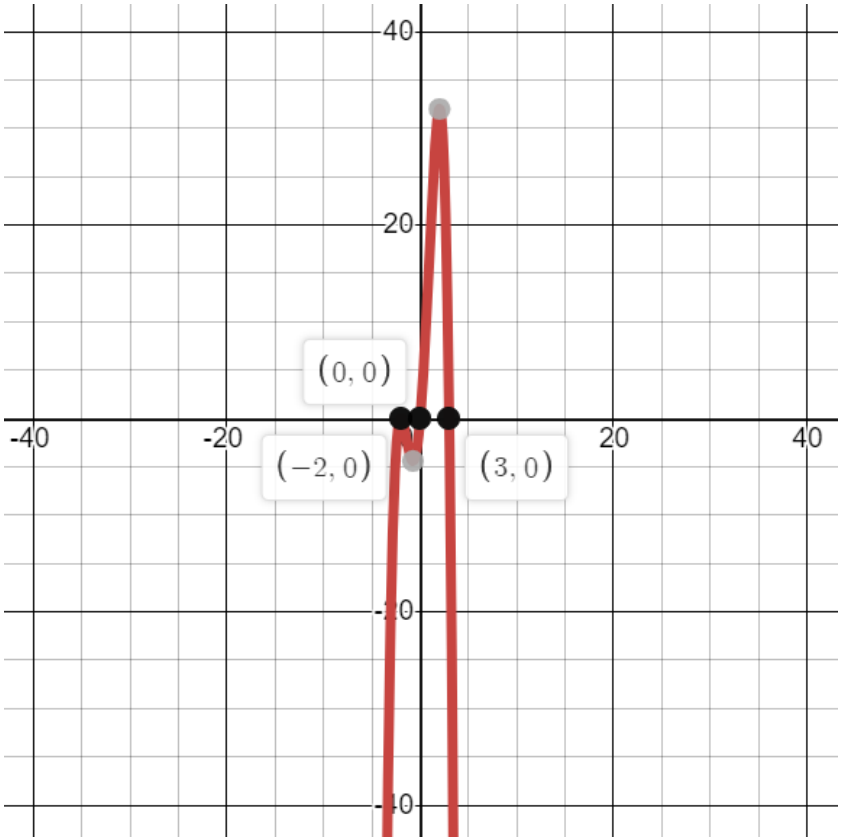
Sketch the graph of:

$$y = (x+1)(x+2)(x-1)(x-2)$$



Sketch the graph of:

$$y = x(x+2)^2(3-x)$$



Worked Example

Sketch the graph of:

$$y = (x + 2)^2(x - 4)^2$$

Sketch the graph of:

$$y = x(x - 4)(x + 5)(x + 6)$$

Your Turn

Sketch the graph of:

$$y = (x-1)^2(x-3)^2$$

Sketch the graph of:

$$y = x(x+1)(x-2)(x-3)$$

Worked Example

Sketch the graph of:

$$y = (x + 4)^2(x - 5)(6 - x)$$

Sketch the graph of:

$$y = (x - 2)(x + 2)^3$$

Your Turn

Sketch the graph of:

$$y = (x-2)^2(x+1)(3-x)$$

Sketch the graph of:

$$y = (x+1)(x-1)^3$$

Worked Example

Sketch the graph of:

$$y = (x + 3)^4$$

Sketch the graph of:

$$y = -(x - 4)(x + 2)^3$$

Your Turn

Sketch the graph of:

$$y = (x - 2)^4$$

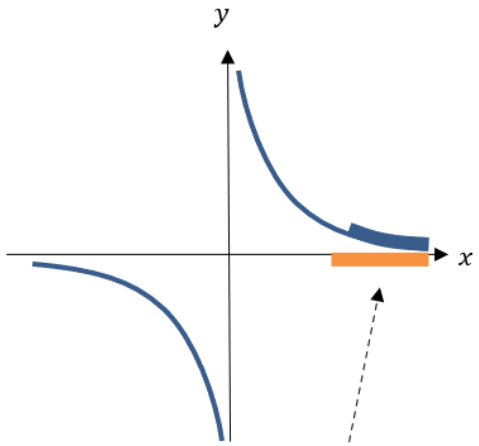
Sketch the graph of:

$$y = -(x + 1)(x - 3)^3$$

T.62 4B: Qs all, P.27 4.2 Qs all

4.3) Reciprocal graphs

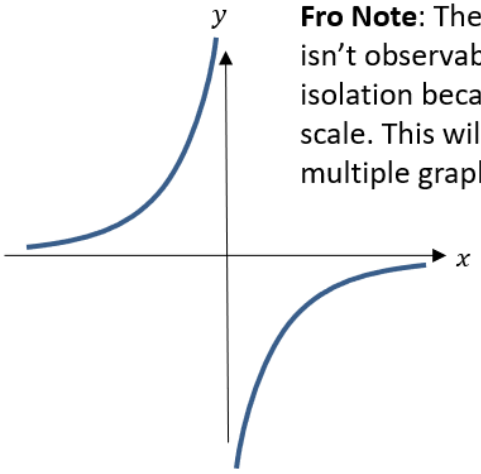
Sketch $y = \frac{1}{x}$



Notice the distance between this line and the x-axis (i.e. the line $y = 0$) gradually decreases as the lines go off towards infinity. The line $y = 0$ is known as an **asymptote** of the graph.

An asymptote is a line which the graph approaches but never reaches.

Sketch $y = -\frac{3}{x}$



Fro Note: The scaling caused by the 3 isn't observable for this graph in isolation because the axes have no scale. This will only be observation for multiple graphs on the same axes.

Asymptotes of $y = \frac{a}{x}$:

$y = 0,$

$x = 0$

T.67 4C: all Qs, P.28 4.3: all Qs

For a graph with equation $y = \frac{ax+b}{cx+d}$

Graph Feature		Technique
x-intercept	$-\frac{b}{a}$	Make y 0 (hence numerator 0).
y-intercept	$\frac{b}{d}$	Make x 0.
Vertical asymptote	$x = -\frac{d}{c}$	Make denominator 0.
Horizontal asymptote	$y = \frac{a}{c}$	Ignore constant terms (b and d).

You **shouldn't need to memorise these results**. Just use of intuition of what each feature means and therefore how we'd determine it.

Worked Example

493a: Identify the x and y intercepts of a more general reciprocal function.

Determine the coordinates of the point where the curve with equation $y = 2 - \dfrac{5}{2x+3}$ crosses the y-axis.

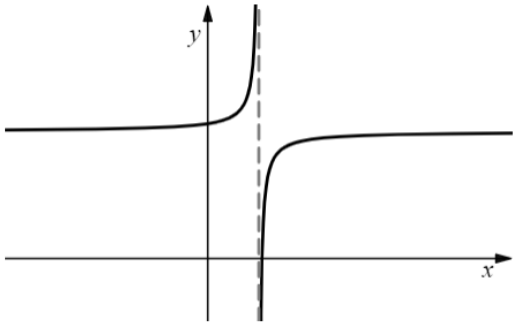
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Only on DFM

Worked Example

493b: **Identify the equation of the vertical asymptote of a more general reciprocal function.**

The graph of $y = 4 - \dfrac{1}{5x - 4}$ is shown below:



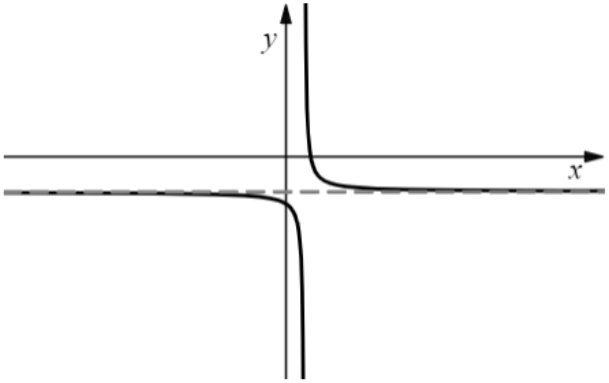
Find the equation of the vertical asymptote.

Only on DFM

Worked Example

493c: **Identify the equation of the horizontal asymptote of a more general reciprocal function.**

The graph of $y = \dfrac{3x-1}{1-4x}$ is shown below:



Find the equation of the horizontal asymptote.

Only on DFM

493g: Determine coefficients in a mo general reciprocal function that lead to a given x or y intercept.

A curve has equation

$$y = \frac{ax - b}{cx + d} \qquad x \neq -\frac{d}{c}$$

where a, b and d are positive prime numbers and c is a positive integer.

The curve crosses the x-axis at the point $\left(\frac{5}{7}, 0\right)$

Find the values of a and b.

Only on DFM

493h: Determine coefficients in a more general reciprocal function that lead to a given vertical or horizontal asymptote.

A curve has equation

$$y = \frac{ax + b}{c - dx} \qquad x \neq \frac{c}{d}$$

where a, c and d are positive prime numbers and b is a positive integer.

An asymptote to the curve has equation $y = -\frac{5}{2}$

Find the values of a and d.

Only on DFM

4.4) Points of intersection

Worked Example

On the same diagram sketch the curves with equations $y = -x^2(5x - a)$ and $y = -\frac{b}{x}$, where a, b are positive constants.

State, giving a reason, the number of real solutions to the equation $x^2(5x - a) + \frac{b}{x} = 0$

Worked Example

On the same diagram sketch the curves with equations $y = \frac{3}{x^2}$ and $y = x^2(x - 4)$.

State, giving a reason, the number of real solutions to the equation $x^4(x - 4) - 3 = 0$

T.69 4D: Qs1-2, P.29 4.4: Qs 2-3

Worked Example

On the same diagram sketch the curves with equations $y = x(x - 5)$ and $y = x(x - 3)^2$, and hence find the coordinates of any points of intersection.

Worked Example

Work out the range of values of a such that the graphs of $y = x^2 + a$ and $3y = x - 2$ have two points of intersection

T.69 4D: Qs3+, P.29 4.4: Qs 1, 4+

5.4 Proportional Graphs *GCSE recap*

1. Direct Proportion

Definition

A variable y is directly proportional to x if

$$y \propto x$$

which means

$$y = kx$$

where k is the constant of proportionality.

Relationship	Equation	Shape
Direct proportion	$y = kx$	Straight line through origin
Square proportion	$y = kx^2$	Parabola
Cube proportion	$y = kx^3$	Cubic curve
Root proportion	$y = k\sqrt{x}$	Increasing curve, flattening out
Cube root proportion	$y = k\sqrt[3]{x}$	S-shaped curve

2. Inverse Proportion

Definition

If y is inversely proportional to x,

$$y \propto \frac{1}{x}$$

so

$$y = \frac{k}{x}$$

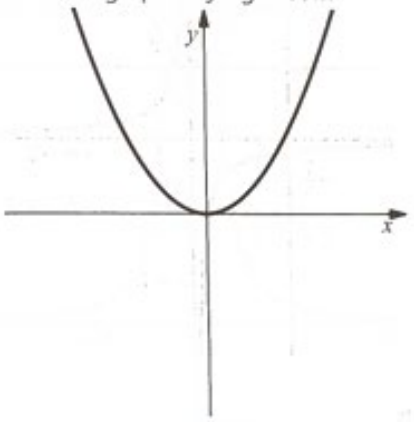
Graph Features

- Reciprocal curve
- Never touches either axis
- Approaches both axes asymptotically

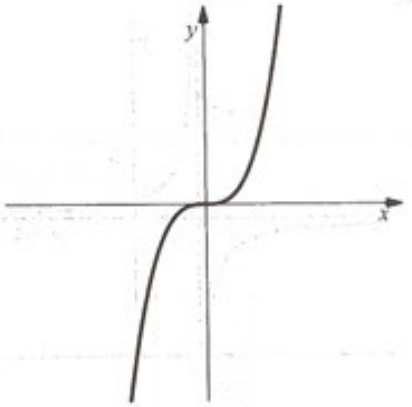
Relationship	Formula
Inverse proportion	$y = \frac{k}{x}$
Inversely proportional to x^2	$y = \frac{k}{x^2}$
Inversely proportional to x^3	$y = \frac{k}{x^3}$
Inversely proportional to $\sqrt{x}$	$y = \frac{k}{\sqrt{x}}$

1) It is given that y is inversely proportional to x^3. State which of the following diagrams could represent the graph of y against x.

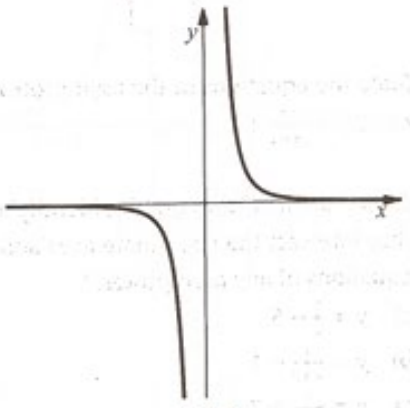
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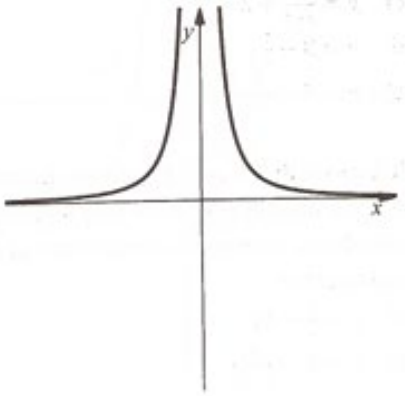
B



C



D



2) In each of these cases, write down an equation for y in terms of x and k, where k is a constant of proportionality:
   a) y is directly proportional to the square of x
   b) y is inversely proportional to x
   c) y is directly proportional to the cube root of x
   d) y is inversely proportional to the cube of $x + 1$

3) Given that y is directly proportional to x^2, and when $x = 5$, $y = 100$, find y when $x = 10$.

4) Given that y is inversely proportional to $\sqrt[4]{x}$, and when $x = 81$, $y = 18$, find x when $y = 180$.

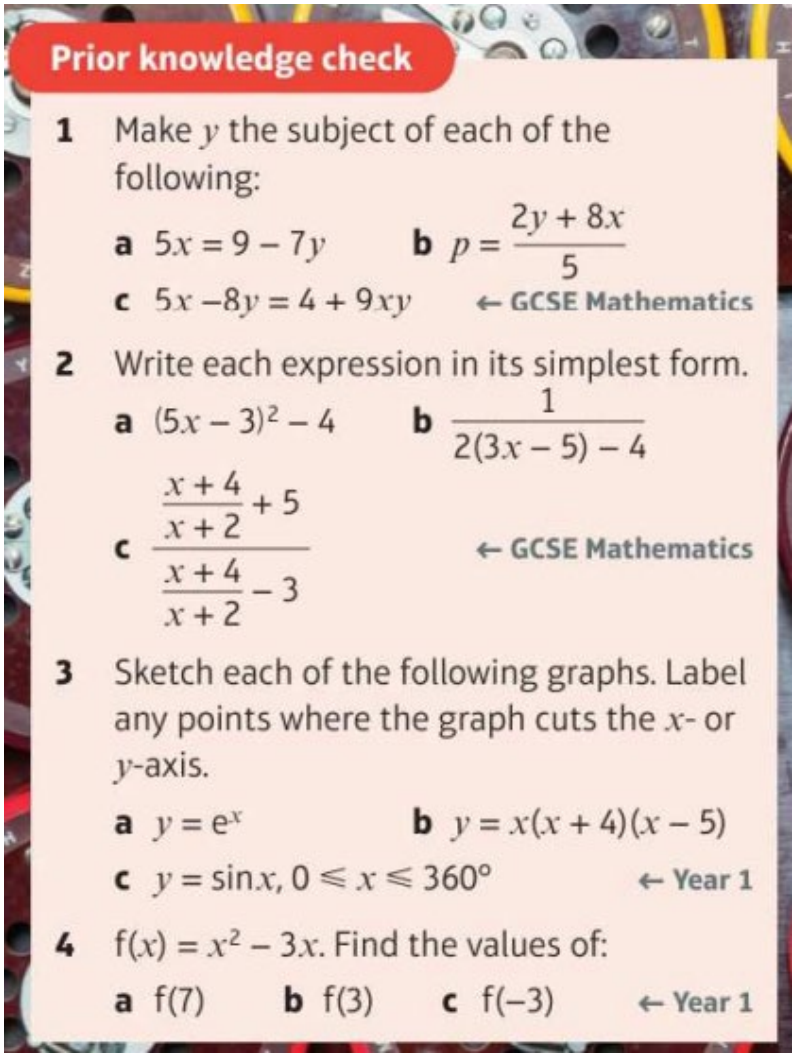
5) The volume of gas, V litres, that a gas canister can hold is directly proportional to the temperature, T kelvin. It is given that when $T = 300$, $V = 24$. Determine the volume of gas when the temperature is 400 Kelvin.

6) The time taken, t hours, to complete a job is inversely proportional to the number of workers, n, employed. It is given that when $n = 5$, $t = 8$. Determine the number of workers required to complete the job in 4 hours.

7) The rate of reaction, R mol/s, of a certain chemical reaction is directly proportional to the concentration C mol/L of a reactant. It is given that when $C = 0.5$, $R = 2$. Determine the concentration that will give a reaction rate of 6 mol/s.

8) The intensity of light, I lumens, from a light source is inversely proportional to the square of the distance d metres from the source. It is given that when $d = 2$, $I = 400$. Determine the distance that will give a light intensity of 100 lumens.

9) It is given that y is directly proportional to $\sqrt{x}$, and that z is inversely proportional to y^{10}. Given that $z = 2$ when $x = 4$, find x when $z = \frac{1}{16}$.



Prior knowledge check

1 Make y the subject of each of the following:

a $5x = 9 - 7y$ **b** $p = \dfrac{2y + 8x}{5}$

c $5x - 8y = 4 + 9xy$ ← GCSE Mathematics

2 Write each expression in its simplest form.

a $(5x - 3)^2 - 4$ **b** $\dfrac{1}{2(3x - 5) - 4}$

c $\dfrac{\dfrac{x+4}{x+2} + 5}{\dfrac{x+4}{x+2} - 3}$ ← GCSE Mathematics

3 Sketch each of the following graphs. Label any points where the graph cuts the x- or y-axis.

a $y = e^x$ **b** $y = x(x + 4)(x - 5)$

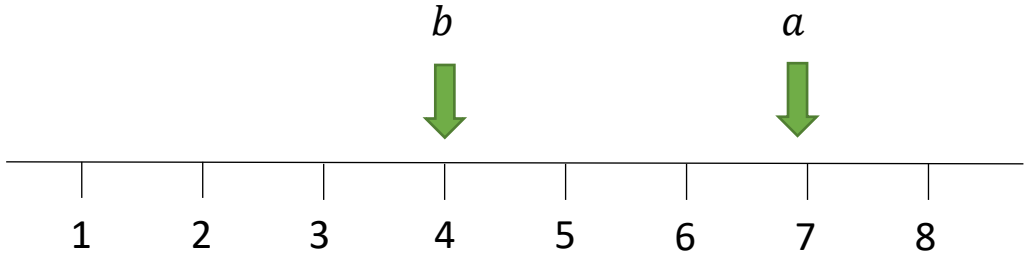
c $y = \sin x$, $0 \leqslant x \leqslant 360°$ ← Year 1

4 $f(x) = x^2 - 3x$. Find the values of:

a f(7) **b** f(3) **c** f(−3) ← Year 1

2.1) The modulus function

The modulus of a number a, written $|a|$, is its **non-negative** numerical value.
e.g. $|6| = 6$ and $|-7.1| = 7.1$



The modulus function is particularly useful in expressing a **difference**. We generally like to quote differences as positive values, but $b - a$ may be negative if b is smaller than a. By using $|b - a|$, we get round this problem!

More fundamentally, the modulus of a value gives us its '**magnitude**', i.e. size; from Mechanics, you should also be used to the notion the distances and speeds are quoted as positive values.
And in Pure Year 1 we saw the same notation used for vectors: $|\boldsymbol{a}|$ gives us the magnitude/length of the vector $\boldsymbol{a}$. It's the same function!

Notes

Worked Example

Solve:

$$|3x - 2| = 7$$

Your Turn

Solve:

a) $|5x - 2| = 3 - \frac{1}{3}x$

b) *hence solve*: $|5x - 2| < 3 - \frac{1}{3}x$

T.26 : 2A Qs 11-13, P.6: 2.1 Qs 5-6

Worked Example

Solve:

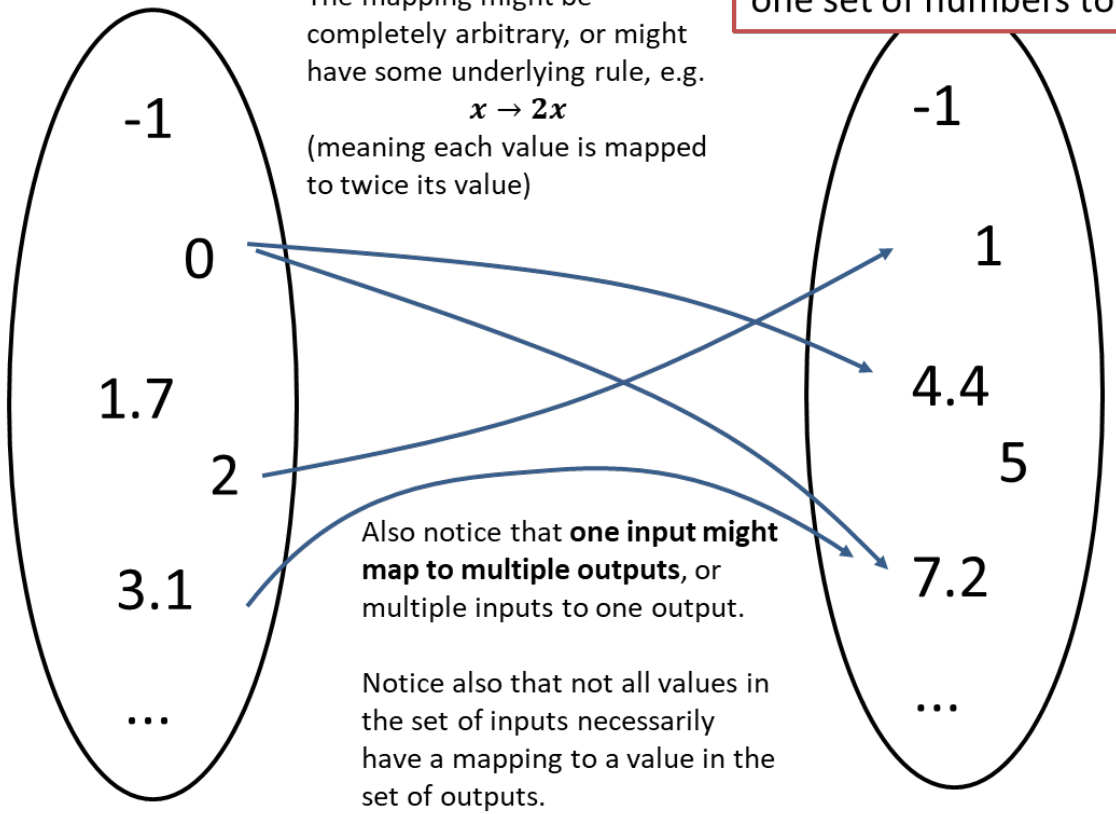
$$|x + 3| = 5x + 2$$

T.26 : 2A Qs 5+, P.6: 2.1 Qs 4,7+

2.2) Functions and mappings

What is a mapping?

A **mapping** is something which maps one set of numbers to another.



The mapping might be completely arbitrary, or might have some underlying rule, e.g.

$$x \rightarrow 2x$$

(meaning each value is mapped to twice its value)

Also notice that **one input might map to multiple outputs**, or multiple inputs to one output.

Notice also that not all values in the set of inputs necessarily have a mapping to a value in the set of outputs.

The **domain** is the set of possible inputs.

The **range** is the set of possible outputs.

Notes

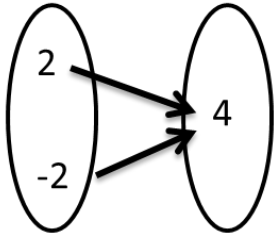
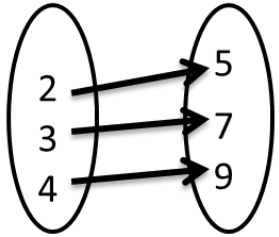
A function is: a mapping such that every element of the domain is mapped to exactly one element of the range.

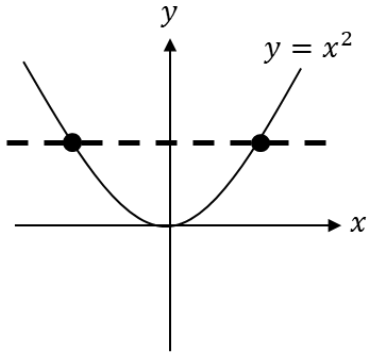
Notation: $f(x) = 2x + 1$ $\quad$ $f: x \to 2x + 1$

$f(x)$ refers to the **output** of the function.

One-to-one vs Many-to-one

While functions permit an input only to be mapped to one output, there's nothing stopping multiple different inputs mapping to the same output.

Type	Description	Example
Many-to-one function	Multiple inputs can map to the same output. 	$f(x) = x^2$ e.g. $f(2) = 4$ $f(-2) = 4$
One-to-one function	Each output has one input and vice versa. 	$f(x) = 2x + 1$



You can use the 'horizontal ray test' to see if a function is one-to-one or many-to-one.

Summary of Domain/Range

It is important that you can identify the range for **<u>common graphs</u>**, **using a suitable sketch**:

$f(x) = x^2, \quad x \in \mathbb{R}$
Range: $\boldsymbol{f(x) \geq 0}$

$f(x) = \frac{1}{x}, \quad x \in \mathbb{R}, x \neq 0$
Range: $\boldsymbol{f(x) \neq 0}$

$f(x) = \ln x, \quad x \in \mathbb{R}, x > 0$
Range: $\boldsymbol{f(x) \in \mathbb{R}}$

$f(x) = e^x, \quad x \in \mathbb{R},$
Range: $\boldsymbol{f(x) > 0}$

$f(x) = x^2 + 2x + 9, \quad x \in \mathbb{R}$
Range: $\boldsymbol{f(x) \geq 8}$

Be careful in noting the domain – it may be 'restricted', which similarly restricts the range. Again, use a sketch!

$f(x) = x^2, \quad x \in \mathbb{R}, -1 \leq x \leq 4$
Range: $\boldsymbol{0 \leq f(x) \leq 16}$

Exercise

State whether:

- the mapping is one-to-one, many-to-one, or one-to-many
- the mapping is a function

$f(x) = 2x - 3, \quad x \in \mathbb{R}$

$g(x) = x^2, \quad x \in \mathbb{R}$

$h(x) = \frac{1}{x}, \quad x \in \mathbb{R}$

$i(x) = \sqrt{x}, \quad x \in \mathbb{R}$

$p(x) = x^3, \quad x \in \mathbb{R}$

$q(x) = \left|\frac{1}{x}\right|, \quad x \in \mathbb{R}$

$r(x) = \sqrt{x}, x \in \mathbb{R}, x \geq 0, \quad x \in \mathbb{R}$

$s(x) = \pm\sqrt{x}, x \in \mathbb{R}, x \geq 0$

T.30 : 2B Qs 1-2, P.7: 2.2 Qs 1

Worked Example

564i: Determine the range of a rational function $f(x) = \frac{ax+b}{cx+d}$

$h(x) = \dfrac{3x+6}{x-3}, \quad x \geq 8$

Worked Example

Find the range of the following functions:

$g(x) = e^x - 4, \qquad x \in \mathbb{R}, x > 0$

$f(x) = \ln x + 5, \qquad x \in \mathbb{R}, x > 0$

$h(x) = -e^x - 3, \qquad x \in \mathbb{R}, x \leq 0$

$g(x) = \ln x - 4, \qquad x \in \mathbb{R}, x > 0$

Worked Example

The function $f(x)$ is defined by

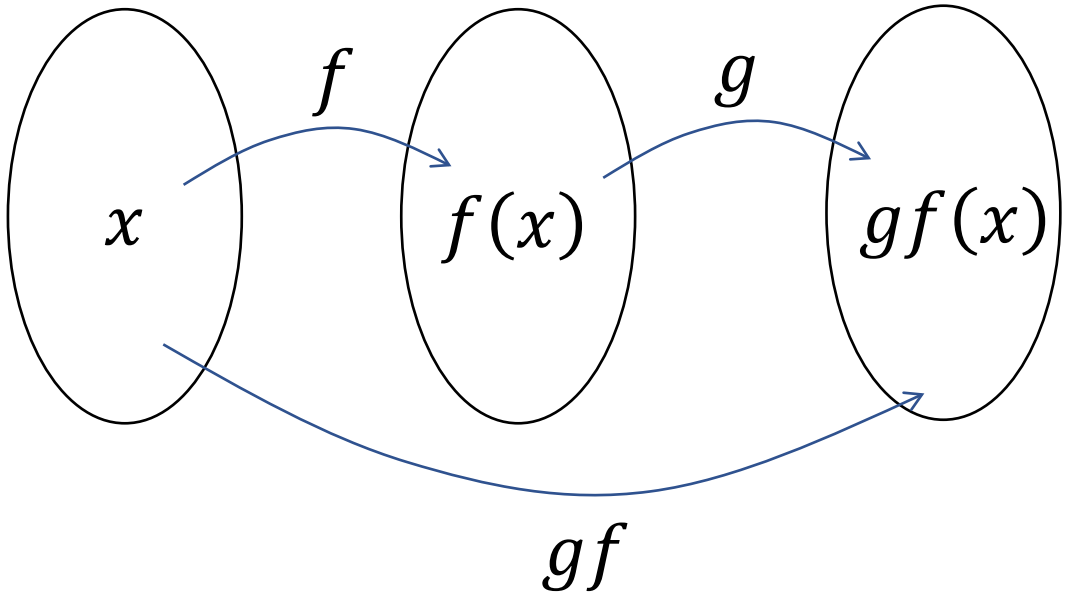
$$f: x \to \begin{cases} 2-5x, & x < 1 \\ x^2 - 3, & x \geq 1 \end{cases}$$

a) Sketch $y = f(x)$, and state the range of $f(x)$.

b) Solve $f(x) = 22$

T.30 : 2B Qs 6+, P.7: 2.2 Qs 4+

2.3) Composite functions



$gf(x)$ means $g(f(x))$, i.e. f is applied first, then g.

Sometimes we may apply multiple functions in succession to an input.
These combined functions are known as a **composite function**.

Worked Example

The functions f and g are defined by

$$f: x \to |3x - 12|$$
$$g: x \to \frac{x+2}{3}$$

a) Find $fg(2)$

b) Solve $fg(x) = x$

T.34 : 2C Qs 3-5, P.8: 2.3 Qs 3-4

Worked Example

The function g is defined by

$$g: x \to 4 - 3x, \qquad x \in \mathbb{R}$$

Solve the equation

$$g^2(x) + [g(x)]^2 = 0$$

Worked Example

The functions f and g are defined by

$f: x \to e^{3x} - 2, \qquad x \in \mathbb{R}$

$g: x \to 4\ln(x+1), \qquad x > -1$

Find $fg(x)$, giving your answer in its simplest form.

Worked Example

The functions f and g are defined by

$f: x \to 3^{2x} - 1, \qquad x \in \mathbb{R}$

$g: x \to 4\log_3(x+5), \qquad x > -5$

Find $fg(x)$, giving your answer in its simplest form.

Worked Example

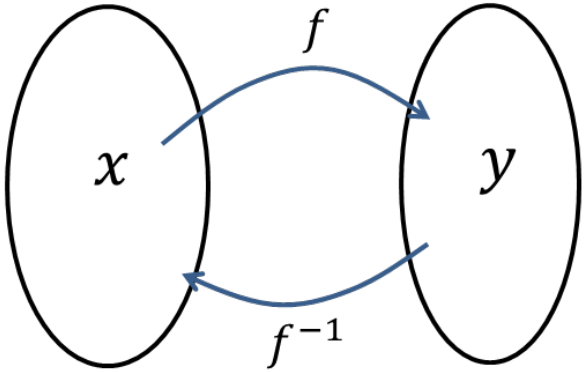
$$f(x) = \frac{1}{x-1}, x \neq 1$$

Find an expression for $f^2(x)$ and $f^3(x)$

Worked Example

A function f has domain $-3 \leq x \leq 12$ and is linear from $(-3, 9)$ to $(0, 6)$ and from $(0, 6)$ to $(12,10)$.
Find the value of $f^2(0)$

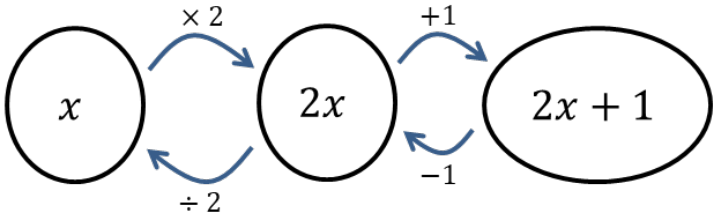
2.4) Inverse functions



Notation: Just like f^2 means "apply f twice", f^{-1} means "apply f -1 times", i.e. once backwards!
This is why we write $\sin^{-1}(x)$ to mean "inverse sin".

An inverse function f^{-1} **does the opposite of the original function**. For example, if $f(4) = 2$, then $f^{-1}(2) = 4$.

If $f(x) = 2x + 1$, we could do the opposite operations within the function in reverse order to get back to the original input:



Thus $f^{-1}(x) = \frac{x-1}{2}$

This has appeared in exams before.

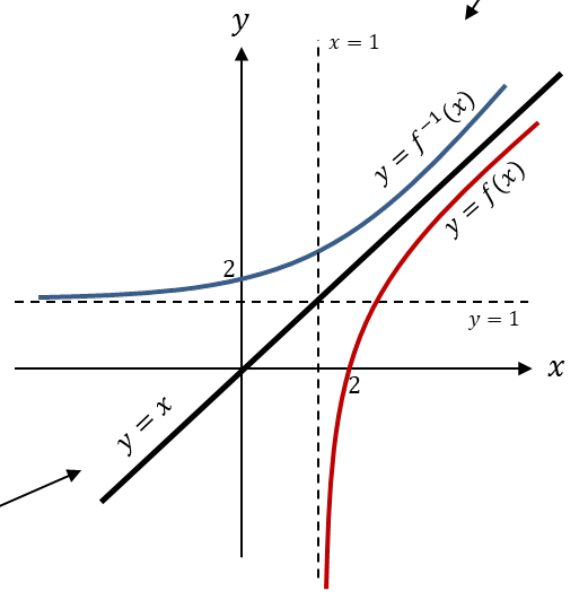
Explain why the function must be one-to-one for an inverse function to exist:
If the mapping was many-to-one, then the inverse mapping would be one-to-many.
But this is not a function!

Graphing an Inverse Function

We saw that the inverse function effectively swaps the input x and output y. Thus the $\boldsymbol{x}$ **and** $\boldsymbol{y}$ **axis are swapped** when sketching the original function and its inverse.

And since the set of inputs and set of outputs is swapped…

The domain of $f(x)$ is the range of $f^{-1}(x)$ and vice versa.



Notice that x-intercepts become y-intercepts, and vertical asymptotes become horizontal ones.

$y = f(x)$ and $y = f^{-1}(x)$ have the line $y = x$ as a line of symmetry.

Domain of f: $\boldsymbol{x > 1}$

Range of f^{-1}: $\boldsymbol{f^{-1}(x) > 1}$

The domain of the function is the same as the range of the inverse, but remember that we write a domain in terms of x, but a range in terms of $f(x)$ or $f^{-1}(x)$.

Worked Example

Find the inverse functions:

$f(x) = \frac{2}{x-5}, \qquad x \in \mathbb{R}, x \neq 5$

$g(x) = \frac{7}{x+2}, \qquad x \in \mathbb{R}, x \neq -2$

Worked Example

Find the inverse functions:

$$f(x) = \frac{x-2}{2x+1}, \qquad x \neq \frac{1}{2}$$

$$g(x) = \frac{2x+3}{4x-5}, \qquad x \neq \frac{5}{4}$$

Worked Example

$$f(x) = \sqrt{x-3}\ \{x \in \mathbb{R}, x \geq 3\}$$

a) State the range of $f(x)$

b) Find the function $f^{-1}(x)$ and state its domain and range

c) Sketch $y = f(x)$, $y = f^{-1}(x)$ and $y = x$

T.38: 2D Qs 5+ P.9 2.4 Qs 2+

4.5) Translating graphs

This is all you need to remember when considering how transforming your function transforms your graph...

	Affects which axis?	What we expect or opposite?
Change **inside** $f(\,)$	x	Opposite
Change **outside** $f(\,)$	y	Direct

Therefore...

$y = f(x-3)$ ⟹ Translation by $\begin{pmatrix}3\\0\end{pmatrix}$

$y = f(x)+4$ ⟹ Translation by $\begin{pmatrix}0\\4\end{pmatrix}$

Effect of transformation on specific points

$\boldsymbol{y = f(x)}$	$\boldsymbol{(4,3)}$	$\boldsymbol{(1,0)}$	$\boldsymbol{(6,-4)}$
$y = f(x+1)$	$(3,3)$	$(0,0)$	$(5,-4)$
$y = f(x)-1$	$(4,2)$	$(1,-1)$	$(6,-5)$

T.74 4E: Qs 1,2, P.30 4.5: Qs 1,2

Worked Example

The point with coordinates $(-1.5, 0)$ lies on the curve with equation

$$y = (x + a)^3 + 6(x + a)^2 + 9(x + a)$$

where a is a constant. Find the two possible values of a

Worked Example

Sketch $y = x(x - 3)$. On the same axes, sketch $y = (x + a)(x + a - 3)$, where $a > 3$.

T.74 4E: Qs 3+, P.30 4.5: Qs 3+

4.6) Stretching graphs

This is all you need to remember when considering how transforming your function transforms your graph...

	Affects which axis?	What we expect or opposite?
Change **inside** $f(\,)$	x	Opposite
Change **outside** $f(\,)$	y	Direct

Therefore...

$y = f(5x)$ ⟹ Stretch in x-direction by scale factor $\frac{1}{5}$

$y = 2f(x)$ ⟹ Stretch in y-direction by scale factor 2

$y = f(x)$	$(4, 3)$	$(1, 0)$	$(6, -4)$
$y = f(2x)$	$(2,3)$	$(0.5, 0)$	$(3, -4)$
$y = 3f(x)$	$(4,9)$	$(1,0)$	$(6, -12)$
$y = f(-x)$	$(-4,3)$	$(-1,0)$	$(-6, -4)$
$y = -f(x)$	$(4, -3)$	$(1,0)$	$(6,4)$

Worked Example KS 450i

The graph of $y = f(x)$ is shown in Figure 1.

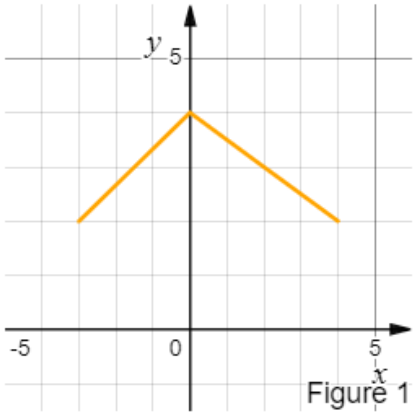


Figure 1

The graph of $y = af(x)$ is shown in Figure 2.

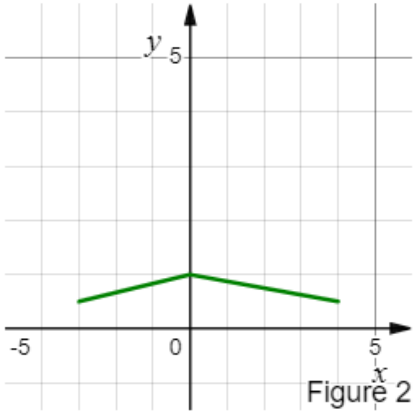


Figure 2

Determine the value of a.

T.78 4F: Qs 4+, P.32 4.6: Qs 4+

AS 2022 Q7

The curve C has equation

$$y = \frac{k^2}{x} + 1 \qquad x \in \mathbb{R},\ x \neq 0$$

where k is a constant.

(a) Sketch C stating the equation of the horizontal asymptote.

(3)

The line l has equation $y = -2x + 5$

(b) Show that the x coordinate of any point of intersection of l with C is given by a solution of the equation

$$2x^2 - 4x + k^2 = 0$$

(2)

(c) Hence find the exact values of k for which l is a tangent to C.

(3)

Your Turn

7. (*a*) Factorise completely $4x - x^3$

(2)

The curve C has equation

$$y = 4x - x^3$$

(*b*) Sketch C showing the coordinates of the points at which the curve cuts the x-axis.

(2)

The line l has equation $y = k$ where k is a constant.

Given that C and l intersect at 3 distinct points,

(*c*) find the range of values for k, writing your answer in set notation.

Solutions relying on calculator technology are not acceptable.

(3)

(Total for Question 7 is 7 marks)

4.7) Transforming functions

Important FACTS to LEARN:

- OUTSIDE THE BRACKET is DIRECT and applies to the $x-axis/direction$
- INSIDE THE BRACKET is OPPOSITE and applies to the y $-$ $axis/direction$
- Summary/examples:

4 The graph of $y = f(x) + a$ is a translation of the graph $y = f(x)$ by the vector $\begin{pmatrix} 0 \\ a \end{pmatrix}$.

5 The graph of $y = f(x + a)$ is a translation of the graph $y = f(x)$ by the vector $\begin{pmatrix} -a \\ 0 \end{pmatrix}$.

6 When you translate a function, any asymptotes are also translated.

7 The graph of $y = af(x)$ is a stretch of the graph $y = f(x)$ by a scale factor of a in the vertical direction.

8 The graph of $y = f(ax)$ is a stretch of the graph $y = f(x)$ by a scale factor of $\frac{1}{a}$ in the horizontal direction.

9 The graph of $y = -f(x)$ is a reflection of the graph of $y = f(x)$ in the x-axis.

10 The graph of $y = f(-x)$ is a reflection of the graph of $y = f(x)$ in the y-axis.

- *This is also true about ORDER of operations!* i.e. $f(2x-1)$ is a translation $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ followed by a stretch scale factor $\frac{1}{2}$ in the x direction.
- Where as $2f(x)-1$ is a stretch scale factor 2 in the y direction followed by a translation $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$

Worked Example

Find the new coordinates under the transformations

$\boldsymbol{y = f(x)}$	$\boldsymbol{(-6, 4)}$	$\boldsymbol{(0, 1)}$
$y = f(x + 2)$		
$y = f(x) - 2$		
$y = f(3x)$		
$y = 4f(x)$		
$y = f\left(\frac{x}{5}\right)$		
$y = 6f(x)$		
$y = -f(x)$		
$y = f(-x)$		
$y = f(1 - x)$		

Worked Example

The point $A(2, 5)$ is the minimum of the curve with equation $y = f(x)$. Write the new coordinates of the new minimum of the curve:

a) $y = 2f(x) + 3$
b) $y = 3f(x) - 2$
c) $y = f(2x) + 3$
d) $y = f(3x) - 2$
e) $y = -f(x) + 3$
f) $y = -f(x) - 2$
g) $y = f(-x) + 3$
h) $y = f(-x) - 2$

Worked Example

The point $A(2, 5)$ is the minimum of the curve with equation $y = f(x)$. Write the new coordinates of the new minimum of the curve:

a) $y = -2f(x) + 3$
b) $y = -3f(x) - 2$
c) $y = 2f(-x) + 3$
d) $y = 3f(-x) - 2$
e) $y = -2f(-x) + 3$
f) $y = -3f(-x) - 2$
g) $y = 3f(2x) + 7$
h) $y = 7f(5x) - 2$
i) $y = -3f(2x) + 7$
j) $y = -7f(5x) - 2$
k) $y = 3f(x) + 7$
l) $y = -(3x + 1) - 2$

T.80 4G: all Qs, P.33 4.7 all Qs

AS 2019

11. $f(x) = 2x^3 - 13x^2 + 8x + 48$

(a) Prove that $(x - 4)$ is a factor of $f(x)$.

(2)

(b) Hence, using algebra, show that the equation $f(x) = 0$ has only two distinct roots.

(4)

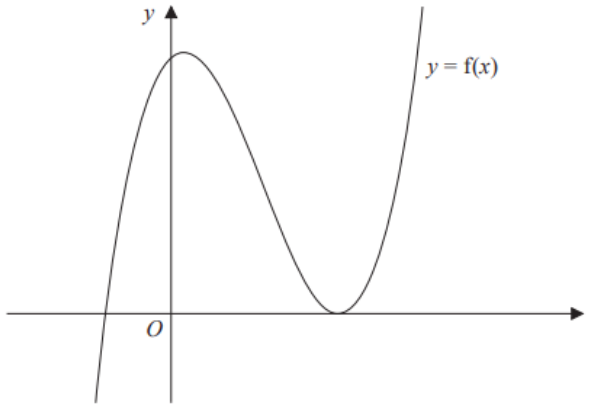


Figure 2

Figure 2 shows a sketch of part of the curve with equation $y = f(x)$.

(c) Deduce, giving reasons for your answer, the number of real roots of the equation

$$2x^3 - 13x^2 + 8x + 46 = 0$$

(2)

Given that k is a constant and the curve with equation $y = f(x + k)$ passes through the origin,

(d) find the two possible values of k.

(2)

Your Turn

11.

$$f(x) = x^3 + x^2 - 8x - 12$$

(a) Prove that $(x - 3)$ is a factor of $f(x)$.

(2)

(b) Hence, using algebra, show that the equation $f(x) = 0$ has only 2 distinct roots.

(3)

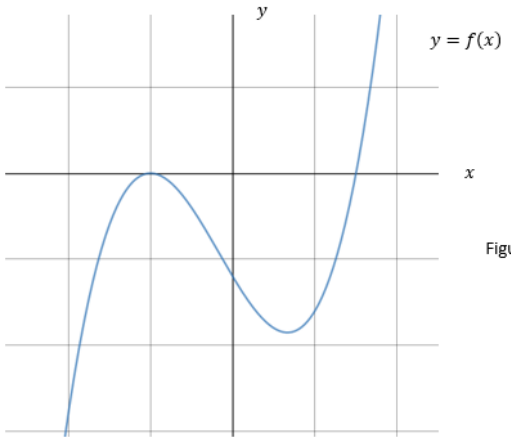


Figure B

Figure B shows a sketch of part of the curve $y = f(x)$.

(c) Deduce, giving reasons for your answer, the number of real roots of the equation

$$2x^3 + 2x^2 - 16x - 50 = 0$$

(3)

Given that k is a constant and the curve with equation $y = f(kx)$ passes through (1,0),

(d) Find the two possible values of k

(2)

9.6 Transforming Trigonometric Graphs

Reminders from chapter 4:

The graph of $y = f(x) + a$ is a translation of the graph $y = f(x)$ by the vector $\begin{pmatrix} 0 \\ a \end{pmatrix}$.

The graph of $y = f(x + a)$ is a translation of the graph $y = f(x)$ by the vector $\begin{pmatrix} -a \\ 0 \end{pmatrix}$.

When you translate a function, any asymptotes are also translated.

The graph of $y = af(x)$ is a stretch of the graph $y = f(x)$ by a scale factor of a in the vertical direction.

The graph of $y = f(ax)$ is a stretch of the graph $y = f(x)$ by a scale factor of $\frac{1}{a}$ in the horizontal direction.

The graph of $y = -f(x)$ is a reflection of the graph of $y = f(x)$ in the x-axis.

The graph of $y = f(-x)$ is a reflection of the graph of $y = f(x)$ in the y-axis.

OR SIMPLY:

OUTSIDE to y and <u>DIRECT</u>

INSIDE to x and <u>OPPOSITE</u>

Worked Example

Sketch on separate sets of axes the graphs of:

a) $y = 3\sin x\,, 0 \leq x \leq 360°$

b) $y = -\tan\theta\,, -180° \leq \theta \leq 180°$

Worked Example

Sketch on separate sets of axes the graphs of:

a) $y = -1 + \sin x, 0 \leq x \leq 360°$

b) $y = \frac{1}{2} + \cos x, 0 \leq x \leq 360°$

Worked Example

Sketch on separate sets of axes the graphs of:

a) $y = \tan(\theta + 45°), 0 \leq \theta \leq 360°$

b) $y = \cos(\theta - 90°), -360° \leq \theta \leq 360°$

Worked Example

Sketch on separate sets of axes the graphs of:

a) $y = \sin 2x\,, 0 \leq x \leq 360°$

b) $y = \cos\frac{\theta}{3}, -540° \leq \theta \leq 540°$

c) $y = \tan(-x)\,, -360° \leq x \leq 360°$

T.197 9G: Qs all, P.74 9.6: Qs all

7. A parallelogram $PQRS$ has area $50\,\text{cm}^2$

Given

- PQ has length $14\,\text{cm}$
- QR has length $7\,\text{cm}$
- angle SPQ is obtuse

find

(a) the size of angle SPQ, in degrees, to 2 decimal places,

(3)

(b) the length of the diagonal SQ, in cm, to one decimal place.

(2)

Your Turn

7. A parallelogram $PQRS$ has area 40 cm^2

Given

- PQ has length 14 cm
- QR has length 7 cm
- angle SPQ is obtuse

find

(a) the size of angle SPQ, in degrees, to 2 decimal places,

(3)

(b) the length of the diagonal SQ, in cm, to one decimal place.

(2)

(Total for Question 7 is 5 marks)

3.

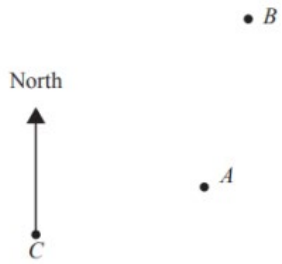


Figure 1

Figure 1 is a sketch showing the position of three phone masts, A, B and C.

The masts are identical and their bases are assumed to lie in the same horizontal plane.

From mast C

- mast A is 8.2 km away on a bearing of 072°
- mast B is 15.6 km away on a bearing of 039°

(a) Find the distance between masts A and B, giving your answer in km to one decimal place.

(3)

An engineer needs to travel from mast A to mast B.

(b) Give a reason why the answer to part (a) is unlikely to be an accurate value for the distance the engineer travels.

(1)

3.

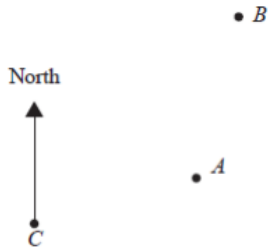


Figure 1

Figure 1 is a sketch showing the position of three lighthouses, A, B and C.

The lighthouses are identical and their bases are assumed to lie in the same horizontal plane.

From lighthouse A, lighthouse C is 10.4 km away on a bearing of 251°

From lighthouse C, lighthouse B is 20.3 km away on a bearing of 042°

(a) Find the distance between lighthouses A and B, giving your answer in km to one decimal place.

(3)

The bearing of lighthouse B from lighthouse A is found to be 018°. A ship needs to travel from lighthouse A to lighthouse B in the most direct route.

(b) Give a reason why the captain of the ship may need to set the ship in a direction with a different bearing to 018°.

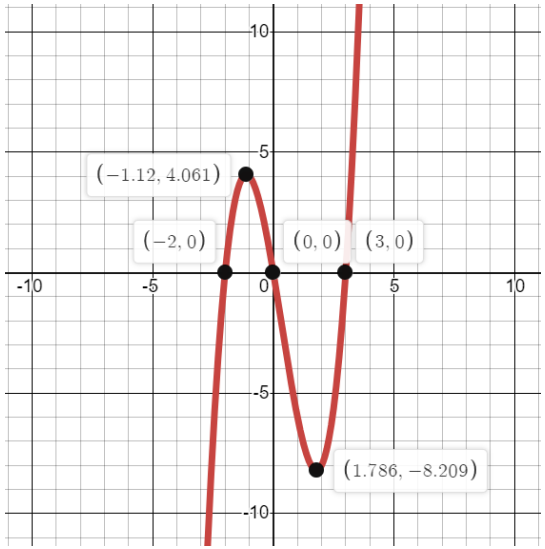
(1)

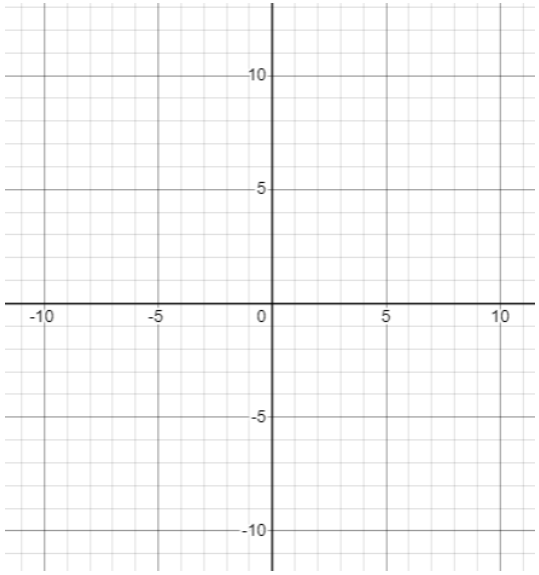
(Total for Question 3 is 4 marks)

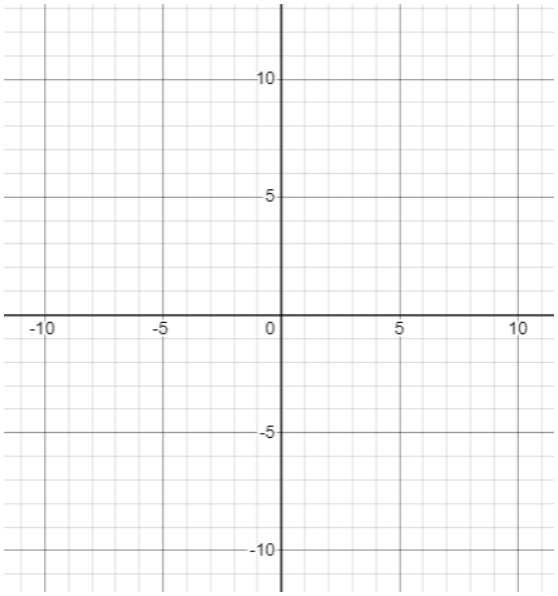
2.5) $y=|f(x)|$ and $y=f(|x|)$

Worked Example

A sketch of $y = f(x)$ is shown.
Sketch $y = |f(x)|$ and $y = f(|x|)$ on separate axes.

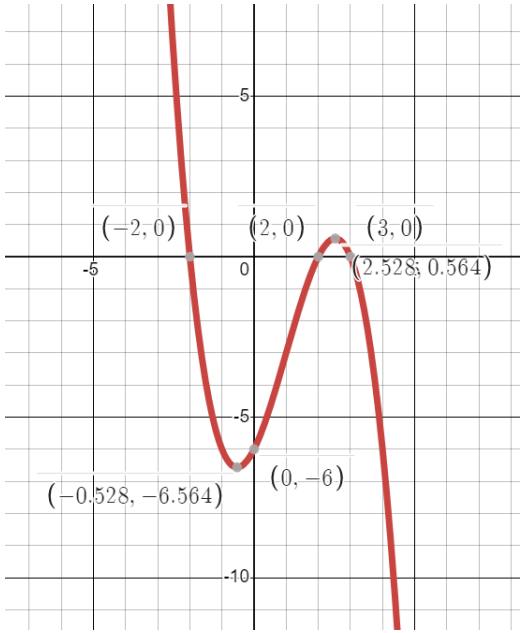


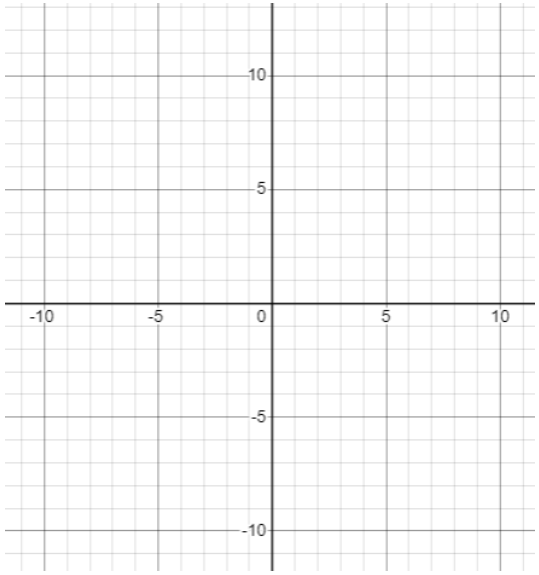


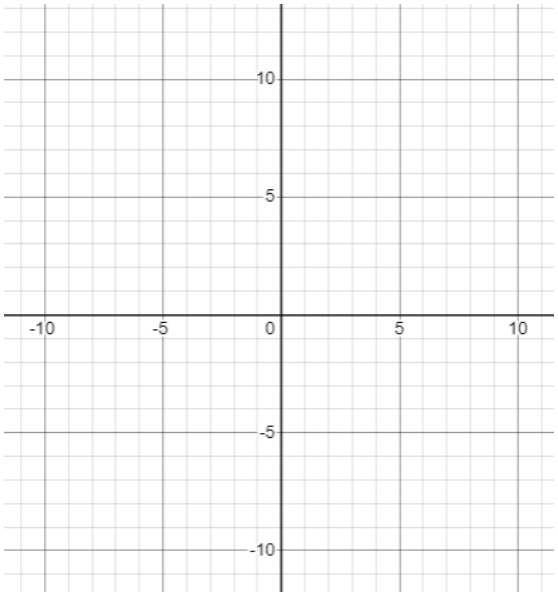


Your Turn

A sketch of $y = f(x)$ is shown.
Sketch $y = |f(x)|$ and $y = f(|x|)$ on separate axes.







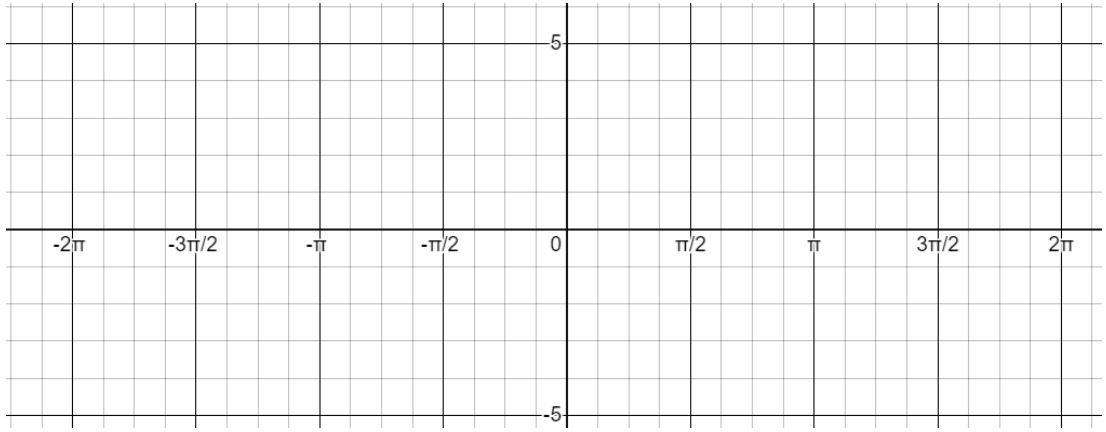
T.42: 2E Qs 1,3-10 P.10 2.5 Qs 3-7

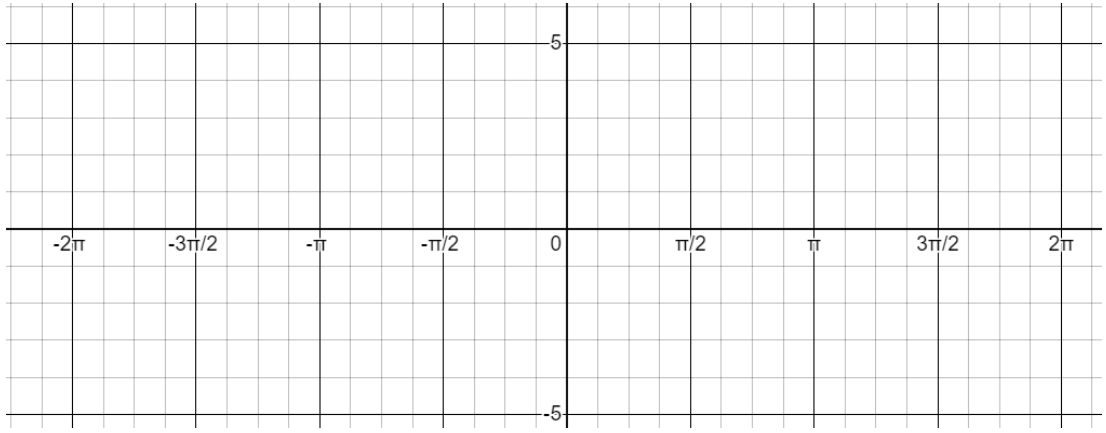
Worked Example

$y = \cos x\,, \qquad -2\pi \leq x \leq 2\pi$

Sketch:

a) $y = |\cos x\,|$ b) $y = \cos|x|$



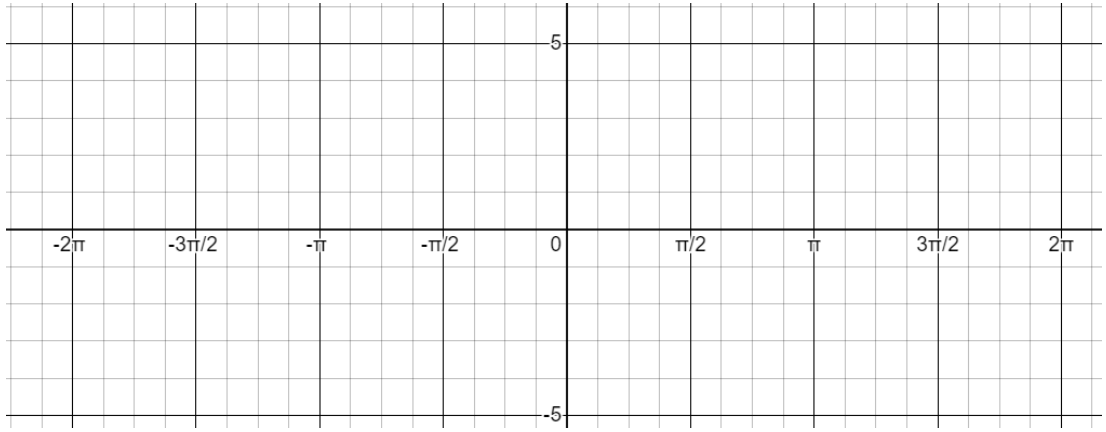


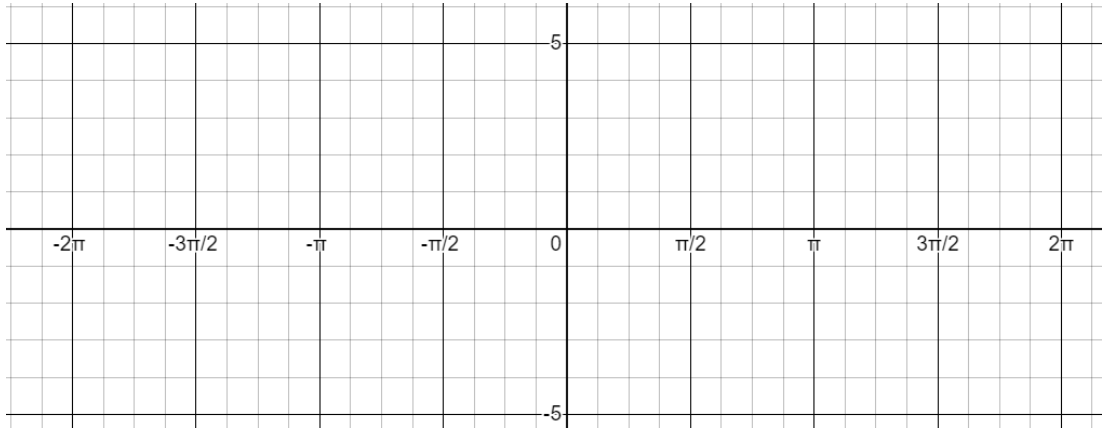
Your Turn

$y = tan\, x\,, \qquad -2\pi \leq x \leq 2\pi$

Sketch:

a) $y = |tanx|$ b) $y = \tan|x|$





T.42: 2E Qs 2 P.10 2.5 Qs 1-2

2.6) Combining transformations

	Affects which axis?	What we expect or opposite?
Change **inside** $f(\,)$	x	Opposite
Change **outside** $f(\,)$	y	What we expect

What if two x changes or two y changes?

$$y = 2f(x) + 1$$

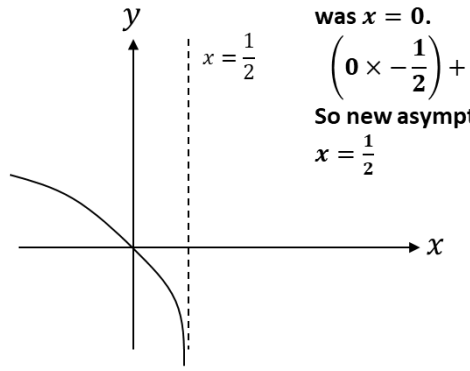
The y values are multiplied by 2, and <u>then</u> 1 is added.

$$y = f(2x + 1)$$

The easiest way is the think of the **inverse function** of $2x + 1$, i.e. $\frac{x-1}{2}$.
This gives us the changes to the x values, and in the correct order! In this case, we would -1 from the x values (translation 1 left) and then halve the x values (stretch on x-axis of scale factor $\frac{1}{2}$)

Sketch $y = \ln(1 - 2x)$

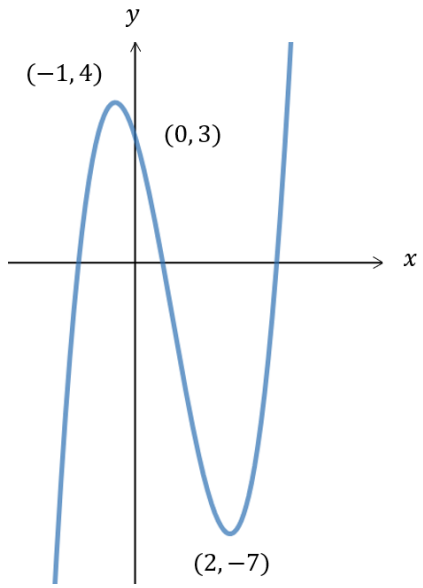
Inverse of $1 - 2x$ is $\frac{1-x}{2} = -\frac{1}{2}x + \frac{1}{2}$
So multiply x values by $-\frac{1}{2}$ and then add $\frac{1}{2}$.



Original asymptote was $x = 0$.
$\left(0 \times -\frac{1}{2}\right) + \frac{1}{2} = \frac{1}{2}$
So new asymptote is $x = \frac{1}{2}$

Worked Example

A sketch of the graph $y = f(x)$ is shown:



Sketch the graph of $y = -f(|x|)$

T.47: 2F Qs all, P.12 2.6 Qs all

2.7) Solving modulus problems

Worked Example

$$f(x) = 2|x + 1| - 3, x \in \mathbb{R}$$

(a) Sketch the graph of $y = f(x)$

(b) State the range of f.

(c) Solve the equation $f(x) = \frac{1}{3}x + 2$

Worked Example

$$f(x) = 6 - 2|x + 3|, x \in \mathbb{R}$$

(a) Sketch the graph of $y = f(x)$

(b) State the range of f.

(c) Solve the inequality $f(x) > 5$

Worked Example

$$f(x) = 6 + 3|x - 2|, x \in \mathbb{R}$$

State the range of values of k for which $f(x) = k$ has:

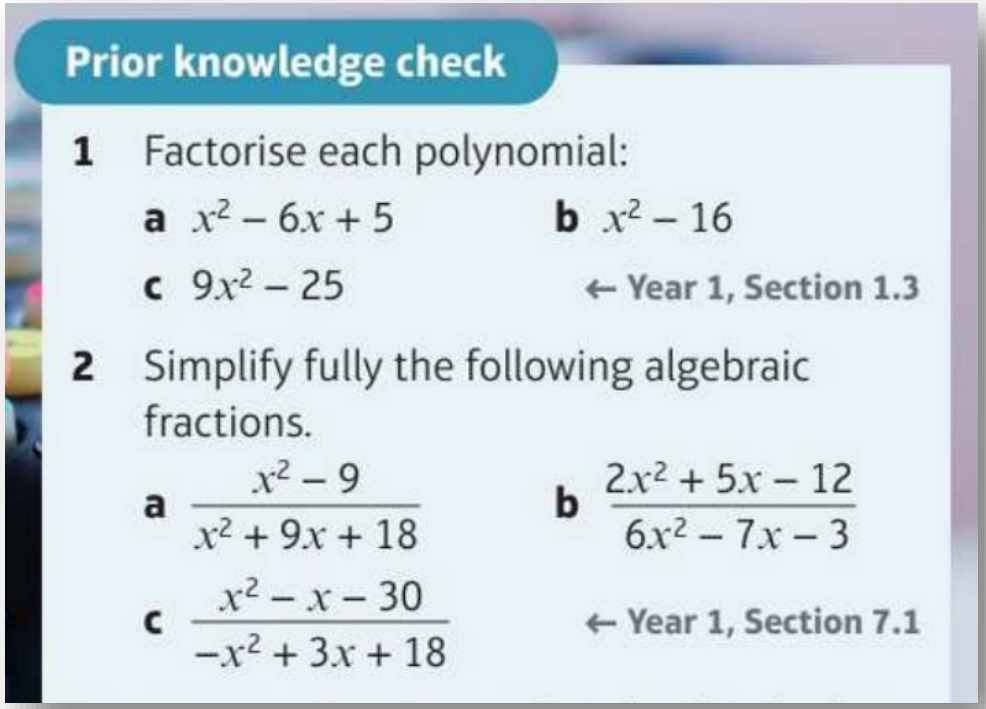
a) no solutions

b) exactly one solution

c) two distinct solutions

T.51: 2G Qs all, P.14 2.7 Qs all

Prior knowledge check



Prior knowledge check

1 Factorise each polynomial:

a $x^2 - 6x + 5$ **b** $x^2 - 16$

c $9x^2 - 25$ ← Year 1, Section 1.3

2 Simplify fully the following algebraic fractions.

a $\dfrac{x^2 - 9}{x^2 + 9x + 18}$ **b** $\dfrac{2x^2 + 5x - 12}{6x^2 - 7x - 3}$

c $\dfrac{x^2 - x - 30}{-x^2 + 3x + 18}$ ← Year 1, Section 7.1

387d: **Simplify algebraic fractions where factorisation of a simple quadratic on the numerator or denominator is required.**

Simplify

$$\frac{x-3}{x^2-4x+3}$$

387e: **Simplify algebraic fractions where factorisation of a simple quadratic on both the numerator and denominator is required.**

Simplify

$$\frac{x^2-7x+10}{x^2-6x+5}$$

1.2) Algebraic fractions

Worked Example

Write as a single fraction:

$$3 - (x - 4) \div \frac{x^2 - 16}{x - 5}$$

Worked Example

390e: **Solve an equation with fractions requiring prior factorisation of one of the fractions.**

Solve the following equation:

$$\frac{4}{2x^2+x-15}-\frac{1}{2x-5}=3$$

Give your answer in exact form.

1.3) Partial fractions

If the **denominator is a product of a linear terms**, it can be split into the sum of 'partial fractions', where **each denominator is a single linear term**.

$$\frac{6x-2}{(x-3)(x+1)} \equiv \frac{A}{x-3} + \frac{B}{x+1}$$

Notation reminder: $\equiv$ means 'equivalent/identical to', and indicates that both sides are equal for **all** values of x.

Worked Example

559a: **Decompose a rational function into partial fractions:** $\frac{ax+b}{(cx\pm d)(ex\pm f)}$

Express

$$\frac{5x-1}{(3x-2)(2x+1)}$$

in the form

$$\frac{A}{3x-2}+\frac{B}{2x+1}$$

Worked Example

Given that $\frac{-6x^2-5x+2}{x(x-1)(2x+1)} \equiv \frac{A}{x} + \frac{B}{x-1} + \frac{C}{2x+1}$, find the values of the constants A, B, C.

Worked Example

Express as partial fractions:

$$\frac{6x^2 + 14x - 12}{x^3 - 4x}$$

1.4) Repeated factors

Suppose we wished to express $\frac{2x+1}{(x+1)^2}$ as $\frac{A}{x+1}+\frac{B}{x+1}$. What's the problem?

Because the denominators are the same, we'd get $\frac{A+B}{x+1}$. There's no constant values of A and B we can choose such that $\frac{2x+1}{(x+1)^2} \equiv \frac{A+B}{x+1}$ because the denominators will still be different.

Notes

Worked Example

Express as partial fractions:

$$\frac{4x}{(x-4)^2}$$

559d: Decompose a rational function into partial fractions: $\frac{ax \pm b}{(cx \pm d)(ex \pm f)^2}$

Express

$$\frac{4z+5}{(z-1)(z-4)^2}$$

in the form

$$\frac{A}{z-1}+\frac{B}{z-4}+\frac{C}{(z-4)^2}$$

Worked Example

Express as partial fractions:

$$\frac{11x^2 - 22x + 9}{(x-1)^2(2x-1)}$$

Worked Example

Express as partial fractions:

$$\frac{5x^2 + 4x + 1}{x^3 + x^2}$$

Worked Example

Express as partial fractions:

$$\frac{15x^2 - 5x + 2}{9x^3 - 6x^2 + x}$$

1.5) Algebraic division

In Pure Year 1, we saw that the '**degree**' of a polynomial is the highest power, e.g. a quadratic has degree 2.

An algebraic fraction is **improper** if the degree of the numerator is **at least** the degree of the denominator.

$$\frac{x^2-3}{x+2} \qquad \frac{x+1}{x-1} \qquad \frac{x^3-x^2+3}{x^2-x}$$

A partial fraction is still improper if the degree is the same top and bottom.

Questions might take one of two forms:

- Do the division to express as a quotient and a remainder, e.g. $\frac{x+1}{x-1} \rightarrow 1+\frac{2}{x-1}$
- Express as partial fractions, e.g. $\frac{x^2+x}{(x+1)(x-2)} = A+\frac{B}{x+1}+\frac{C}{x-2}$

Notes

Worked Example

Find the values of A, B and C:

$$\frac{x^2 + 9x - 5}{x + 3} = Ax + B + \frac{C}{x + 3}$$

Worked Example

Find the values of A, B and C:

$$\frac{x^2 - 5x + 9}{x - 2} = Ax + B + \frac{C}{x - 2}$$

Worked Example

Simplify:

$$\frac{x^4 - 81}{x + 3}$$

Worked Example

Split into partial fractions:

$$\frac{9x^2 + 20x - 10}{(x-2)(3x+1)}$$

Worked Example

Split into partial fractions:

$$\frac{9x^2 + 16}{9x^2 - 16}$$

Worked Example

Find the values of A, B, C and D:

$$\frac{x^3 - x^2 + 7}{x - 2} = Ax^2 + Bx + C + \frac{D}{x - 2}$$

Worked Example

Find the values of A, B, C, D and E:

$$\frac{x^4 - x^3 - x + 10}{x^2 - 2x - 3} = Ax^2 + Bx + C + \frac{Dx + E}{x^2 + 2x - 3}$$

Summary of Key Points P1 chapter 4

Summary of key points

1. If p is a root of the function $f(x)$, then the graph of $y = f(x)$ touches or crosses the x-axis at the point $(p, 0)$.
2. The graphs of $y = \frac{k}{x}$ and $y = \frac{k}{x^2}$, where k is a real constant, have asymptotes at $x = 0$ and $y = 0$.
3. The x-coordinate(s) at the points of intersection of the curves with equations $y = f(x)$ and $y = g(x)$ are the solution(s) to the equation $f(x) = g(x)$.
4. The graph of $y = f(x) + a$ is a translation of the graph $y = f(x)$ by the vector $\begin{pmatrix} 0 \\ a \end{pmatrix}$.
5. The graph of $y = f(x + a)$ is a translation of the graph $y = f(x)$ by the vector $\begin{pmatrix} -a \\ 0 \end{pmatrix}$.
6. When you translate a function, any asymptotes are also translated.
7. The graph of $y = af(x)$ is a stretch of the graph $y = f(x)$ by a scale factor of a in the vertical direction.
8. The graph of $y = f(ax)$ is a stretch of the graph $y = f(x)$ by a scale factor of $\frac{1}{a}$ in the horizontal direction.
9. The graph of $y = -f(x)$ is a reflection of the graph of $y = f(x)$ in the x-axis.
10. The graph of $y = f(-x)$ is a reflection of the graph of $y = f(x)$ in the y-axis.

T.82 mixed ex P.35 BSG

Summary of Key Points P2 chapter 1

3 To multiply fractions, cancel any common factors, then multiply the numerators and multiply the denominators.

4 To divide two fractions, multiply the first fraction by the reciprocal of the second fraction.

5 To add or subtract two fractions, find a common denominator.

6 A single fraction with two distinct linear factors in the denominator can be split into two separate fractions with linear denominators. This is called splitting it into **partial fractions**:

$$\frac{5}{(x+1)(x-4)} = \frac{A}{x+1} + \frac{B}{x-4}$$

7 The method of partial fractions can also be used when there are more than two distinct linear factors in the denominator:

$$\frac{7}{(x-2)(x+6)(x+3)} = \frac{A}{x-2} + \frac{B}{x+6} + \frac{C}{x+3}$$

8 A single fraction with a repeated linear factor in the denominator can be split into two or more separate fractions:

$$\frac{2x+9}{(x-5)(x+3)^2} = \frac{A}{x-5} + \frac{B}{x+3} + \frac{C}{(x+3)^2}$$

9 An improper algebraic fraction is one whose numerator has a degree equal to or larger than the denominator. An improper fraction must be converted to a mixed fraction before you can express it in partial fractions.

10 You can either use:

- algebraic division
- or the relationship $F(x) = Q(x) \times \text{divisor} + \text{remainder}$

to convert an improper fraction into a mixed fraction.

T.19 mixed ex. , P.5 BSG

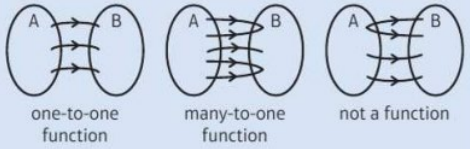
Summary of Key Points P2 Chapter 2

1 A modulus function is, in general, a function of the type $y = |f(x)|$.

- When $f(x) \geqslant 0$, $|f(x)| = f(x)$
- When $f(x) < 0$, $|f(x)| = -f(x)$

2 To sketch the graph of $y = |ax + b|$, sketch $y = ax + b$ then reflect the section of the graph below the x-axis in the x-axis.

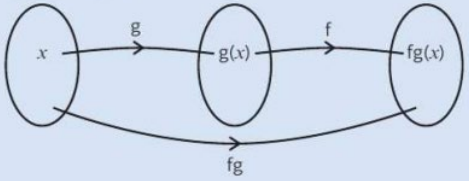
3 A mapping is a **function** if every input has a distinct output. Functions can either be **one-to-one** or **many-to-one**.



one-to-one function | many-to-one function | not a function

4 $fg(x)$ means apply g first, then apply f.

$fg(x) = f(g(x))$



5 Functions $f(x)$ and $f^{-1}(x)$ are inverses of each other. $ff^{-1}(x) = x$ and $f^{-1}f(x) = x$.

6 The graphs of $y = f(x)$ and $y = f^{-1}(x)$ are reflections of each another in the line $y = x$.

7 The domain of $f(x)$ is the range of $f^{-1}(x)$.

8 The range of $f(x)$ is the domain of $f^{-1}(x)$.

9 To sketch the graph of $y = |f(x)|$

- Sketch the graph of $y = f(x)$
- Reflect any parts where $f(x) < 0$ (parts below the x-axis) in the x-axis
- Delete the parts below the x-axis

10 To sketch the graph of $y = f(|x|)$

- Sketch the graph of $y = f(x)$ for $x \geqslant 0$
- Reflect this in the y-axis

11 $f(x + a)$ is a horizontal translation of $-a$.

12 $f(x) + a$ is a vertical translation of $+a$.

13 $f(ax)$ is a horizontal stretch of scale factor $\frac{1}{a}$

14 $af(x)$ is a vertical stretch of scale factor a.

15 $f(-x)$ reflects $f(x)$ in the y-axis.

16 $-f(x)$ reflects $f(x)$ in the x-axis.

T.53: mixed, P.15: BSG