KING EDWARD VI
HANDSWORTH GRAMMAR
SCHOOL FOR BOYS

KING EDWARD VI
ACADEMY TRUST
BIRMINGHAM

Year 12
Pure Mathematics
P1 11 Vectors Booklet

HGS Maths

Dr Frost Course

Name: ________________________________

Class: ________________________________

Contents

Prior knowledge check

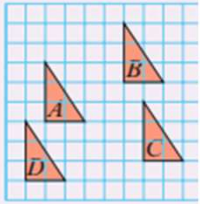
Prior knowledge check

1 Write the column vector for the translation of shape

a A to B

b A to C

c A to D



← GCSE Mathematics

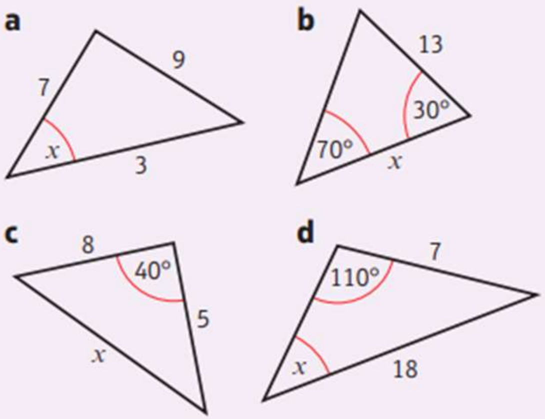
2 P divides the line AB in the ratio $AP : PB = 7 : 2$.

Find:

a $\frac{AP}{AB}$ **b** $\frac{PB}{AB}$ **c** $\frac{AP}{PB}$

← GCSE Mathematics

3 Find x to one decimal place.



← Sections 9.1, 9.2

11.1) Vectors

A Whereas a **coordinate** represents a **position** in space, a **vector** represents a **displacement** in space.

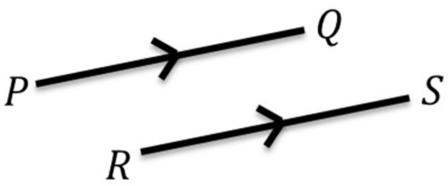
A vector has 2 properties:

- Direction
- Magnitude (i.e. length)

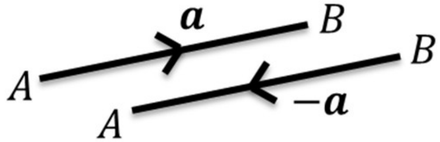
If P and Q are points then $\overrightarrow{PQ}$ is the vector between them.

B If two vectors $\overrightarrow{PQ}$ and $\overrightarrow{RS}$ have the same magnitude and direction, **they're the same vector** and are **parallel**.

This might seem obvious, but students sometimes think the vector is different because the movement occurred at a different point in space. Nope!

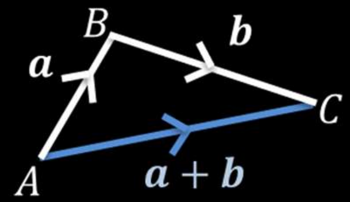


C $\overrightarrow{AB} = -\overrightarrow{BA}$ and the two vectors are parallel, equal in magnitude but in **opposite directions**.



D Triangle Law for vector addition:

$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$$



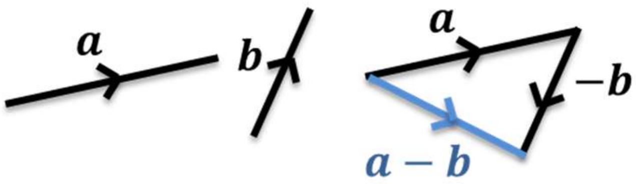
The vector of multiple vectors is known as the **resultant vector.**

(you will encounter this term in Mechanics)

Notes

E Vector **subtraction** is defined using vector addition and negation:

$$\boldsymbol{a} - \boldsymbol{b} = \boldsymbol{a} + (-\boldsymbol{b})$$



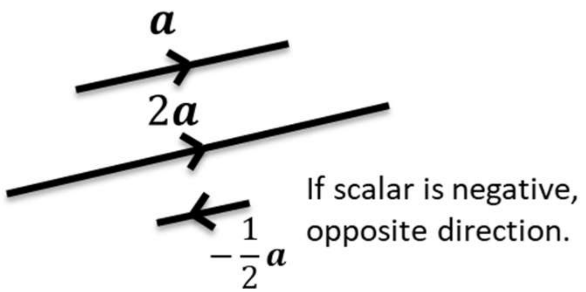
F The zero vector $\mathbf{0}$ (a bold 0), represents no movement.

$$\overrightarrow{PQ} + \overrightarrow{QP} = \mathbf{0}$$

In 2D: $\mathbf{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

G A **scalar** is a normal number, which can be used to 'scale' a vector.

- The **direction** will be the **same**.
- But the **magnitude** will be **different** (unless the scalar is 1).



H Any vector parallel to the vector $\boldsymbol{a}$ can be written as $\lambda\boldsymbol{a}$, where λ is a scalar.

The implication is that if we can write one vector **as a multiple of** another, then we can show they are parallel.

"Show $2\boldsymbol{a} + 4\boldsymbol{b}$ and $3\boldsymbol{a} + 6\boldsymbol{b}$ are parallel".

$3\boldsymbol{a} + 6\boldsymbol{b} = \frac{3}{2}(\boldsymbol{a} + 2\boldsymbol{b})$ ∴ parallel

Worked Example

$PQRS$ is a parallelogram.

N is the point on SQ such that $SN : NQ = 3 : 4$

$\overrightarrow{PQ} = \boldsymbol{b}$ and $\overrightarrow{PS} = \boldsymbol{a}$

Express $\overrightarrow{NR}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$

Worked Example

OAB is a triangle.

$\overrightarrow{OA} = \boldsymbol{b}$ and $\overrightarrow{OB} = \boldsymbol{a}$

P is the point on AB such that $AP:PB = 2:3$.

Find $\overrightarrow{OP}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$

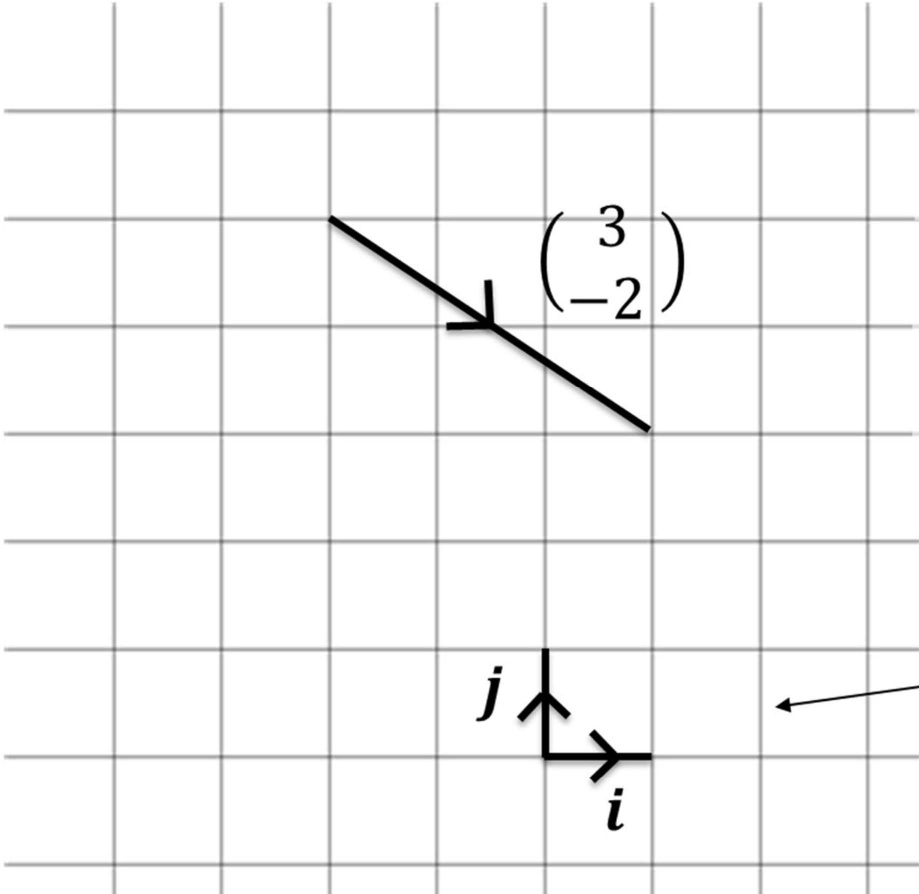
Worked Example

Show that the vectors are parallel:

$3\boldsymbol{a} + 4\boldsymbol{b}$ and $15\boldsymbol{a} + 20\boldsymbol{b}$

$3\boldsymbol{a} + 4\boldsymbol{b}$ and $-0.75\boldsymbol{a} - \boldsymbol{b}$

11.1) Vectors



You should already be familiar that the value of a vector is the **displacement** in the x and y direction (if in 2D).

$$\boldsymbol{a} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}, \quad \boldsymbol{b} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$$\boldsymbol{a} + \boldsymbol{b} = \begin{pmatrix} 3 \\ -3 \end{pmatrix}$$

$$2\boldsymbol{a} = \begin{pmatrix} 6 \\ -4 \end{pmatrix}$$

✎ A **unit vector** is a vector of magnitude 1. $\boldsymbol{i}$ and $\boldsymbol{j}$ are unit vectors in the x-axis and y-axis respectively.

$$\boldsymbol{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \boldsymbol{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

e.g. $\begin{pmatrix} 4 \\ 3 \end{pmatrix} = 4\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 3\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 4\boldsymbol{i} + 3\boldsymbol{j}$

Side Notes: This allows us to write any vector algebraically without using vector notation. **Any point in 2D space,** as a vector from the origin, can be obtained using a linear combination of $\boldsymbol{i}$ and $\boldsymbol{j}$, e.g. if $P(5, -1)$, $\overrightarrow{OP} = 5\boldsymbol{i} - \boldsymbol{j}$. For this reason, $\boldsymbol{i}$ and $\boldsymbol{j}$ are known as **basis vectors** of 2D coordinate space. In fact, any two non-parallel/non-zero vectors can be used as basis vectors, e.g. if $\boldsymbol{a} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$, it's possible to get any vector $\begin{pmatrix} x \\ y \end{pmatrix}$ using a linear combination of these, i.e. we can always find scalars p and q such that $\begin{pmatrix} x \\ y \end{pmatrix} = p\begin{pmatrix} 5 \\ 2 \end{pmatrix} + q\begin{pmatrix} 3 \\ 0 \end{pmatrix}$.

Notes

Worked Example

Draw a diagram to represent the vector:

$2\boldsymbol{i} + 3\boldsymbol{j}$

$2\boldsymbol{i} - 3\boldsymbol{j}$

$-3\boldsymbol{i} + 2\boldsymbol{j}$

$-2\boldsymbol{i} - 3\boldsymbol{j}$

Worked Example

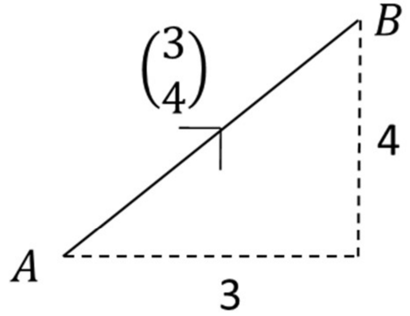
Given $\boldsymbol{a} = 8\boldsymbol{i} - 6\boldsymbol{j}$ and $\boldsymbol{b} = 9\boldsymbol{i} + 7\boldsymbol{j}$, find:

- $4\boldsymbol{b} - 2\boldsymbol{a}$
- $-\boldsymbol{b} + \frac{1}{4}\boldsymbol{a}$

11.1) Vectors

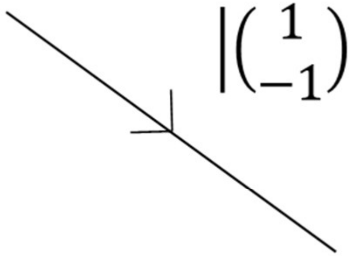
The magnitude $|a|$ of a vector a is its length.

✏ If $\boldsymbol{a} = \begin{pmatrix} x \\ y \end{pmatrix}$ $\quad |\boldsymbol{a}| = \sqrt{x^2 + y^2}$

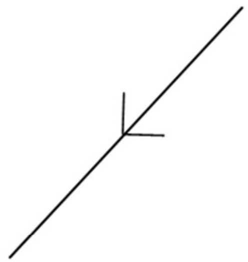


$\begin{pmatrix} 3 \\ 4 \end{pmatrix}$

$$\left|\overrightarrow{AB}\right| = \sqrt{3^2 + 4^2} = 5$$



$$\left|\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right| = \sqrt{2}$$



$$\left|\begin{pmatrix} -5 \\ -12 \end{pmatrix}\right| = 13$$

$$\boldsymbol{a} = \begin{pmatrix} 4 \\ -1 \end{pmatrix} \quad |\boldsymbol{a}| = \sqrt{4^2 + 1^2} = \sqrt{17}$$

$$\boldsymbol{b} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \quad |\boldsymbol{b}| = \sqrt{2^2 + 0^2} = 2$$

A unit vector is a vector whose magnitude is 1

There's certain operations on vectors that require the vectors to be 'unit' vectors. We just scale the vector so that its magnitude is now 1.

$$\boldsymbol{a} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

$$|\boldsymbol{a}| = \sqrt{3^2 + 4^2} = 5$$

$$\hat{\boldsymbol{a}} = \begin{pmatrix} 3/5 \\ 4/5 \end{pmatrix}$$

If $\boldsymbol{a}$ is a vector, then the unit vector $\hat{\boldsymbol{a}}$ in the same direction is

$$\hat{\boldsymbol{a}} = \frac{\boldsymbol{a}}{|\boldsymbol{a}|}$$

Worked Example

472b: **Calculate the magnitude of a 2D column vector.**

Given that:

$$\overrightarrow{AB} = \begin{pmatrix} 7 \\ -5 \end{pmatrix}$$

Find the exact magnitude of $\overrightarrow{AB}$.

Worked Example

Find a unit vector in the direction of:

$$\boldsymbol{a} = 8\boldsymbol{i} + 15\boldsymbol{j}$$

$$\boldsymbol{b} = -9\boldsymbol{i} + 12\boldsymbol{j}$$

Worked Example

Given $\boldsymbol{a} = 8\boldsymbol{i} - 6\boldsymbol{j}$ and $\boldsymbol{b} = 9\boldsymbol{i} + 7\boldsymbol{j}$, find

$$|2\boldsymbol{b} - 3\boldsymbol{a}|$$

Worked Example

506a: **Determine the angle between a vector and the x or y axis.**

Find the angle between the vector $\begin{pmatrix} 13 \\ -12 \end{pmatrix}$ and the positive x-axis.
Give your answer correct to 1 decimal place.

Worked Example

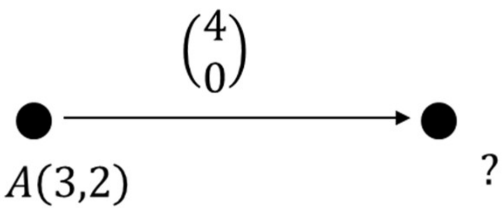
Vector $\boldsymbol{a}$ has magnitude 5 and make an angle of $60°$ with $\boldsymbol{i}$.
Find $\boldsymbol{a}$ in $\boldsymbol{i}, \boldsymbol{j}$ and column vector format.

Worked Example

In triangle PQR, $\overrightarrow{PQ} = \boldsymbol{i} + 2\boldsymbol{j}$ and
$\overrightarrow{PR} = 8\boldsymbol{i} - 15\boldsymbol{j}$.
Find the area of triangle PQR

11.4) Position vectors

Suppose we started at a point $(3,2)$ and translated by the vector $\begin{pmatrix}4\\0\end{pmatrix}$:

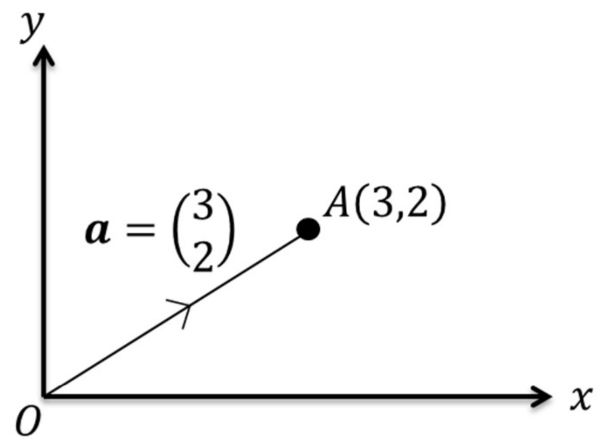


You might think we can do something like:

$$(3,2) + \begin{pmatrix}4\\0\end{pmatrix} = (7,2)$$

But only vectors can be added to other vectors. **If we treated the point $(3,2)$ as a vector**, then this solves the problem:

$$\begin{pmatrix}3\\2\end{pmatrix} + \begin{pmatrix}4\\0\end{pmatrix} = \begin{pmatrix}7\\2\end{pmatrix}$$



A vector used to represent a position is unsurprisingly known as a **position vector**.
A position can be thought of as a translation from the origin, as per above. It enables us to use positions in all sorts of vector (and matrix!) calculations.

The position vector of a point A is the vector $\overrightarrow{OA}$, where O is the origin. $\overrightarrow{OA}$ is usually written as $\boldsymbol{a}$.

Notes

Worked Example

The points A and B have coordinates $(2,5)$ and $(6,13)$ respectively.

Find, in terms of $\boldsymbol{i}$ and $\boldsymbol{j}$:

a) The position vector of A

b) The position vector of B

c) The vector $\overrightarrow{AB}$

Worked Example

$\overrightarrow{OA} = 4\boldsymbol{i} + 3\boldsymbol{j}$ and $\overrightarrow{AB} = 2\boldsymbol{i} - 5\boldsymbol{j}$. Find:

a) The position vector of B.

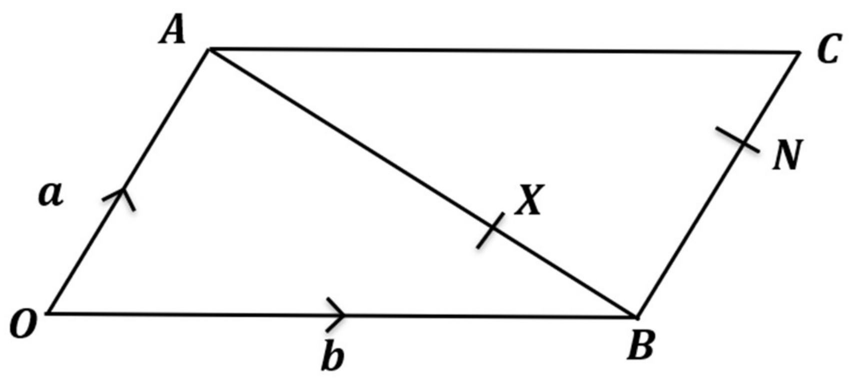
b) The exact value of $\left|\overrightarrow{OB}\right|$ in simplified surd form.

11.5) Solving geometric problems

Notes

Worked Example

$OACB$ is a parallelogram.
X is a point on AB such that $AX:XB = 2:1$. N is the point such that NC is half of BN.
Show that $\overrightarrow{XN}$ is parallel to $\overrightarrow{OC}$.

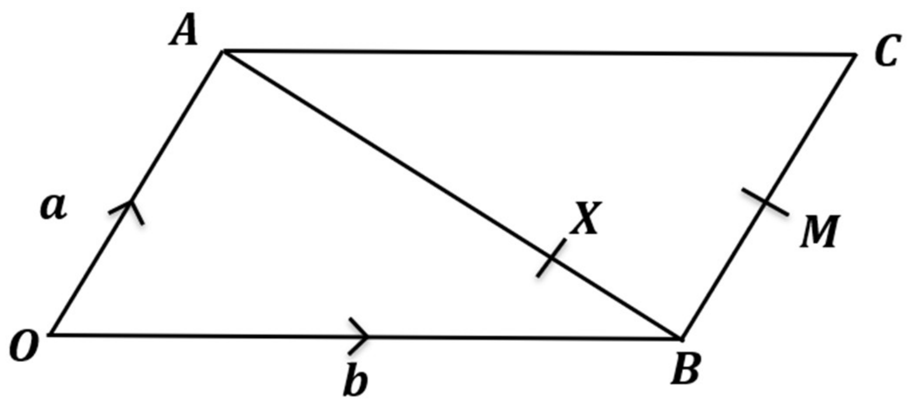


Your Turn

$OACB$ is a parallelogram.

X is a point on AB such that $AX : XB = 3 : 1$. M is the midpoint of BC.

Show that $\overrightarrow{XM}$ is parallel to $\overrightarrow{OC}$.



Worked Example

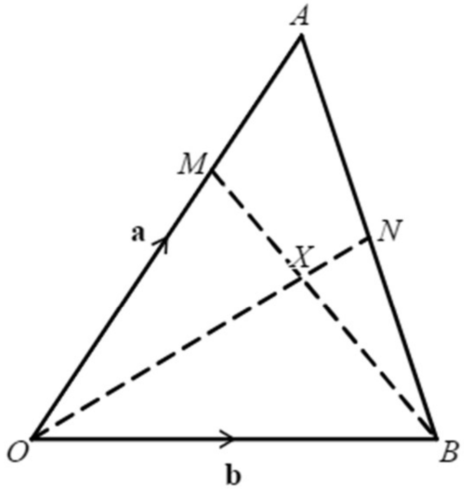
$\overrightarrow{AB} = 2\boldsymbol{i} - 5\boldsymbol{j}$ and $\overrightarrow{AC} = 3\boldsymbol{i} - 7\boldsymbol{j}$.

Determine $\angle BAC$.

Worked Example

475a: **Determine a vector by equating coefficients for two different scalars/routes.**

OAB is a triangle.



The vector $\overrightarrow{OA} = \mathbf{a}$.

The vector $\overrightarrow{OB} = \mathbf{b}$.

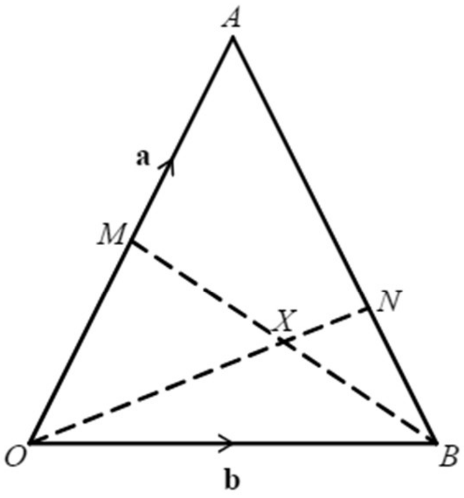
The ratio $OM : MA$ is $2 : 1$.
N is the midpoint of AB.

Given that $\overrightarrow{OX} = \lambda\overrightarrow{ON}$ and $\overrightarrow{MX} = \mu\overrightarrow{MB}$
use a vector method to find the value of λ and μ.

Worked Example

475b: Determine a ratio by writing a vector using two different scalars/routes.

OAB is a triangle.



The vector $\overrightarrow{OA} = \mathbf{a}$.

The vector $\overrightarrow{OB} = \mathbf{b}$.

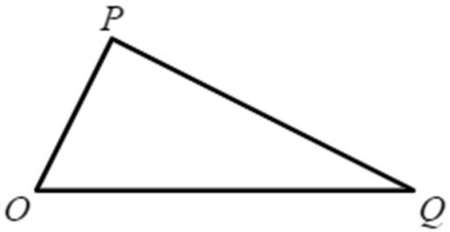
M is the midpoint of OA.
The ratio $BN : NA$ is $1 : 2$.

Find the ratio $OX : XN$.
Give your answer in its simplest form.

Worked Example

475c: **Determine a ratio of vectors by extending out a vector.**

The diagram below shows a sketch of triangle OPQ



The point R is such that $OP : PR = 2 : 3$

The point M is such that $PM : MQ = 3 : 2$

The straight line through R to M cuts OQ at the point N

Let $\vec{OP} = \mathbf{a}$ and $\vec{OQ} = \mathbf{b}$

By first finding $\vec{RM}$ in terms of $\mathbf{a}$ and $\mathbf{b}$, and letting $\vec{RN} = \lambda\vec{RM}$, find $ON : NQ$.

Extension

1 *[STEP 2010 Q7]*

Relative to a fixed origin O, the points A and B have position vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, respectively. (The points O, A and B are not collinear.) The point C has position vector $\boldsymbol{c}$ given by

$$\boldsymbol{c} = \alpha\boldsymbol{a} + \beta\boldsymbol{b},$$

where α and β are positive constants with $\alpha + \beta < 1$. The lines OA and BC meet at the point P with position vector $\boldsymbol{p}$ and the lines OB and AC meet at the point Q with position vector $\boldsymbol{q}$. Show that

$$\boldsymbol{p} = \frac{\alpha a}{1-\beta}$$

and write down q in terms of α, β and b.

Show further that the point R with position vector r given by

$$\boldsymbol{r} = \frac{\alpha\boldsymbol{a} + \beta\boldsymbol{b}}{\alpha + \beta},$$

lies on the lines OC and AB.

The lines OB and PR intersect at the point S. Prove that $\frac{OQ}{BQ} = \frac{OS}{BS}$.

Click here for the solution: http://www.mathshelper.co.uk/STEP%202010%20Solutions.pdf (go to Q7)

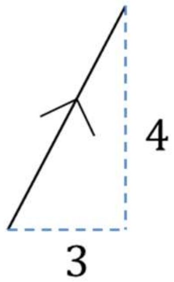
11.6) Modelling with vectors

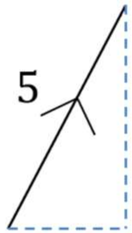
In Mechanics, you will see certain things can be represented as a simple number (without direction), or as a vector (with direction):

Remember a 'scalar' just means a normal number (in the context of vectors). It can be obtained using the **magnitude** of the vector.

Vector Quantity	**Equivalent Scalar Quantity**
Velocity e.g. $\begin{pmatrix} 3 \\ 4 \end{pmatrix}\ km/h$	Speed $= 5\ km/h$
Displacement e.g. $\begin{pmatrix} -5 \\ 12 \end{pmatrix}\ km$	Distance $= 13\ km$

This means the position vector of the object changes by $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ each hour.





...which is equivalent to moving 5km each hour.

Notes

Worked Example

A girl walks 6 km due east from a fixed point O to A, and then 4 km due south from A to B. Find:

a) the total distance travelled

b) the position vector of B relative to O

c) $|\overrightarrow{OB}|$

d) The bearing of B from O.

Worked Example

In an orienteering exercise, a cadet leaves the starting point O and walks 30 km on a bearing of $150°$ to reach A, the first checkpoint.
From A she walks 18 km on a bearing of $210°$ to the second checkpoint, at B.
From B she returns directly to O.
Find:

a) the position vector of A relative to O

b) $|\overrightarrow{OB}|$

c) the bearing of B from O

d) the position vector of B relative O.

2. Relative to a fixed origin O,

the point A has position vector $(2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k})$,

the point B has position vector $(4\mathbf{i} - 2\mathbf{j} + 3\mathbf{k})$,

and the point C has position vector $(a\mathbf{i} + 5\mathbf{j} - 2\mathbf{k})$, where a is a constant and $a < 0$

D is the point such that $\overrightarrow{AB} = \overrightarrow{BD}$.

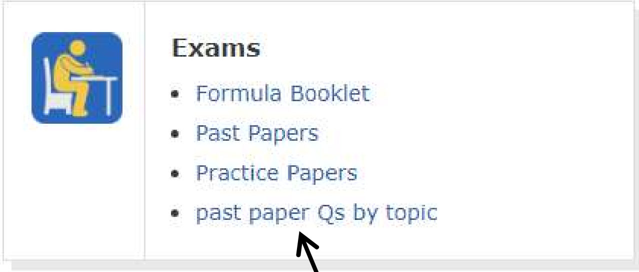
(a) Find the position vector of D.

(2)

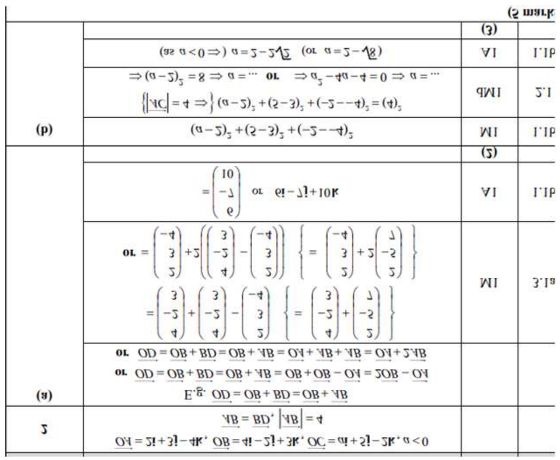
Given $|\overrightarrow{AC}| = 4$

(b) find the value of a.

(3)



Past paper practice by topic. Both new and old specification can be found via this link on hgsmaths.com



Summary of Key Points

Summary of key points

1 If $\overrightarrow{PQ} = \overrightarrow{RS}$ then the line segments PQ and RS are equal in length and are parallel.

2 $\overrightarrow{AB} = -\overrightarrow{BA}$ as the line segment AB is equal in length, parallel and in the opposite directi
to BA.

3 **Triangle law for vector addition:** $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$

If $\overrightarrow{AB} = \mathbf{a}$, $\overrightarrow{BC} = \mathbf{b}$ and $\overrightarrow{AC} = \mathbf{c}$, then $\mathbf{a} + \mathbf{b} = \mathbf{c}$

4 Subtracting a vector is equivalent to 'adding a negative vector': $\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b})$

5 Adding the vectors $\overrightarrow{PQ}$ and $\overrightarrow{QP}$ gives the zero vector **0**: $\overrightarrow{PQ} + \overrightarrow{QP} = 0$.

6 Any vector parallel to the vector **a** may be written as $\lambda\mathbf{a}$, where λ is a non-zero scalar.

7 To multiply a column vector by a scalar, multiply each component by the scalar: $\lambda\begin{pmatrix} p \\ q \end{pmatrix} = ($

8 To add two column vectors, add the x-components and the y-components $\begin{pmatrix} p \\ q \end{pmatrix} + \begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix}$

9 A unit vector is a vector of length 1. The unit vectors along the x- and y-axes are usually denoted by **i** and **j** respectively. $\mathbf{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\mathbf{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

10 For any two-dimensional vector: $\begin{pmatrix} p \\ q \end{pmatrix} = p\mathbf{i} + q\mathbf{j}$

11 For the vector $\mathbf{a} = x\mathbf{i} + y\mathbf{j} = \begin{pmatrix} x \\ y \end{pmatrix}$, the magnitude of the vector is given by: $|\mathbf{a}| = \sqrt{x^2 + y^2}$

12 A unit vector in the direction of **a** is $\frac{\mathbf{a}}{|\mathbf{a}|}$

13 In general, a point P with coordinates (p, q) has position vector:

$$\overrightarrow{OP} = p\mathbf{i} + q\mathbf{j} = \begin{pmatrix} p \\ q \end{pmatrix}$$

14 $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$, where $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are the position vectors of A and B respectively.

15 If the point P divides the line segment AB in the ratio $\lambda : \mu$, then

$$\overrightarrow{OP} = \overrightarrow{OA} + \frac{\lambda}{\lambda + \mu}\overrightarrow{AB}$$

$$= \overrightarrow{OA} + \frac{\lambda}{\lambda + \mu}(\overrightarrow{OB} - \overrightarrow{OA})$$