KING EDWARD VI
HANDSWORTH GRAMMAR
SCHOOL FOR BOYS

KING EDWARD VI
ACADEMY TRUST
BIRMINGHAM

Year 12
Pure Mathematics
P1 10 Trigonometric Identities and Equations
Booklet

HGS Maths

Dr Frost Course

Name: ______________________________

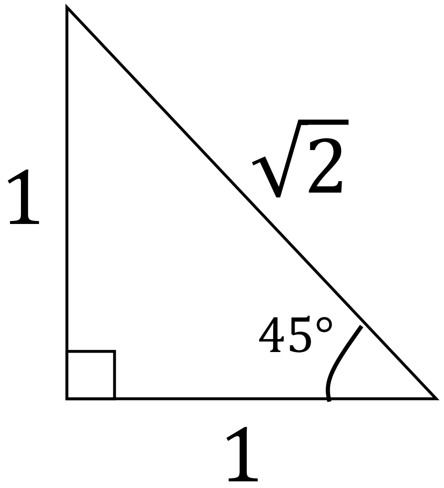
Class: ______________________________

Contents

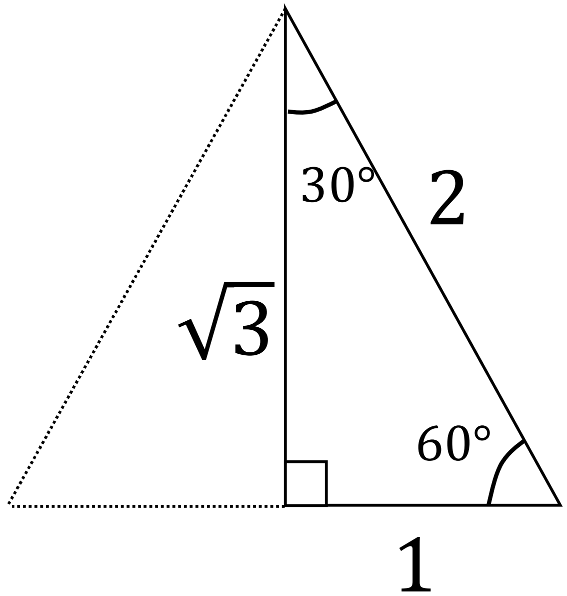
Prerequisite Knowledge

sin/cos/tan of $30°, 45°, 60°$

You will frequently encounter angles of $30°, 60°, 45°$ in geometric problems. Why?
We see these angles in equilateral triangles and half squares:



$$\sin(45°) = \frac{1}{\sqrt{2}}$$
$$\cos(45°) = \frac{1}{\sqrt{2}}$$
$$\tan(45°) = 1$$

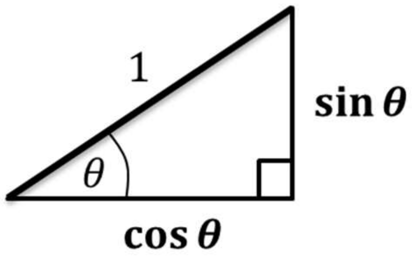


$$\sin(30°) = \frac{1}{2}$$
$$\cos(30°) = \frac{\sqrt{3}}{2}$$
$$\tan(30°) = \frac{1}{\sqrt{3}}$$
$$\sin(60°) = \frac{\sqrt{3}}{2}$$
$$\cos(60°) = \frac{1}{2}$$
$$\tan(60°) = \sqrt{3}$$

Notes

The Unit Circle and Trigonometry

For values of θ in the range $0 < \theta < 90°$, you know that $\sin\theta$ and $\cos\theta$ are lengths on a right-angled triangle:



And what would be the **gradient** of the bold line?

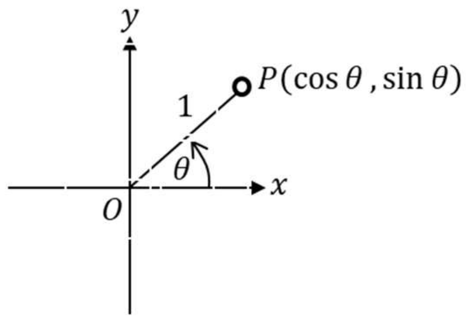
$$m = \frac{\Delta y}{\Delta x} = \frac{\sin\theta}{\cos\theta}$$

But also: $\tan\theta = \frac{opp}{adj} = \frac{\sin\theta}{\cos\theta}$

$$\therefore m = \tan\theta$$

But how do we get the rest of the graph for sin, cos and tan when $90° \leq \theta \leq 360°$?

The point P on a unit circle, such that OP makes an angle θ with the positive x-axis, has coordinates $(\cos\theta, \sin\theta)$.
OP has gradient $\tan\theta$.



Angles are always measured **anticlockwise**.
(Further Mathematicians will encounter the same when they get to Complex Numbers)

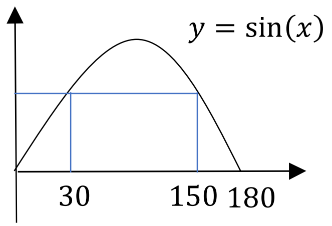
We can consider the coordinate $(\cos\theta, \sin\theta)$ as θ increases from 0 to 360°...

Notes

A Few Trigonometric Angle Laws

1 $\sin(x) = \sin(180° - x)$

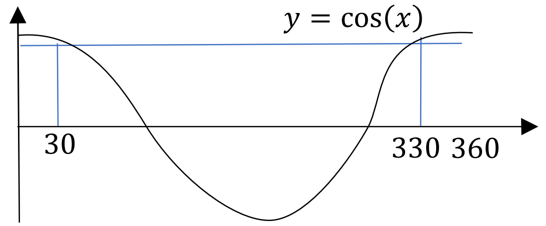
e.g. $\sin(150°) = \sin(30°)$



We saw this in the previous chapter when covering the 'ambiguous case' when using the sine rule.

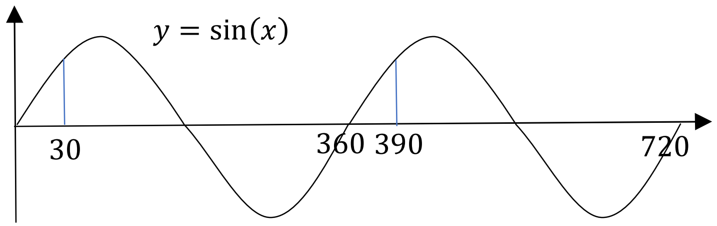
2 $\cos(x) = \cos(360° - x)$

e.g. $\cos(330°) = \cos(30°)$



3 *sin* and *cos* repeat every 360° but *tan* every 180°

e.g. $\sin(390°) = \sin(30°)$



4 $\sin(x) = \cos(90° - x)$

e.g. $\sin(50°) = \cos(40°)$

Remember from the previous chapter that "cosine" by definition is the sine of the "complementary" angle. This was/is never covered in the textbook but caught everyone by surprise when it came up in a C3 exam.

Examples

$\tan(225°) = \mathbf{\tan(45°)} = \mathbf{1}$

tan repeats every 180° so can subtract 180°

$\tan(210°) = \mathbf{\tan(30°)} = \mathbf{\frac{1}{\sqrt{3}}}$

$\sin(150°) = \mathbf{\sin(30°)} = \mathbf{\frac{1}{2}}$

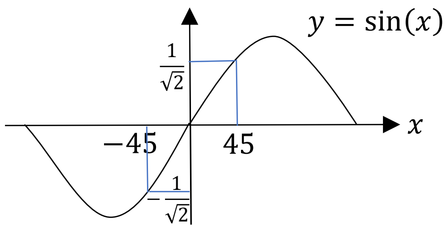
For *sin* we can subtract from 180°.

$\cos(300°) = \mathbf{\cos(60°)} = \mathbf{\frac{1}{2}}$

For *cos* we can subtract from 360°.

$\sin(-45°) = \mathbf{-\sin(45°)} = \mathbf{-\frac{1}{\sqrt{2}}}$

We have to resort to a sketch for this one.

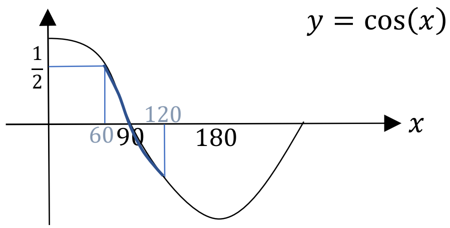


$\cos(750°) = \mathbf{\cos(30°)} = \mathbf{\frac{\sqrt{3}}{2}}$

cos repeats every 360°.

$\cos(120°) = \mathbf{-\cos(60°)} = \mathbf{-\frac{1}{2}}$

Again, let's just use a graph.



The graph is rotationally symmetric about 90°. Since 120° is 30° above 90°, we get the same y value for $90° - 30° = 60°$, except negative.

Use the 'laws' where you can, but otherwise just draw out a quick sketch of the graph.

- $\sin(x) = \sin(180 - x)$
- $\cos(x) = \cos(360 - x)$
- sin, cos repeat every 360° but tan every 180°

reflections: It's not hard to see from the graph that in general, $\sin(-x) = -\sin(x)$. Even more generally, a function f is known as an '**odd function**' if $f(-x) = -f(x)$. tan is similarly 'odd' as $\tan(-x) = -\tan(x)$.

A function is **even** if $f(-x) = f(x)$. Examples are $f(x) = \cos(x)$ and $f(x) = x^2$ as $\cos(-x) = \cos(x)$ and $(-x)^2 = (x)^2$. You do not need to know this for the exam.

Test Your Understanding

Without a calculator, work out the value of each below.

$\cos(315°) =$

$\sin(420°) =$

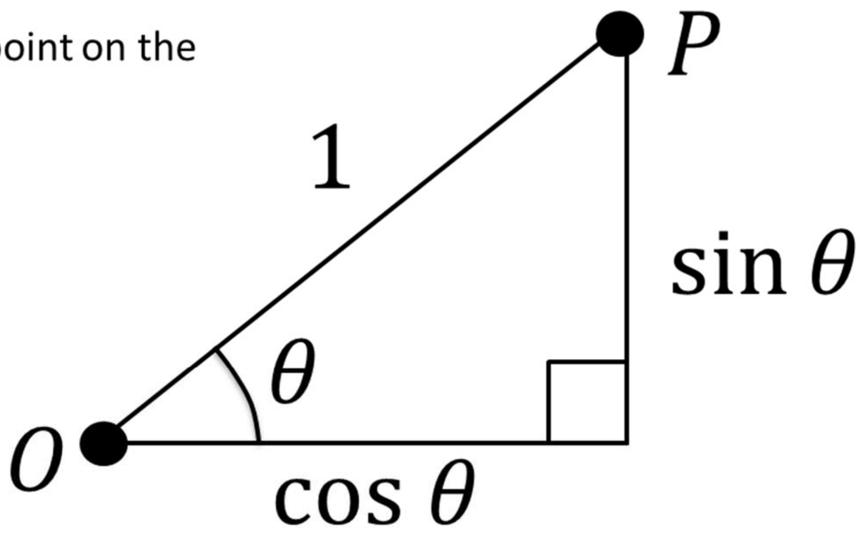
$\tan(-120°) =$

$\tan(-45°) =$

$\sin(135°) =$

10.3 Trigonometric Identities

Returning to our point on the unit circle...



1 **Then** $\boldsymbol{tan\ \theta = \frac{sin\ \theta}{cos\ \theta}}$

2 Pythagoras gives you...

$$\boldsymbol{sin^2\theta + cos^2\theta = 1}$$

$\sin^2\theta$ is a shorthand for $(\sin\theta)^2$. It does NOT mean the sin is being squared – this does not make sense as sin is a function, and not a quantity that we can square!

Notes

Worked Example

Simplify:

$\sin^2 3x + \cos^2 3x$

Worked Example

Prove that $1 - \tan\theta \sin\theta \cos\theta \equiv \cos^2\theta$

Worked Example

Prove that $\tan\theta + \frac{1}{\tan} \equiv \frac{1}{\sin\theta\cos\theta}$

Worked Example

Simplify $5 - 5\sin^2\theta$

Worked Example

Simplify:

$$\frac{\sin 2\theta}{\sqrt{1-\sin^2 2\theta}}$$

Worked Example

Prove that

$$\frac{\cos^4 \theta - \sin^4 \theta}{\cos^2 \theta} \equiv 1 - \tan^2 \theta$$

Worked Example

Prove that

$$\frac{\tan x \cos x}{\sqrt{1-\cos^2 x}} \equiv 1$$

Worked Example

Prove that

$$\tan^2\theta \equiv \frac{1}{\cos^2\theta} - 1$$

Worked Example K502a

Given that $\sin\theta = \frac{2}{5}$ and that θ is obtuse, find the exact value of $\cos\theta$

Worked Example

502b: Determine exact values of $\sin$, $\cos$ and $\tan$ using another known trigonometric ratio, for obtuse or reflex angles.

It is given that B is reflex and that $\sin B = -\frac{1}{2}$

Find the exact value of $\cos B$.

Worked Example

502c: Determine exact values of $\sin$, $\cos$ and $\tan$ using another known trigonometric ratio in terms of an algebraic expression.

Given that

$$\sin\theta = \frac{k-1}{k+1}$$

where k is a positive constant and θ is acute.

Find an expression for $\cos\theta$ in terms of k.

$\cos\theta =$

Worked Example

502f: Write an expression given in the form $a\sin^2 x + b\cos^2 x$ in terms of just $\sin$ or $\cos$

Given that:

$$2\sin^2 3\alpha + 8\cos^2 3\alpha \equiv A + B\sin^2 3\alpha$$

where A and B are integers.

Work out the values of A and B.

Worked Example

502i: Simplify a trigonometric expression by using the identity $\sin^2 x + \cos^2 x \equiv 1$ and $\tan x = \frac{\sin x}{\cos x}$

Simplify

$$1 - \sin\alpha\cos\alpha\tan\alpha$$

giving your answer as a single trigonometric function.

Worked Example

502j: Simplify a trigonometric expression involving $\tan^2 x$ or $\tan^3 x$

Simplify

$$1 - \cos^2\theta\tan^2\theta$$

giving your answer as a single trigonometric function.

Worked Example

502k: Simplify an algebraic fraction in terms of $\sin$ and $\cos$ using quadratic factorisation.

Simplify

$$\frac{12 + 3\sin x}{2\cos^2 x - 5\sin x + 10}$$

Giving your answer in the form $\dfrac{a}{b + c\sin x}$

Worked Example

Given that $p = 3\cos\theta$ and $q = 2\sin\theta$, show that $4p^2 + 9q^2 = 36$

10.4 Simple Trigonometric Equations

Notes

Worked Example

503a: Solve a trigonometric equation given in the form $\sin x = k$ where x is in degrees.

Solve $\sin x = -0.8$ in the interval $0° \leq x \leq 720°$

Give your solution(s) correct to 1 decimal place where appropriate.

$x =$ ____ °

$x =$ ____ °

$x =$ ____ °

$x =$ ____ °

Worked Example

503f: Solve trigonometric equations given in the form $\sin^2 x = a$ where x is in degrees.

Solve $\sin^2 x = 0.49$ in the interval $0° \leq x \leq 360°$

Give your solution(s) correct to 1 decimal place where appropriate.

$x =$ _____ °

$x =$ _____ °

$x =$ _____ °

$x =$ _____ °

10.5 Harder Trigonometric Equations

Notes

Worked Example

503b: Solve a trigonometric equation given in the form $\sin(ax) = k$ where x is in degrees.

Solve $\tan\left(\frac{1}{2}x\right) = 0.4$ in the interval $-180^\circ \leq x \leq 540^\circ$

Give your solution(s) correct to 1 decimal place where appropriate.

$x =$ ____ $^\circ$

$x =$ ____ $^\circ$

Worked Example

503c: State the number of solutions to a trigonometric equation given in the form $\sin(ax) = k$ or $\cos(ax) = k$ or $\tan(ax) = k$ for a given range.

Determine the number of solutions of the equation $\cos(2x) = -0.8$ in the interval $-360° \leq x \leq 315°$

Number of solutions =

Worked Example

503d: Solve a trigonometric equation given in the form $\sin(ax + b) = k$ where x is in degrees and k is positive.

Solve $\sin\left(\frac{1}{3}\theta - 150\right) = 0.9$ in the interval $-180^\circ \leq \theta \leq 900^\circ$

Give your solution(s) correct to 1 decimal place where appropriate.

$\theta =$ ____ $^\circ$

$\theta =$ ____ $^\circ$

503e: Solve a trigonometric equation given in the form $\sin(ax+b)=k$ where x is in degrees and k is negative.

Solve $\sin(2x+90)=-0.4$ in the interval $0^\circ \leq x \leq 360^\circ$

Give your solution(s) correct to 1 decimal place where appropriate.

$x=$ $^\circ$

$x=$ $^\circ$

$x=$ $^\circ$

$x=$ $^\circ$

Worked Example

503i: Solve a trigonometric equation given in the form $p\sin(ax+b) = q\cos(ax+b)$

Solve $4\sin(\frac{1}{2}\theta + 60) = \sqrt{3}\cos(\frac{1}{2}\theta + 60)$ in the interval $0^\circ < \theta < 720^\circ$

Give your solutions correct to 2 decimal places where appropriate.

$\theta =$ ______ $^\circ$

$\theta =$ ______ $^\circ$

Worked Example

503n: Solve a trigonometric equation from a modelled scenario.

The depth of water, H meters, in a harbour on a particular day is modelled by the formula

$$H = 10 + 2\sin\left(\tfrac{1}{2}\theta\right), \qquad 0 \leq \theta \leq 1440$$

where θ is the number of minutes after midnight.

Freddie needs to leave the harbour by 1 pm at the latest.
Find the last time the water is at a depth of the water is 11.96 meters before Freddie needs to leave.
Give your answer to the nearest minute in 24 hour clock.

☐ : ☐

10.6 Equations and Identities

Worked Example

Solve in the interval $0 \leq x < 360°$:

$5\sin^2 x + 3\sin x - 2 = 0$

Worked Example

503j: Solve a quadratic equation involving a single trigonometric function.

Solve $2\cos^2 x = 15\cos x - 7$ in the interval $0^\circ < x < 360^\circ$

Give your solution(s) correct to 2 decimal places where appropriate.

$x =$ ______ $^\circ$

$x =$ ______ $^\circ$

Worked Example

503k: Solve a trigonometric equation involving a mixture of $\sin$ and $\cos$ where one is squared.

Solve $2\cos^2\theta = 3(-3\sin\theta + 2)$ in the interval $0^\circ \leq \theta < 360^\circ$

Give your solution(s) correct to 2 decimal places where appropriate.

$\theta =$ ____ $^\circ$

$\theta =$ ____ $^\circ$

Worked Example

Solve in the interval $0 \leq x \leq 360°$:

$$\sin^2(x - 30°) = \frac{1}{2}$$

12. (a) Solve, for $-180° \leqslant x < 180°$, the equation

$$3\sin^2 x + \sin x + 8 = 9\cos^2 x$$

giving your answers to 2 decimal places.

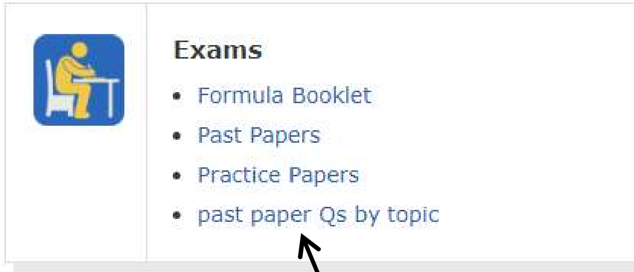
(6)

(b) Hence find the smallest positive solution of the equation

$$3\sin^2(2\theta - 30°) + \sin(2\theta - 30°) + 8 = 9\cos^2(2\theta - 30°)$$

giving your answer to 2 decimal places.

(2)



Past paper practice by topic. Both new and old specification can be found via this link on hgsmaths.com

12(a)	Uses $\cos^2 x = 1 - \sin^2 x \Rightarrow 3\sin^2 x + \sin x + 8 = 9(1 - \sin^2 x)$	M1	3.1
	$\Rightarrow 12\sin^2 x + \sin x - 1 = 0$	A1	1.1
	$\Rightarrow (4\sin x - 1)(3\sin x + 1) = 0$	M1	1.1
	$\Rightarrow \sin x = \frac{1}{4}, -\frac{1}{3}$	A1	1.1
	Uses arcsin to obtain two correct values	M1	1.1
	All four of $x = 14.48°, 165.52°, -19.47°, -160.53°$	A1	1.1
		(6)	
(b)	Attempts $2\theta - 30° = -19.47°$	M1	3.1
	$\Rightarrow \theta = 5.26°$	A1ft	1.1
		(2)	

(8 marks)

Summary of Key Points

5 For all values of θ, $\sin^2 \theta + \cos^2 \theta \equiv 1$

6 For all values of θ such that $\cos \theta \neq 0$, $\tan \theta \equiv \dfrac{\sin \theta}{\cos \theta}$

7
- Solutions to $\sin \theta = k$ and $\cos \theta = k$ only exist when $-1 \leqslant k \leqslant 1$
- Solutions to $\tan \theta = p$ exist for all values of p.

8 When you use the inverse trigonometric functions on your calculator, the angle you get is called the **principal value**.

9 Your calculator will give principal values in the following ranges:
- $\cos^{-1}$ in the range $0 \leqslant \theta \leqslant 180°$
- $\sin^{-1}$ in the range $-90° \leqslant \theta \leqslant 90°$
- $\tan^{-1}$ in the range $-90° \leqslant \theta \leqslant 90°$