KING EDWARD VI
HANDSWORTH GRAMMAR
SCHOOL FOR BOYS

KING EDWARD VI
ACADEMY TRUST
BIRMINGHAM

# Year 12
# Pure Mathematics
# 8 The Binomial Expansion
# Booklet

HGS Maths

Dr Frost Course

## Name: ______________________________

## Class: ______________________________

## Contents

# Prior knowledge check

## Prior knowledge check

1 Expand and simplify where possible:

a $(2x - 3y)^2$  b $(x - y)^3$  c $(2 + x)^3$

← Section 1.2

2 Simplify

a $(-2x)^3$  b $(3x)^{-4}$

c $\left(\frac{2}{5}x\right)^2$  d $\left(\frac{1}{3}x\right)^{-3}$

← Sections 1.1, 1.4

3 Simplify

a $(25x)^{\frac{1}{2}}$  b $(64x)^{-\frac{2}{3}}$

c $\left(\frac{9}{100}x\right)^{-\frac{1}{2}}$  d $\left(\frac{8}{27}x\right)^{\frac{4}{3}}$

← Section 1.4

# 8.3 The Binomial Expansion

## Pascal's Triangle

The second number of each row tells us what row we should use for an expansion.

So if we were expanding $(2 + x)^4$, the power is 4, so we use this row.

In Pascal's Triangle, each term (except for the 1s) is the sum of the two terms above.

**Memorise this**

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 1 1

# Notes

# Worked Example

## 501a: Expand $(x+b)^n$, where $n$ is a positive integer up to 6, using Pascal's triangle.

Use Pascal's triangle to find the binomial expansion of $(y+9)^3$.

## Worked Example

**501b: Expand $(x-b)^n$, where $n$ is a positive integer up to 6, using Pascal's triangle.**

Use Pascal's triangle to find the binomial expansion of $(y-8)^4$.

Give your answer in its simplest form.

# Worked Example

## 501c: Expand $(ax \pm b)^n$, where $n$ is a positive integer up to 6, using Pascal's triangle.

Use Pascal's triangle to find the binomial expansion of $(1 + 5y)^4$.

Give your answer in its simplest form.

## Worked Example

### 501g: Expand $(ax \pm b)^n$ for positive integers $n$, where $b$ is fractional.

Find the first three terms, in ascending powers of $y$, of the binomial expansion of $\left(4y - \frac{2}{3}\right)^6$.



*T.164: 8C QS1-6, P.63: 8.3 Qs 1-6*

# Worked Example

## 501i: Use a binomial expansion to reason about terms in the expansion of $(a+bx)^n(c+dx)$

The first three terms, in ascending powers of $y$, of the expansion of $(2y-1)^7$ are:

$$-1+14y-84y^2$$

Hence find the coefficient of $y^2$ in the expansion of $(4y+5)(2y-1)^7$.

## Worked Example

**501h: Expand $(a + bx)^n$ for positive integers $n$, where $b$ is algebraic.**

Find the first three terms, in ascending powers of $x$, of the binomial expansion of $(px + 2)^9$.

## Worked Example

Find the binomial expansion of $\left(x+\frac{1}{x}\right)^5$ giving each term in its simplest form.

*T.164: 8C Q7, P.63: 8.3 Qs 7+*

# 8.4 Solving Binomial Problems

# Factorial and Choose Function

$n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1$
said "$n$ factorial", is the number of ways of arranging $n$ objects in a line.

For example, suppose you had three letters, A, B and C, and wanted to arrange them in a line to form a 'word', e.g. ACB or BAC.

- There are 3 choices for the first letter.
- There are then 2 choices left for the second letter.
- There is then only 1 choice left for the last letter.

There are therefore $3 \times 2 \times 1 = 3! = 6$ possible combinations.
**Your calculator can calculate a factorial using the $x!$ button.**

${}^{n}Cr = \binom{n}{r} = \frac{n!}{r!(n-r)!}$
said "$n$ choose $r$", is the number of ways of 'choosing' $r$ things from $n$, such that the order in our selection does not matter.
These are also known as <u>**binomial coefficients**</u>.

For example, if you a football team captain and need to choose 4 people from amongst 10 in your class, there are $\binom{10}{4} = \frac{10!}{4!6!} = 210$ possible selections.
*(Note: the $\binom{10}{4}$ notation is preferable to ${}^{10}C_4$)*
**Use the $nCr$ button on your calculator (your calculator input should display "10C4")**

## Examples

Calculate the following <u>WITHOUT</u> a calculator

a) $\binom{5}{3}$

b) $\binom{n}{n-1}$

c) $\binom{n}{n-3}$

d) $\binom{2n}{n}$

e) $\binom{n}{1}$

f) $\binom{n}{n}$

*(EXT: f and g are equivalent to ?)*

## Exercise

$$\binom{n}{r} = {}^{n}C_r = \frac{n!}{r!\,(n-r)!}$$

1. Simplify each of these:
    a. $\binom{n}{n-1}$
    b. $\binom{n}{3}$
    c. $\binom{n}{n-3}$
    d. $\binom{n+1}{n-1}$
    e. $\binom{n-2}{3}$
    f. $\binom{2n}{2}$
    g. $\binom{3n-1}{2}$
    h. $\binom{4n+5}{3}$
    i. $\binom{5n+4}{5n+2}$
    j. $\binom{2n-4}{2n-6}$

2. Solve the following equations:
    a. $3 \times {}^{n}C_6 = 11 \times {}^{n}C_4$
    b. $10 \times {}^{n}C_5 = 21 \times {}^{n}C_3$
    c. $33 \times {}^{n}C_2 = 2 \times {}^{n}C_5$
    d. $5 \times {}^{n}C_5 = 7 \times {}^{n}C_7$
    e. $28 \times {}^{n}C_4 = 15 \times {}^{n}C_6$
    f. $82 \times {}^{n}C_4 = 205 \times {}^{n}C_2$
    g. $7 \times {}^{n}C_7 = 117 \times {}^{n}C_5$
    h. $4 \times {}^{n}C_8 = 85 \times {}^{n}C_6$
    i. $5 \times {}^{n}C_6 = 506 \times {}^{n}C_3$
    j. $32509 \times {}^{n}C_4 = 35 \times {}^{n}C_7$

## Worked Example

### 501f: Determine the value of a coefficient in a binomial expansion with a positive integer power.

Find the coefficient of $y^4$ in the expansion of $(2y-2)^7$.

*T.166: 8D Q1, P.64: 8.4 Qs 1,2*

## Worked Example

$g(x) = (1 + kx)^{10}$, where $k$ is a constant.
Given that the coefficient of $x^3$ in the binomial expansion of $g(x)$ is $15$, find the value of $k$.

*T.166: 8D Q2,3, P.64: 8.4 Qs 1,2*

## Extra Exercise

3. In the binomial expansion of $(1+x)^n$, where $n \geq 5$, the coefficient of $x^2$ and the coefficient of $x^5$ is the same. Find the value of $n$.

4. In the binomial expansion of $(1+x)^n$, where $n \geq 4$, the coefficient of $x^5$ is 2 times the coefficient of $x^4$. Find the value of $n$.

5. In the binomial expansion of $(1+x)^n$, where $n \geq 4$, the coefficient of $x^4$ is 13 times the coefficient of $x^2$. Find the value of $n$.

6. In the binomial expansion of $(1+x)^n$, where $n \geq 6$, the coefficient of $x^6$ is 14 times the coefficient of $x^2$. Find the value of $n$.

7. In the binomial expansion of $(1+x)^n$, where $n \geq 4$, the coefficient of $x^4$ is the sum of 3 times the coefficient of $x^3$ and 4 times the coefficient of $x^2$. Find the value of $n$.

8. In the binomial expansion of $(1+x)^n$, where $n \geq 4$, the coefficient of $x^3$ is the sum of the coefficient of $x^2$ and 80 times the coefficient of $x$. Find the value of $n$.

9. In the binomial expansion of $(1+x)^n$, where $n \geq 7$, the coefficient of $x^7$ is the difference between 20 times the coefficient of $x^4$ and 17 times the coefficient of $x^3$. Find the value of $n$.

10. In the binomial expansion of $(1+x)^n$, where $n \geq 4$, the coefficient of $x^4$ is the difference between 2 times the coefficient of $x^3$ and 3 times the coefficient of $x$. Find the two possible values of $n$.

11. In the binomial expansion of $(1+2x)^n$, where $n \geq 2$, the coefficient of $x^2$ is 10 times the coefficient of $x$. Find the value of $n$.

## Extra Exercise

12. In the binomial expansion of $(1 + 2x)^n$, where $n \geq 4$, the coefficient of $x^4$ is 70 times the coefficient of $x$. Find the value of $n$.

13. In the binomial expansion of $(1 + 2x)^n$, where $n \geq 5$, the coefficient of $x^5$ is the difference between 4400 times the coefficient of $x$ and 1024. Find the value of $n$.

14. In the binomial expansion of $\left(1 + \frac{x}{2}\right)^n$, where $n \geq 5$, the coefficient of $x^4$ is 2 times the coefficient of $x^5$. Find the value of $n$.

15. In the binomial expansion of $\left(1 + \frac{x}{2}\right)^n$, where $n \geq 5$, the coefficient of $x^4$ is equal to the coefficient of $x^5$. Find the value of $n$. [No Title]

16. In the binomial expansion of $(1 + 3x)^n$, where $n \geq 5$, the coefficient of $x^5$ is 1512 times the coefficient of $x^2$. Find the value of $n$.

17. In the binomial expansion of $(1 + 3x)^n$, where $n \geq 3$, the coefficient of $x^3$ is the sum of 6 times the coefficient of $x^2$ and 27 times the coefficient of $x$. Find the value of $n$.

18. In the binomial expansion of $(1 + 5x)^n$, where $n \geq 3$, the coefficient of $x^3$ is the difference between 32 times the coefficient of $x^2$ and 95 times the coefficient of $x$. Find the value of $n$.

# Worked Example

## 501k: Reason about the $b$ and $n$ in $(a+bx)^n$ given known coefficients of terms in the binomial expansion.

In the expansion of $\left(\frac{4}{5}y+p\right)^5$, where $p$ is a constant, the coefficient of $y$ is $0.25$.

Hence find the value of $p$.

## Worked Example

a) Write down the first three terms, in ascending powers of $x$, of the binomial expansion of $(1 + qx)^8$, where $q$ is a non-zero constant.

b) Given that, in the expansion of $(1 + qx)^8$, the coefficient of $x$ is $-r$ and the coefficient of $x^2$ is $7r$, find the value of $q$ and the value of $r$.

*T.166: 8D Qs 3+, P.64: 8.4 Qs 3+*

# 8.5 Binomial Estimation

# Notes

## Worked Example

### 501j: Use a binomial expansion to estimate the value of $k^n$

The first three terms, in ascending powers of $y$, of the expansion of $\left(2-\frac{y}{2}\right)^5$ are:

$$32-40y+20y^2$$

Hence find an approximate value of $(1.995)^5$.

Give your answer as a fraction in its simplest form.

## Worked Example

a) Find the first three terms of the binomial expansion, in ascending powers of $x$, of $\left(7 - \frac{x}{5}\right)^9$.

b) Use your expansion to estimate the value of $6.991^8$, giving your answer to 4 significant figures.



*T.168: 8E all Qs, P.65: 8.5 all Qs*

# Extract from Formulae booklet

**Binomial series**

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots + b^n \quad (n \in \mathbb{N})$$

where $\binom{n}{r} = {}^nC_r = \dfrac{n!}{r!(n-r)!}$

**6.** (a) Find the first 4 terms, in ascending powers of $x$, in the binomial expansion of

$$(1 + kx)^{10}$$

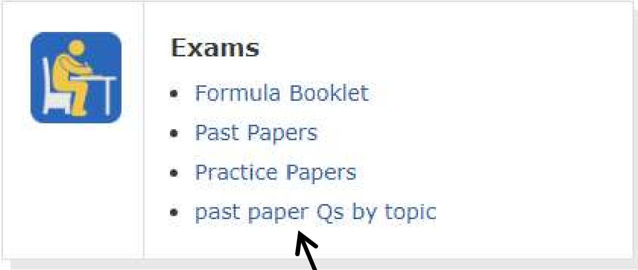
where $k$ is a non-zero constant. Write each coefficient as simply as possible.

**(3)**

Given that in the expansion of $(1 + kx)^{10}$ the coefficient $x^3$ is 3 times the coefficient of $x$,

(b) find the possible values of $k$.

**(3)**




Past paper practice by topic. Both new and old specification can be found via this link on hgsmaths.com

| Question | Scheme | Marks | AOs |
|---|---|---|---|
| 6 (a) | $(1+kx)^{10} = 1 + \binom{10}{1}(kx)^1 + \binom{10}{2}(kx)^2 + \binom{10}{3}(kx)^3 \ldots$ | M1<br>A1 | 1.1b<br>1.1b |
| | $= 1 + 10kx + 45k^2x^2 + 120k^3x^3 \ldots$ | A1 | 1.1b |
| | | (3) | |
| (b) | Sets $120k^3 = 3 \times 10k$ | B1 | 1.2 |
| | $4k^2 = 1 \Rightarrow k = \ldots$ | M1 | 1.1b |
| | $k = \pm\frac{1}{2}$ | A1 | 1.1b |
| | | (3) | |
| (6 marks) | | | |

## Summary of Key Points

**1** Pascal's triangle is formed by adding adjacent pairs of numbers to find the numbers on the next row.

**2** The $(n + 1)$th row of Pascal's triangle gives the coefficients in the expansion of $(a + b)^n$.

**3** $n! = n \times (n - 1) \times (n - 2) \times \ldots \times 3 \times 2 \times 1$.

**4** You can use factorial notation and your calculator to find entries in Pascal's triangle quickly.

- The number of ways of choosing $r$ items from a group of $n$ items is written as ${}^nC_r$ or $\binom{n}{r}$: ${}^nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$
- The $r$th entry in the $n$th row of Pascal's triangle is given by ${}^{n-1}C_{r-1} = \binom{n-1}{r-1}$.

**5** The binomial expansion is:

$$(a + b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots + b^n \ (n \in \mathbb{N})$$

where $\binom{n}{r} = {}^nC_r = \frac{n!}{r!(n-r)!}$

**6** In the expansion of $(a + b)^n$ the general term is given by $\binom{n}{r}a^{n-r}b^r$.

**7** If $x$ is small, the first few terms in the binomial expansion can be used to find an approximate value for a complicated expression.