KING EDWARD VI HANDSWORTH GRAMMAR SCHOOL FOR BOYS

KING EDWARD VI ACADEMY TRUST BIRMINGHAM

Year 12
Pure Mathematics
P1 2.3-2.6 Quadratics Booklet

HGS Maths

Dr Frost Course

Name: ______________________________

Class: ______________________________

Contents

Prior knowledge check

you should have completed the summer task booklet before starting this chapter as it covers 2.1 – 2.3

Prior knowledge check

1 Solve the following equations:

a $3x + 6 = x - 4$

b $5(x + 3) = 6(2x - 1)$

c $4x^2 = 100$

d $(x - 8)^2 = 64$

← GCSE Mathematics

2 Factorise the following expressions:

a $x^2 + 8x + 15$ **b** $x^2 + 3x - 10$

c $3x^2 - 14x - 5$ **d** $x^2 - 400$

← Section 1.3

3 Sketch the graphs of the following equations, labelling the points where each graph crosses the axes:

a $y = 3x - 6$ **b** $y = 10 - 2x$

c $x + 2y = 18$ **d** $y = x^2$

← GCSE Mathematics

4 Solve the following inequalities:

a $x + 8 < 11$ **b** $2x - 5 \geqslant 13$

c $4x - 7 \leqslant 2(x - 1)$ **d** $4 - x < 11$

← GCSE Mathematics

2.3) Functions

*Hidden quadratics only**

Notes

Worked Example

365h: Factorise an expression using the substitution of a linear expression to reduce to a standard quadratic form.

Factorise fully

$$12(x+2)^2+29\,(x+2)+15$$

Worked Example

485a: Solve a hidden quadratic equation given in the form x^n by substitution.

Solve the equation

$$2y^6 = 15y^3 + 27$$

Worked Example

485b: Solve a hidden quadratic equation containing roots by substitution.

Solve the equation

$$4z = 8\sqrt{z} - 3$$

Worked Example

485c: Solve a hidden quadratic equation given in the form x^n by substitution, that requires multiplying through by a power of x.

Solve the equation

$$2y^2 - \frac{15}{y^2} + 1 = 0$$

Worked Example

485e: Solve a quadratic equation in terms of an exponential term of the form a^x (no logarithms).

Find the exact solutions of

$$4 \times 2^{2x} - 65 \times 2^{x} + 16 = 0.$$

Exercise

Question 1

Skill involved: 485a: Solve a hidden quadratic equation given in the form x^n by substitution.

Solve the equation

$4y^6 + 8y^3 + 3 = 0$

$y =$ ....

or $y =$ ....

Question 2

Skill involved: 485a: Solve a hidden quadratic equation given in the form x^n by substitution.

Solve the equation

$x^6 - x^3 = 56$

$x =$ ....

or $x =$ ....

Question 3

Skill involved: 485a: Solve a hidden quadratic equation given in the form x^n by substitution.

Solve the equation

$4z^6 + 8z^3 = 45$

$z =$ ....

or $z =$ ....

Question 4

Skill involved: 485a: Solve a hidden quadratic equation given in the form x^n by substitution.

Solve the equation

$2x^6 - 9x^3 = -7$

$x =$ ....

or $x =$ ....

Question 5

Skill involved: 485a: Solve a hidden quadratic equation given in the form x^n by substitution.

Solve the equation

$z^6 + 9z^3 + 18 = 0$

$z =$ ....

or $z =$ ....

Question 6

Skill involved: 485b: Solve a hidden quadratic equation containing roots by substitution.

Solve the equation

$4y + 12\sqrt{y} - 7 = 0$

$y =$ ....

Question 7

Skill involved: 485b: Solve a hidden quadratic equation containing roots by substitution.

Solve the equation

$x - 6\sqrt{x} - 16 = 0$

$x =$ ....

Question 8

Skill involved: 485b: Solve a hidden quadratic equation containing roots by substitution.

Solve the equation

$x + 3\sqrt{x} = 54$

$x =$ ....

Question 9

Skill involved: 485b: Solve a hidden quadratic equation containing roots by substitution.

Exercise

Solve the equation

$4z - 24\sqrt{z} = -27$

$z =$ ....

or $z =$ ....

Question 10

Skill involved: 485b: Solve a hidden quadratic equation containing roots by substitution.

Solve the equation

$2x = 25\sqrt{x} - 72$

$x =$ ....

or $x =$ ....

Question 11

Skill involved: 485c: Solve a hidden quadratic equation given in the form x^n by substitution, that requires multiplying through by a power of x.

Solve the equation

$2x^2 + \frac{36}{x^2} = 17$

$x =$ ....

or $x =$ ....

or $x =$ ....

or $x =$ ....

Question 12

Skill involved: 485c: Solve a hidden quadratic equation given in the form x^n by substitution, that requires multiplying through by a power of x.

Solve the equation

$2x^2 + \frac{24}{x^2} - 19 = 0$

$x =$ ....

or $x =$ ....

or $x =$ ....

or $x =$ ....

Question 13

Skill involved: 485c: Solve a hidden quadratic equation given in the form x^n by substitution, that requires multiplying through by a power of x.

Solve the equation

$y^2 - \frac{54}{y^2} = 3$

$y =$ ....

or $y =$ ....

Question 14

Skill involved: 485c: Solve a hidden quadratic equation given in the form x^n by substitution, that requires multiplying through by a power of x.

Solve the equation

$2x^2 = -\frac{25}{x^2} + 15$

$x =$ ....

or $x =$ ....

or $x =$ ....

or $x =$ ....

Exercise

Question 15

Skill involved: 485c: Solve a hidden quadratic equation given in the form x^n by substitution, that requires multiplying through by a power of x.

Solve the equation

$x^2 - \frac{6}{x^2} = 5$

$x =$ ....

or $x =$ ....

Question 16

Skill involved: 485e: Solve a quadratic equation in terms of an exponential term of the form a^x (no logarithms).

Find the exact solution of

$3^{2x} - 2 \times 3^x - 3 = 0.$

$x =$ ....

Question 17

Skill involved: 485e: Solve a quadratic equation in terms of an exponential term of the form a^x (no logarithms).

Solve the equation

$9 \times 3^{2x-1} - 4 \times 3^x + 1 = 0$

$x =$ ....

or $x =$ ....

Question 18

Skill involved: 485e: Solve a quadratic equation in terms of an exponential term of the form a^x (no logarithms).

Find the exact solution of

$9^x - 80 \times 3^x - 81 = 0.$

$x =$ ....

Question 19

Skill involved: 485e: Solve a quadratic equation in terms of an exponential term of the form a^x (no logarithms).

Solve the equation

$3^{2x+1} - 2 \times 3^x - 1 = 0$

$x =$ ....

Question 20

Skill involved: 485e: Solve a quadratic equation in terms of an exponential term of the form a^x (no logarithms).

Solve the equation

$4 \times 4^x = 3 \times 2^x + 1$

$x =$ ....

2.4) Quadratic graphs

Notes

Worked Example

3680: Determine the line of symmetry of a quadratic graph.

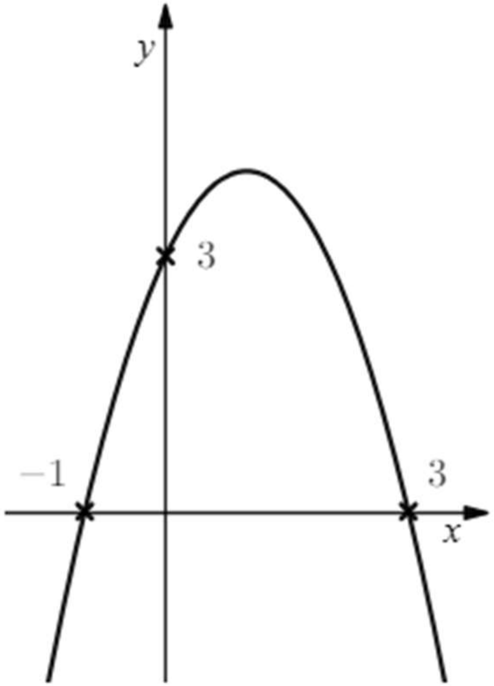
A graph has equation $y = x^2 + 3x - 2$

Find the equation of its line of symmetry.

Worked Example

368p: Determine the coordinate of the turning point of a quadratic using symmetry and a sketch of the graph.

The graph below has equation $y = -x^2 + 2x + 3$.

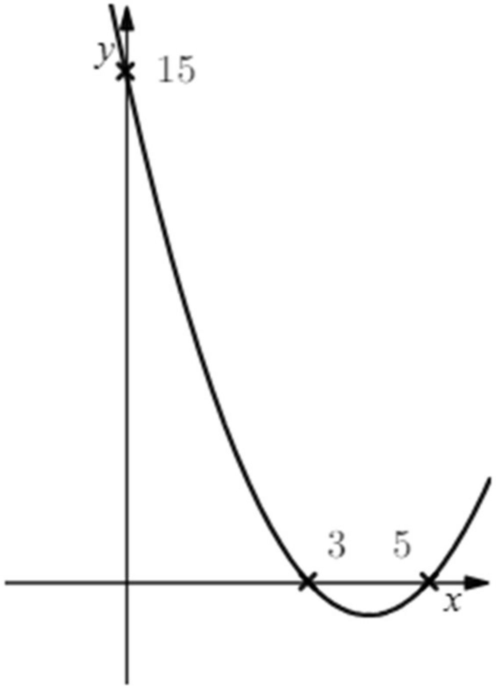


Find the coordinate of the turning point.

Worked Example

368q: Determine the equation of a quadratic function from its graph.

The graph below shows a quadratic curve.

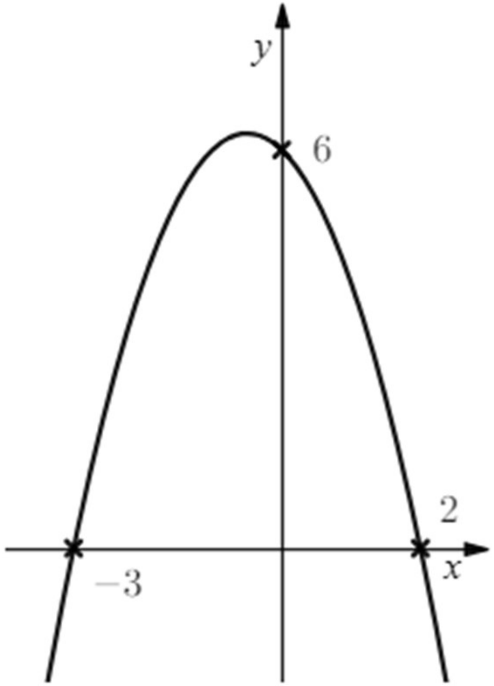


Find the equation of the curve, giving your answer in the form $y = ax^2 + bx + c$
where a, b and c are integers.

Worked Example

368q: Determine the equation of a quadratic function from its graph.

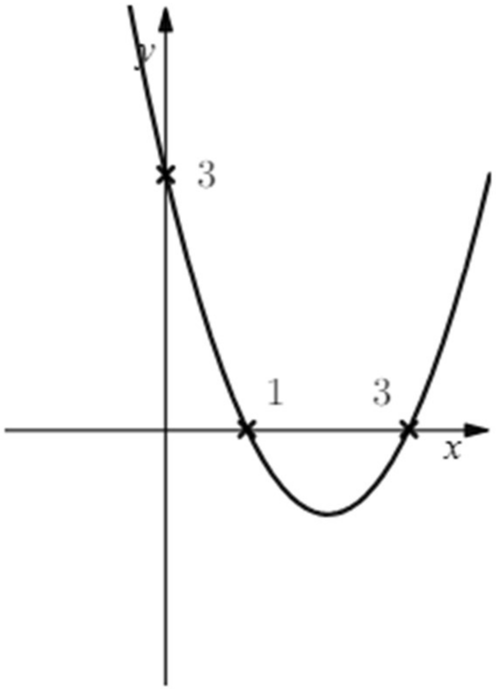
The graph below shows a quadratic curve.



Find the equation of the curve, giving your answer in the form $y = ax^2 + bx + c$
where a, b and c are integers.

368r: Determine an unknown consta in a quadratic function given a point on the curve and a sketch.

The graph below shows a quadratic curve.



The equation of the curve is $y = x^2 + bx + 3$

Find the value of b.

Worked Example

The graph of $y = ax^2 + bx + c$ has a minimum at $(3, -5)$ and passes through $(4, 0)$. Find the values of a, b and c

Worked Example

Sketch $y = x^2 + 6x + 8$, labelling the intercepts with the axes and the turning points.

Worked Example

Sketch $y = x^2 + 6x + 9$, labelling the intercepts with the axes and the turning points.

Worked Example

Sketch $y = x^2 + 6x + 10$, labelling the intercepts with the axes and the turning points.

Worked Example

Sketch $y = x^2 + 6x - 7$, labelling the intercepts with the axes and the turning points.

Worked Example

Sketch $y = x^2 + 6x$, labelling the intercepts with the axes and the turning points.

Worked Example

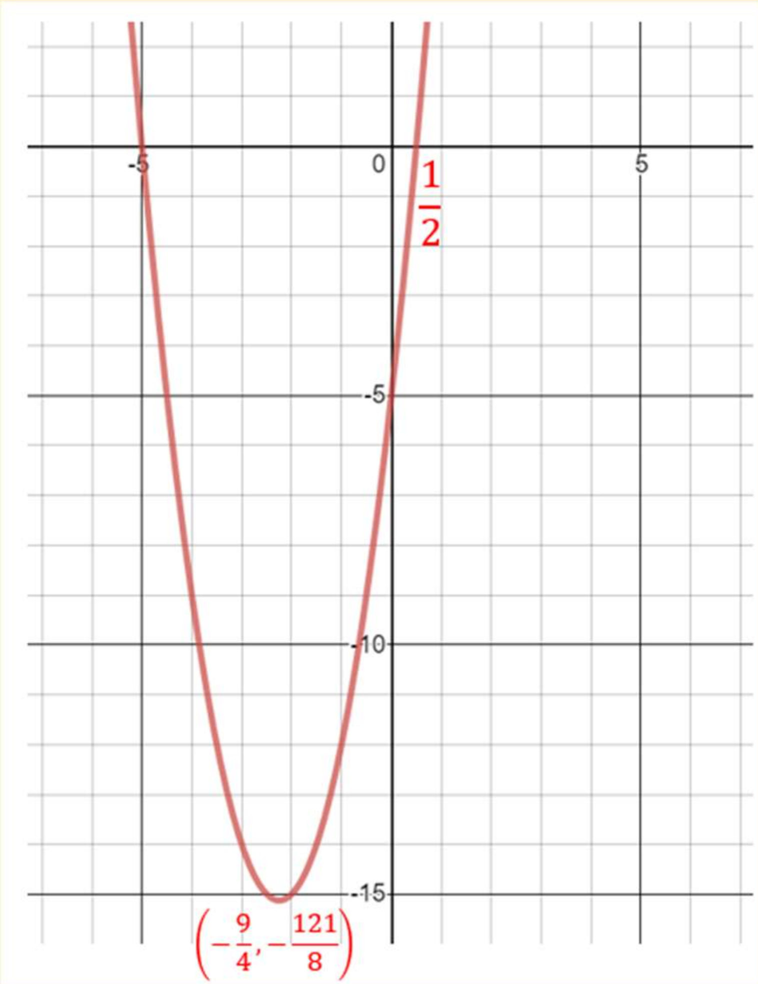
Sketch $y = -x^2 + 3x - 2$, labelling the intercepts with the axes and the turning points.

Worked Example

Sketch $y = 2x^2 + 5x - 3$, labelling the intercepts with the axes and the turning points.

Your Turn

Sketch $y = 2x^2 + 9x - 5$, labelling the intercepts with the axes and the turning points.



Exercise 2F

1 Sketch the graphs of the following equations. For each graph, show the coordinates of the point(s) where the graph crosses the coordinate axes, and write down the coordinate of the turning point and the equation of the line of symmetry.

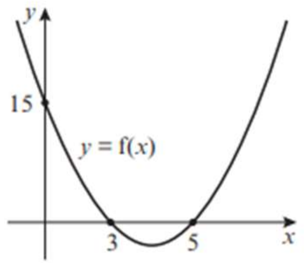
a $y = x^2 - 6x + 8$ **b** $y = x^2 + 2x - 15$ **c** $y = 25 - x^2$ **d** $y = x^2 + 3x + 2$

e $y = -x^2 + 6x + 7$ **f** $y = 2x^2 + 4x + 10$ **g** $y = 2x^2 + 7x - 15$ **h** $y = 6x^2 - 19x + 10$

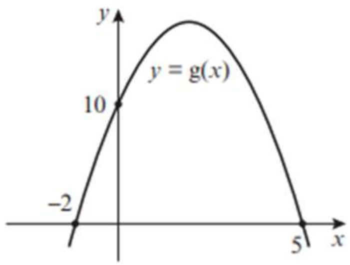
i $y = 4 - 7x - 2x^2$ **j** $y = 0.5x^2 + 0.2x + 0.02$

(P) 2 These sketches are graphs of quadratic functions of the form $ax^2 + bx + c$.
Find the values of a, b and c for each function.

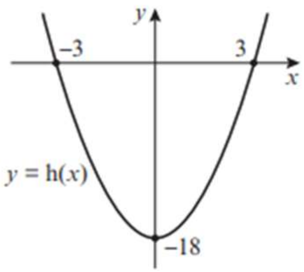
a



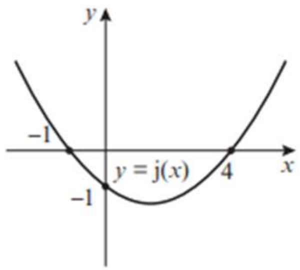
b



c



d



Problem-solving

Check your answers by substituting values into the function. In part **c** the graph passes through (0, −18), so h(0) should be −18.

(E/P) 3 The graph of $y = ax^2 + bx + c$ has a minimum at (5, −3) and passes through (4, 0).
Find the values of a, b and c. **(3 marks)**

Exercise

2.4 Quadratic graphs

1 Sketch graphs of each of the following equations, showing the coordinates of the points where the graph crosses the coordinate axes.

a $y = x^2 + 11x + 18$ **b** $y = 4x^2 - 16$
c $y = -6x^2 + 2x$

Hint Factorise each equation to find the points where $y = 0$. These are the values of x at the points where the graph crosses the x-axis. To find the y-intercept, substitute $x = 0$ into the equation.

2 Find the coordinates of the turning point on each of these graphs:

a $y = (x - 1)^2 + 9$ **b** $y = x^2 + x - 6$
c $y = -x^2 - 13x - 42$

Hint You can find coordinates of the turning point on a quadratic curve by completing the square. The curve with equation $y = (x - a)^2 + b$ will have a turning point at (a, b).

3 Sketch the graphs of the following equations. For each graph, indicate where the graph crosses the coordinate axes, and write down the coordinates of the turning point and the equation of the line of symmetry.

a $y = x^2 - 6x + 20$ **b** $y = -2x^2 - 5x - 2$ **c** $4x^2 - y = 4x + 3$

Hint A quadratic graph has a vertical line of symmetry that passes through its turning point.

4 Sketch the graphs of the following equations. For each graph, indicate where the graph crosses the coordinate axes, leaving your answer in surd form. Write down the coordinates of the turning point and the equation of the line of symmetry.

a $y = x^2 + 7x + 5$ **b** $y = -5x^2 - 12x - 3$ **c** $y = 2x^2 + 7x + 4$

Hint After drawing your axes, write in all the required coordinate points and then draw a smooth curve through these points.

(E) 5 The expression $8x - 7 - x^2$ can be written in the form $q - (x - p)^2$, where p and q are integers.

a Find the value of p and the value of q. **(3 marks)**

b Sketch the curve with equation $y = 8x - 7 - x^2$, showing clearly the coordinates of any points where the curve crosses the coordinate axes. **(3 marks)**

(E) 6 $f(x) = x^2 + 6x + 4, x \in \mathbb{R}$

a Express f(x) in the form $(x + a)^2 + b$, where a and b are constants. **(2 marks)**

The curve C with equation $y = f(x)$ crosses the y-axis at point P and has a minimum point at the point Q.

b Sketch the graph of C, showing the coordinates of points P and Q. **(3 marks)**

c Explain why the equation $f(x) = -6$ has no real solutions. **(1 mark)**

(E/P) 7 $p(x) = 3 - 2x$, $q(x) = x^2 - 9x - 10$, $x \in \mathbb{R}$

a Solve the equation $q(x) = 0$. **(2 marks)**

b Sketch the graphs of $y = p(x)$ and $y = q(x)$ on the same set of axes. Label all points where the curves intersect the coordinate axes. **(4 marks)**

(E/P) 8 The graph of $y = ax^2 + bx + c$ has a minimum at $(2, -5)$ and passes through $(3, 0)$. Find the values of a, b and c. **(4 marks)**

2.5) The discriminant

Notes

Fill in the blanks

Equation	Discriminant	Number of Distinct Real Roots
$x^2 + 3x + 4 = 0$		
$x^2 - 4x + 1 = 0$		
$x^2 - 4x + 4 = 0$		
$2x^2 - 6x - 3 = 0$		
$x - 4 - 3x^2 = 0$		
$1 - x^2 = 0$		

Worked Example

492b: Use the discriminant of a simple quadratic function to determine the number of solutions to an equation.

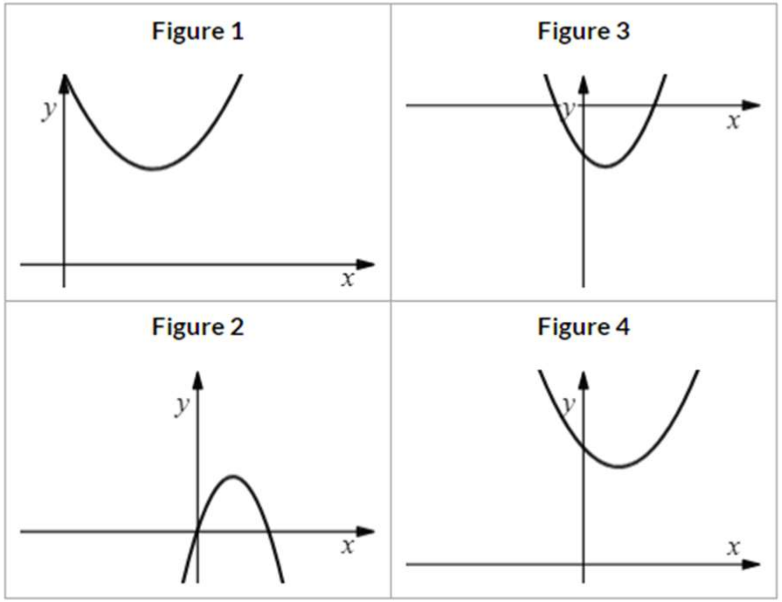
By calculating the discriminant, work out the number of real solutions to the equation $3x - 3x^2 - 2 = 0$

Discriminant =

Number of real solutions =

492c: Understand the relationship between the discriminant of a quadratic function and its sketch.

Calculate the discriminant and, hence, identify which figure represents the graph with equation $y = 4x^2 - 5x - 6$.



Discriminant =

Enter which figure corresponds to the graph:

492d: Determine the discriminant of a quadratic function with algebraic coefficients.

Find the discriminant of $3x^2 + (k+3)x - 3$.

Give your answer in the form $ak^2 + bk + c$, where a, b, and c are constants to be found.

492e: Use the discriminant to reason about the number of solutions of a factorised cubic.

Given that

$$4x^3 - x^2 + 4x - 7$$

can be factorised into the form

$$(x-1)(4x^2 + 3x + 7)$$

Find the discriminant and determine the number of real solutions of

$$4x^3 - x^2 + 4x - 7 = 0$$

Discriminant =

Number of real solutions =

Worked Example

492f: Use the discriminant to determine the values for algebraic coefficients in a quadratic function, where there are equal roots.

The equation $(k+1)x - 3x^2 - 2k - 2 = 0$ has equal roots.

Find the possible values of k.

$k =$ ☐

or $k =$ ☐

Worked Example

492g: Use the discriminant to determine the values for algebraic coefficients in a quadratic function, where there are no roots or distinct roots.

The equation $(3k-3)x-k^2x^2-1=0$ has real roots.

Find the possible range of values of k.

Worked Example

492h: Use the discriminant to determine the values for algebraic coefficients in a quadratic function, requiring rearrangement.

The equation $3x^2 - x - 1 = 2kx + 2k$ has real roots.

Find the possible values of k.

Worked Example

492i: Use the discriminant to determine when a straight line is tangent to a quadratic graph.

The straight line with equation $y = 9x - 6p$ touches the curve with equation $y = 3x^2 - 9px + 10$, where p is a constant.

Find the set of possible values of p.

$p =$ [ ]

or $p =$ [ ]

Exercises

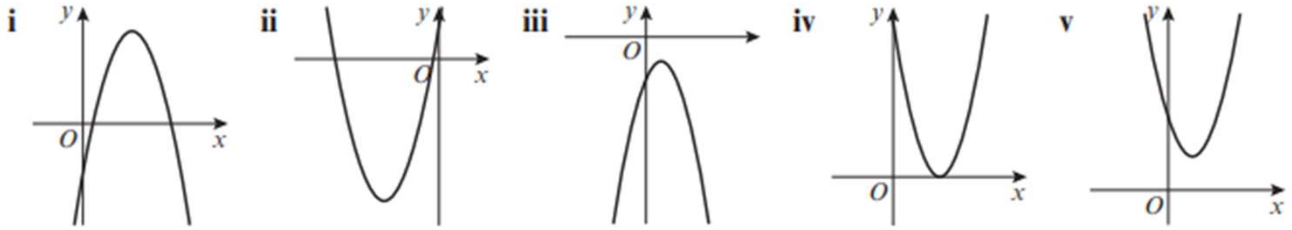
Exercise 2G

1 a Calculate the value of the discriminant for each of these five functions:

i $f(x) = x^2 + 8x + 3$ ii $g(x) = 2x^2 - 3x + 4$ iii $h(x) = -x^2 + 7x - 3$

iv $j(x) = x^2 - 8x + 16$ v $k(x) = 2x - 3x^2 - 4$

b Using your answers to part **a**, match the same five functions to these sketch graphs.



E/P 2 Find the values of k for which $x^2 + 6x + k = 0$ has two real solutions. **(2 marks)**

E/P 3 Find the value of t for which $2x^2 - 3x + t = 0$ has exactly one solution. **(2 marks)**

E/P 4 Given that the function $f(x) = sx^2 + 8x + s$ has equal roots, find the value of the positive constant s. **(2 marks)**

E/P 5 Find the range of values of k for which $3x^2 - 4x + k = 0$ has no real solutions. **(2 marks)**

E/P 6 The function $g(x) = x^2 + 3px + (14p - 3)$, where p is an integer, has two equal roots.

a Find the value of p. **(2 marks)**

b For this value of p, solve the equation $x^2 + 3px + (14p - 3) = 0$. **(2 marks)**

E/P 7 $h(x) = 2x^2 + (k + 4)x + k$, where k is a real constant.

a Find the discriminant of $h(x)$ in terms of k. **(3 marks)**

b Hence or otherwise, prove that $h(x)$ has two distinct real roots for all values of k. **(3 marks)**

Problem-solving

If a question part says 'hence or otherwise' it is usually easier to use your answer to the previous question part.

2.5 The discriminant

1 State the condition for which the function $f(x) = ax^2 + bx + c$ has:

a 2 distinct, real roots **b** 1 repeated root

c no real roots

Hint The discriminant of the quadratic function $f(x) = ax^2 + bx + c$ is $b^2 - 4ac$.

2 For each of these functions:

i calculate the value of the discriminant

ii write down the number of real roots of the function.

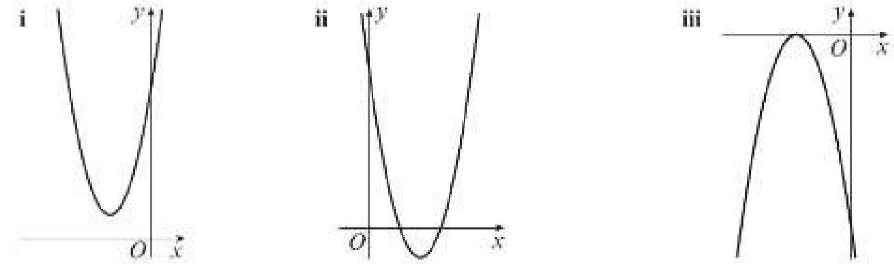
a $f(x) = -2x^2 - 11x - 12$ **b** $f(x) = x^2 + 6x + 9$ **c** $f(x) = 3x^2 - 12x + 18$

Hint Be careful with negative signs. In part **a**, $a = -2$, $b = -11$ and $c = -12$.

3 Calculate the value of the discriminant for each of these functions and match them to the sketch graphs.

Hint A quadratic graph with one repeated real root will have its turning point on the x-axis.

a $f(x) = x^2 - 10x + 21$ **b** $f(x) = -x^2 - 10x - 25$ **c** $f(x) = x^2 + 8x + 19$



P **4** $f(x) = x^2 + kx + 25$, $x \in \mathbb{R}$

a Find the discriminant of $f(x)$ in terms of k. **(2 marks)**

b Given that the equation $f(x) = 0$ has one repeated root, find the possible values of k. **(2 marks)**

P **5** $f(x) = x^2 + (k + 3)x + k$, where k is a real constant and $x \in \mathbb{R}$.

a Find the discriminant of $f(x)$ in terms of k. **(2 marks)**

b Show that the discriminant of $f(x)$ can be expressed in the form $(k + a)^2 + b$, where a and b are constants to be found. **(2 marks)**

c Show that, for all values of k, the equation $f(x) = 0$ has distinct real roots. **(2 marks)**

P **6** The equation $kx^2 + 3x - 5 = 0$, where k is a constant, has two distinct real roots. Find the range of possible values of k. **(3 marks)**

P **7** Find the range of values of p for which the equation $2x^2 + 7x + p = 0$ has no real solutions. **(3 marks)**

2.6) Modelling with quadratics

Notes

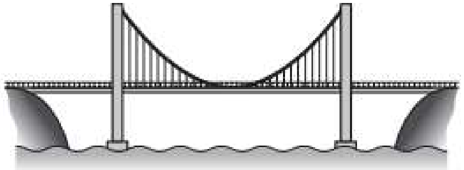
2.6) Modelling with quadratics

A spear is thrown over level ground from the top of a tower.
The height, in metres, of the spear above the ground after t seconds is modelled by the function: $h(t) = 1.65 + 24.5t - 4.9t^2$, $t \geq 0$

a) Interpret the meaning of the constant term 12.25 in the model.
b) After how many seconds does the spear hit the ground?
c) Write $h(t)$ in the form $A - B(t - C)^2$, where A, B and C are constants to be found.
d) Using your answer to part c or otherwise, find the maximum height of the spear above the ground, and the time at which this maximum height is reached?

Exercise 2H

E/P **1** The diagram shows a section of a suspension bridge carrying a road over water.



Problem-solving

For part **a**, make sure your answer is in the context of the model.

The height of the cables above water level in metres can be modelled by the function $h(x) = 0.000\,12x^2 + 200$, where x is the displacement in metres from the centre of the bridge.

a Interpret the meaning of the constant term 200 in the model. **(1 mark)**

b Use the model to find the two values of x at which the height is 346 m. **(3 marks)**

c Given that the towers at each end are 346 m tall, use your answer to part **b** to calculate the length of the bridge to the nearest metre. **(1 mark)**

E/P **2** A car manufacturer uses a model to predict the fuel consumption, y miles per gallon (mpg), for a specific model of car travelling at a speed of x mph.

$$y = -0.01x^2 + 0.975x + 16, \; x > 0$$

a Use the model to find two speeds at which the car has a fuel consumption of 32.5 mpg. **(3 marks)**

b Rewrite y in the form $A - B(x - C)^2$, where A, B and C are constants to be found. **(3 marks)**

c Using your answer to part **b**, find the speed at which the car has the greatest fuel efficiency. **(1 mark)**

d Use the model to calculate the fuel consumption of a car travelling at 120 mph. Comment on the validity of using this model for very high speeds. **(2 marks)**

E/P **3** A fertiliser company uses a model to determine how the amount of fertiliser used, f kilograms per hectare, affects the grain yield g, measured in tonnes per hectare.

$$g = 6 + 0.03f - 0.000\,06f^2$$

a According to the model, how much grain would each hectare yield without any fertiliser? **(1 mark)**

b One farmer currently uses 20 kilograms of fertiliser per hectare. How much more fertiliser would he need to use to increase his grain yield by 1 tonne per hectare? **(4 marks)**

E/P **4** A football stadium has 25 000 seats. The football club know from past experience that they will sell only 10 000 tickets if each ticket costs £30. They also expect to sell 1000 more tickets every time the price goes down by £1.

a The number of tickets sold t can be modelled by the linear equation $t = M - 1000p$, where £p is the price of each ticket and M is a constant. Find the value of M. **(1 mark)**

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Quadratics

The total revenue, £r, can be calculated by multiplying the number of tickets sold by the price of each ticket. This can be written as $r = p(M - 1000p)$.

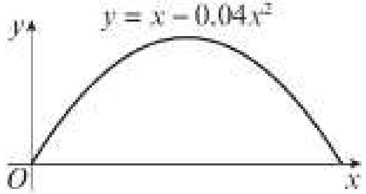
b Rearrange r into the form $A - B(p - C)^2$, where A, B and C are constants to be found. **(3 marks)**

c Using your answer to part **b** or otherwise, work out how much the football club should charge for each ticket if they want to make the maximum amount of money. **(2 marks)**

Exercises

2.6 Modelling with quadratics

(E) 1 A ball is kicked through the air from a level playing field. The path of the ball can be modelled by the equation $y = x - 0.04x^2$, where x is the horizontal distance (in metres) and y is the vertical height (in metres) from the point where it was kicked.



a Find the horizontal distance travelled by the ball from the point where it was kicked to the point where it lands on the playing field again. **(2 marks)**

b Show that the equation for the path of the ball can be written in the form $y = a(x - b)^2 + c$, where a, b and c are constants to be found. **(3 marks)**

c Using your answer to part **b**, or otherwise, find the maximum height the ball reaches above the playing field, and the horizontal distance at which this maximum height is reached. **(2 marks)**

Hint The maximum height of the ball will occur at the turning point of the path.

(E) **2** A company makes a particular type of phone case.
The annual profit made by the company is modelled by the equation

$$P = 73.5x - 5.25x^2 - 107.5$$

where P is the annual profit, measured in thousands of pounds, and x is the selling price of the phone case, in pounds. The company wishes to maximise its annual profit.

a Show that this model can be written in the form $P = A - 5.25(x - B)^2$, where A and B are constants to be determined. **(2 marks)**

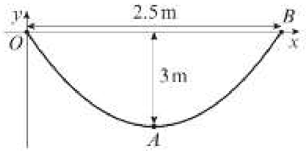
Hint For part **a** you need to complete the square.

b Hence, or otherwise, state, according to the model:

i the maximum possible annual profit

ii the selling price of the phone case that maximises the annual profit. **(2 marks)**

(E/P) **3** The diagram shows the cross-section of a canal. O and B are points on the surface of the water, with $OB = 2.5$ m. At point A the canal is 3 m deep. The cross-section is modelled as a quadratic curve OAB. x and y are the horizontal and vertical distances from O in metres.



a i Write down the coordinates of point A.

ii Hence find an equation for the curve OAB. **(3 marks)**

The cross-section of a narrowboat is modelled as a rectangle of width 2.1 m, with constant depth below water of 1 m.

b Determine whether or not it is possible for the narrowboat to pass through the canal. **(4 marks)**

(E/P) **4** A car manufacturing company uses a model to determine how the number of workers employed, w, affects the number of cars produced per day, q.
The model suggests $q = 0.4w - 0.00008w^2$.

a Suggest a reason why this model does not contain a constant term. **(1 mark)**

On a particular day, 1500 workers are employed.

b Find the number of cars produced on this day. **(1 mark)**

c According to the model, what is the maximum number of cars that can be produced and how many workers are needed for this? **(3 marks)**

(E/P) **5** A ball is thrown from the top of a cliff.
The height, h metres, of the ball above ground level after t seconds is modelled by the function $h(t) = 120 + 12.25t - 4.9t^2$

a Give a physical interpretation of the meaning of the constant term 120 in the model. **(1 mark)**

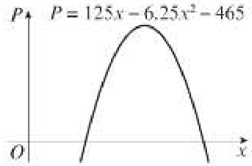
b Write $h(t)$ in the form $A - B(t - C)^2$, where A, B and C are constants to be found. **(3 marks)**

c Using your answer to part **b**, or otherwise, find, with justification:

i the time between the instant the ball is thrown and the instant it reaches ground level

ii the maximum height of the ball above the ground and the time at which this maximum height is reached. **(5 marks)**

(E/P) **6** A company makes a particular type of T-shirt.
The annual profit made by the company is modelled by the equation $P = 125x - 6.25x^2 - 465$, where P is the profit measured in thousands of pounds and x is the selling price of the T-shirt in pounds.
A sketch of P against x is shown in the diagram.



Ellie claims that this model is not valid for a selling price of £4.00, because the value of P is negative.

a Suggest a reason why Ellie may be incorrect. **(1 mark)**

b Write the model in the form $P = c - a(x - b)^2$, where a, b and c are constants to be found. **(3 marks)**

c Given that the company made an annual profit of more than £120 000 find, according to the model, the least possible selling price for the T-shirt. **(3 marks)**

The company wishes to maximise its annual profit.

d State, according to the model:

i the maximum possible annual profit

ii the selling price of the T-shirt that maximises the annual profit. **(2 marks)**

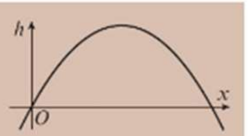
(P) **7** In microeconomics, the market price and quantity of an item are found by relating two functions.
Inverse supply expresses supply price £P_s in terms of the quantity supplied, q thousands.
Inverse demand expresses demand price £P_d in terms of the quantity demanded, q thousands.
Market equilibrium occurs when $P_s = P_d$
In this model:
Inverse supply is $P_s = 8q + 8$
Inverse demand is $P_d = q^2 - 14q + 48 \quad (0 \leqslant q \leqslant 6)$
Find the market equilibrium quantity and price given by this model.

Exercises

Problem solving Set A

Bronze

The sketch shows the path of a stone that is kicked through the air from level ground.



The path of the stone can be modelled by the function $h(x) = 2x - x^2$, where x metres is the horizontal distance the stone travels from the place where it was kicked, and h metres is the vertical height of the stone above ground level.

a Write $h(x)$ in the form $h(x) = A - (x - B)^2$, where A and B are constants to be found. **(3 marks)**

b Using your answer to part **a**, or otherwise, solve $h(x) = 0$ and find the horizontal distance the stone has travelled when it lands on the ground. **(3 marks)**

Silver

A stone is thrown from the top of a cliff.

The path of the stone can be modelled by the function $h(x) = 114 + 10.4x - 5.2x^2$, where x metres is the horizontal distance the stone travels, and h metres is the vertical height of the stone above ground level.

a Give a physical interpretation of the meaning of the constant term 114 in the model. **(1 mark)**

b **i** Show that $h(x)$ can be rearranged to give $h(x) = 114 - 5.2(x^2 - 2x)$.

ii Hence, or otherwise, write $h(x)$ in the form $h(x) = A - 5.2(x - B)^2$, where A and B are constants to be found. **(3 marks)**

c Using your answer to part **b ii**, or otherwise, find, with justification:

i the horizontal distance the stone has travelled when it lands on the ground

ii the maximum height of the stone above the ground and the horizontal distance at which this maximum height is reached. **(5 marks)**

Gold

A stone is thrown from the top of a cliff.

The path of the stone can be modelled by the function $h(x) = 125 + 12.75x - 4.5x^2$, where x metres is the horizontal distance the stone travels, and h metres is the vertical height of the stone above ground level.

a Give a physical interpretation of the meaning of the constant term 125 in the model. **(1 mark)**

b Find, with justification, the maximum height of the stone above the ground. **(3 marks)**

c If measured in a straight line, what is the distance from the point where the stone is thrown to the point where it lands on the ground? **(2 marks)**

Problem solving Set B

Bronze

The equation $2kx^2 + 4x + k = 0$, where k is a constant, has one repeated root.

a Show that $16 - 8k^2 = 0$. **(2 marks)**

b Hence, find two possible values of k. **(1 mark)**

Silver

The equation $3x^2 + px + 2p = 0$, where p is a non-zero constant, has equal roots.

Find the value of p. **(3 marks)**

Gold

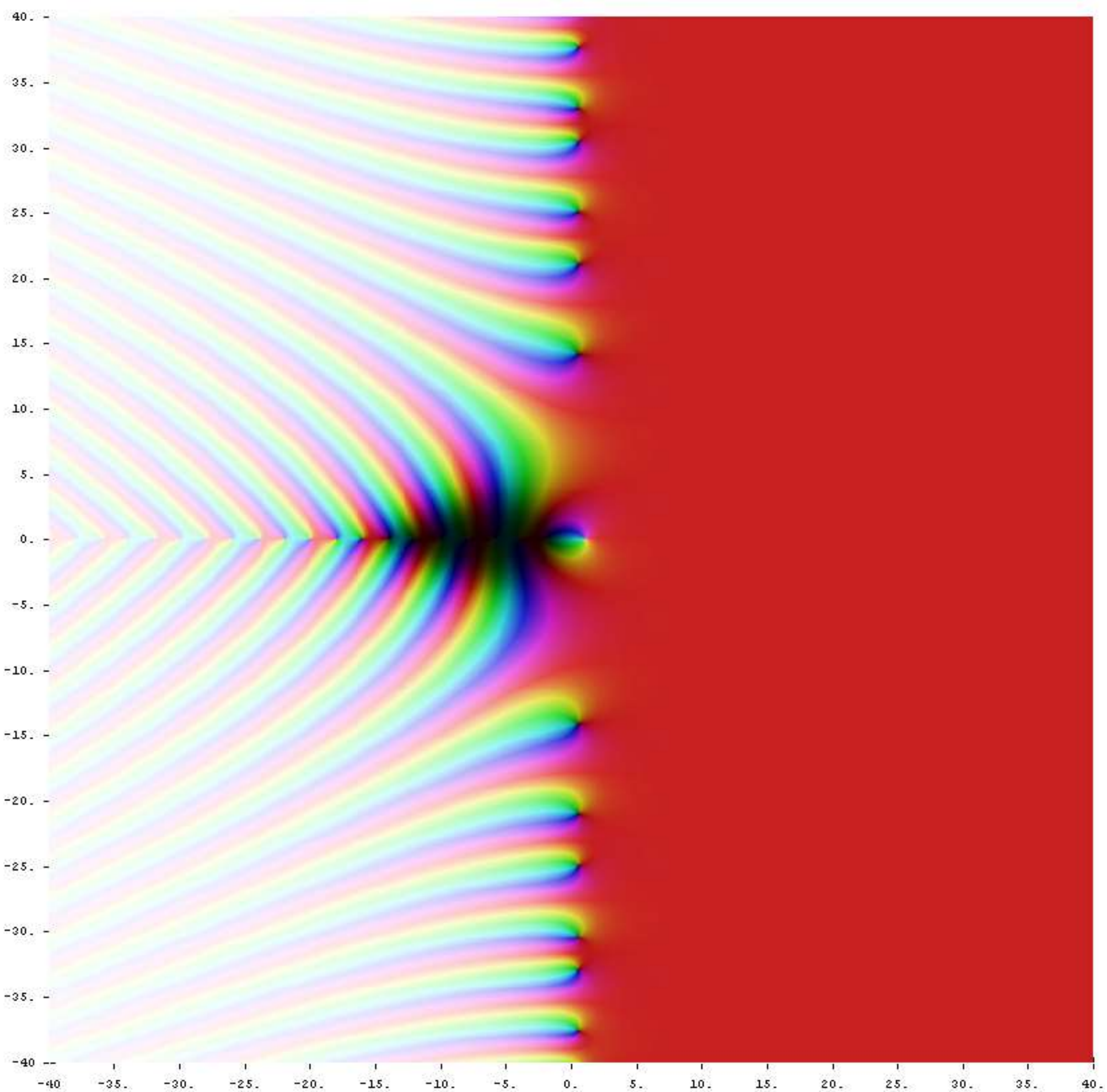
The equation $\frac{3 - x^2}{x + 2} = q$, where q is a constant, has one repeated real root.

Find two possible values for q. **(4 marks)**

Now try this → Exam question bank Q38, Q43, Q52, Q67, Q91

Would you like $1,000,000 for finding roots?

We saw earlier that the roots of a function f are the values x such that $f(x) = 0$.



The **Riemann Zeta Function** is a function that allows you to do the infinite sum of powers of reciprocals, e.g.

$$\zeta(2) = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$$
$$\zeta(3) = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \cdots$$

One of the 8 '**Clay Millennium Problems**' (for which solving any attracts a $1,000,000 prize) is to **showing all roots of this function have some particular form**, i.e. the form of x such that $\zeta(x) = 0$.

Summary of Key Points

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1 To solve a quadratic equation by factorising:

- Write the equation in the form $ax^2 + bx + c = 0$
- Factorise the left-hand side
- Set each factor equal to zero and solve to find the value(s) of x

2 The solutions of the equation $ax^2 + bx + c = 0$ where $a \neq 0$ are given by the formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

3 $x^2 + bx = \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2$

4 $ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 + \left(c - \frac{b^2}{4a^2}\right)$

5 The set of possible inputs for a function is called the **domain**.

The set of possible outputs of a function is called the **range**.

6 The **roots** of a function are the values of x for which $f(x) = 0$.

7 You can find the coordinates of a **turning point** of a quadratic graph by completing the square. If $f(x) = a(x + p)^2 + q$, the graph of $y = f(x)$ has a turning point at $(-p, q)$.

8 For the quadratic function $f(x) = ax^2 + bx + c = 0$, the expression $b^2 - 4ac$ is called the **discriminant**. The value of the discriminant shows how many roots $f(x)$ has:

- If $b^2 - 4ac > 0$ then a quadratic function has two distinct real roots.
- If $b^2 - 4ac = 0$ then a quadratic function has one repeated real root.
- If $b^2 - 4ac < 0$ then a quadratic function has no real roots

9 Quadratics can be used to model real-life situations.