KING EDWARD VI
HANDSWORTH GRAMMAR
SCHOOL FOR BOYS

KING EDWARD VI
ACADEMY TRUST
BIRMINGHAM

Year 12
Pure Mathematics
P1 4 Graphs and Transformations
Booklet

HGS Maths

Dr Frost Course

Name: ________________________________

Class: ________________________________

Contents

Prior knowledge check

Prior knowledge check

1 Factorise these quadratic expressions:

a $x^2 + 6x + 5$ **b** $x^2 - 4x + 3$

← GCSE Mathematics

2 Sketch the graphs of the following functions:

a $y = (x + 2)(x - 3)$ **b** $y = x^2 - 6x - 7$

← Section 2.4

3 **a** Copy and complete the table of values for the function $y = x^3 + x - 2$.

x	-2	-1.5	-1	-0.5	0	0.5	1	1.5	2
y	-12	-6.875			-2	-1.375			

b Use your table of values to draw the graph of $y = x^3 + x - 2$.

← GCSE Mathematics

4 Solve each pair of simultaneous equations:

a $y = 2x$
$x + y = 6$

b $y = x^2$
$y = 2x - 1$

← Sections 3.1, 3.2

4.1) Cubic graphs

Polynomial Graphs

In Chapter 2 we briefly saw that a **polynomial** expression is of the form:

$$a + bx + cx^2 + dx^3 + ex^3 + \cdots$$

where $a, b, c, d, e, \ldots$ are constants (which could be 0).

The **order** of a polynomial is its highest power.

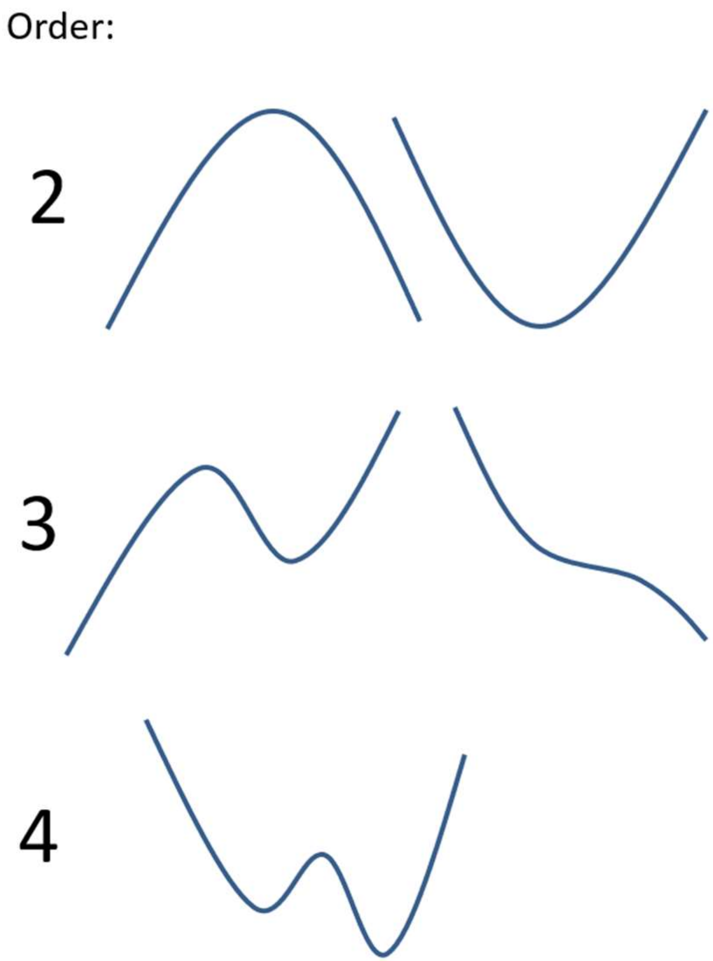
Order	Name
0	Constant (e.g. "4")
1	Linear (e.g. "$2x - 1$")
2	Quadratic (e.g. "$x^2 + 3$")
3	Cubic
4	Quartic
5	Quintic

These are covered in Chapter 5.

Chapter 2 explored the graphs for these.

We will cover these now.

While these are technically beyond the A Level syllabus, we will look at how to sketch polynomials in general.



What property connects the order of the polynomial and the shape?

The number of 'turns' is one less than the order, e.g. a cubic has 2 'turns', a quartic 3 'turns'.

Bro Note: ...Actually this is not strictly true, e.g. consider $y = x^4$, which has a U shape. But this is because multiple turns are being squashed into a single point.

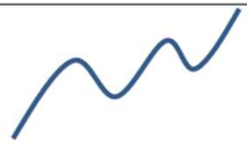
In Chapter 2 how did we tell what way up a quadratic is, and why does this work?

For a quadratic $y = ax^2 + bx + c$, i.f. $a > 0$, we had a 'valley' shape. This is because if x was a large positive value, ax^2 would be large and positive, thus the graph's y value tends towards infinity.

We would write:
"As $x \to \infty, y \to \infty$" where "$\to$" means "tends towards".

Summary

e.g. If $y = 2x^2 + 3$, try a large positive value like $x = 1000$. We can see we'd get a large positive y value. Thus as $x \to \infty$, $y \to \infty$

Equation	If $a > 0$	Resulting Shape	If $a < 0$	Resulting Shape
$y = ax^2 + bx + c$	As $x \to \infty, y \to \infty$ As $x \to -\infty, y \to \infty$		As $x \to \infty, y \to -\infty$ As $x \to -\infty, y \to -\infty$	
$y = ax^3 + bx^2 + cx + d$	As $x \to \infty, y \to \infty$ As $x \to -\infty, y \to -\infty$		As $x \to \infty, y \to -\infty$ As $x \to -\infty, y \to \infty$	
$y = ax^4 + bx^3 + cx^2 + dx + e$	As $x \to \infty, y \to \infty$ As $x \to -\infty, y \to \infty$		As $x \to \infty, y \to -\infty$ As $x \to -\infty, y \to -\infty$	
$y = ax^5 + bx^4 + \cdots$	As $x \to \infty, y \to \infty$ As $x \to -\infty, y \to -\infty$		As $x \to \infty, y \to -\infty$ As $x \to -\infty, y \to \infty$	

If $a > 0$, what therefore can we say about the shape if:

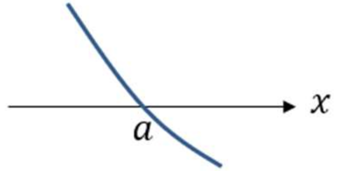
- **The order is odd**: It goes uphill (from left to right)
- **The order is even**: The tails go upwards.

(And we have the opposite if $a < 0$)

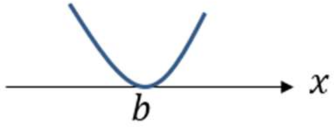
Important notes

If we sketched $y = (x-a)(x-b)^2(x-c)^3$ what happens on the x-axis at:

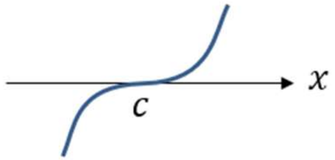
$x = a$: The line **crosses** the axis.



$x = b$: The line **touches** the axis.



$x = c$: **Point of inflection** on the axis.



Notes

Worked Example

Sketch the graph of:

$$y = (x + 1)(x + 2)(x + 3)$$

$$y = (x + 1)(x - 2)(x + 3)$$

Worked Example

Sketch the graph of:

$$y = (x + 1)(x - 2)(3 - x)$$

$$y = (x - 1)(x - 2)(3 - x)$$

Worked Example

Sketch the graph of:

$$y = x(x+3)(x+4)$$

Sketch the graph of:

$$y = (x+2)^2(x-2)$$

Worked Example

Sketch the graph of:

$$y = x^3 - 4x^2 - 5x$$

Sketch the graph of:

$$y = (x + 4)^3$$

Worked Example

Sketch the graph of:

$$y = -(x + 4)^3$$

Sketch the graph of:

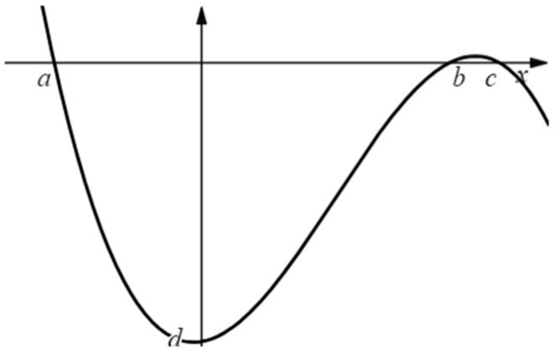
$$y = (4 - x)^3$$

T.62 4A: Qs 1-4, P.26 4.1 Qs 1-3

Worked Example

494b: Identify the intercepts of a cubic with the coordinate axes, where the equation is factorised.

The graph of $y = (x-5)(6-x)(x+3)$ is sketched below.



a, b, c and d are the intercepts between the curve and the axes.

Find the values of a, b, c and d.

Worked Example

A curve is a positive cubic, touches the x-axis at 3 and crosses the x-axis at -2. Write a possible equation for the curve.

T.62 4A: Qs 5+, P.26 4.1 Qs 4+

4.2) Quartic graphs

Notes

Worked Example

Sketch the graph of:

$$y = (x + 3)(x + 4)(x - 3)(x - 4)$$

Sketch the graph of:

$$y = x(x - 3)^2(2 - x)$$

Worked Example

Sketch the graph of:

$$y = (x + 2)^2(x - 4)^2$$

Sketch the graph of:

$$y = x(x - 4)(x + 5)(x + 6)$$

Worked Example

Sketch the graph of:

$$y = (x+4)^2(x-5)(6-x)$$

Sketch the graph of:

$$y = (x-2)(x+2)^3$$

Worked Example

Sketch the graph of:

$$y = (x + 3)^4$$

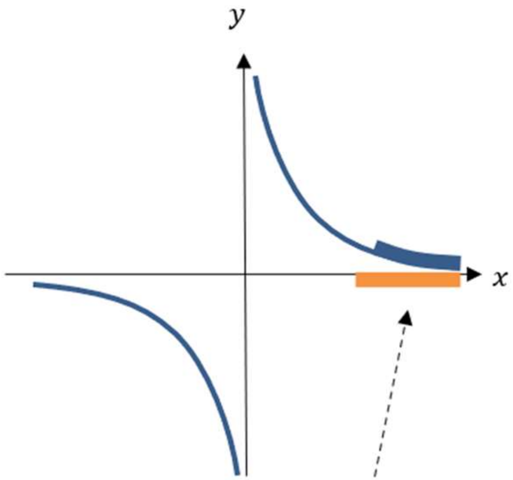
Sketch the graph of:

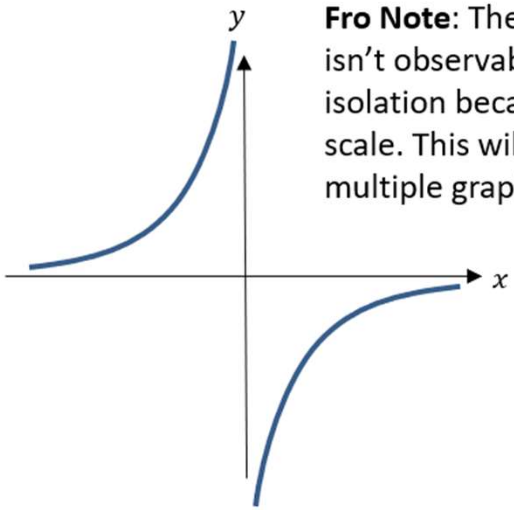
$$y = -(x - 4)(x + 2)^3$$

4.3) Reciprocal graphs

Sketch $y = \frac{1}{x}$

Sketch $y = -\frac{3}{x}$





Fro Note: The scaling caused by the 3 isn't observable for this graph in isolation because the axes have no scale. This will only be observation for multiple graphs on the same axes.

Notice the distance between this line and the x-axis (i.e. the line $y = 0$) gradually decreases as the lines go off towards infinity. The line $y = 0$ is known as an **asymptote** of the graph.

✏ An asymptote is a line which the graph approaches but never reaches.

Asymptotes of $y = \frac{a}{x}$:

$y = 0$,
$x = 0$

T.67 4C: all Qs, P.28 4.3: all Qs

Notes

Worked Example

493a: Identify the x and y intercepts of a more general reciprocal function.

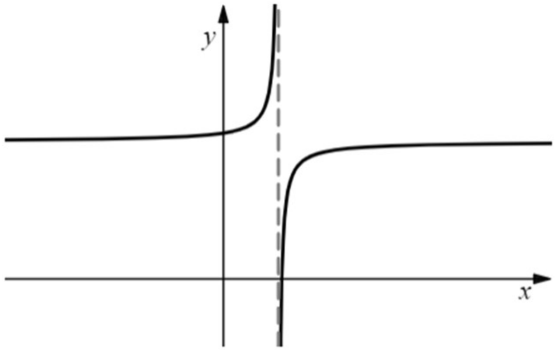
Determine the coordinates of the point where the curve with equation $y = 2 - \dfrac{5}{2x+3}$ crosses the y-axis.

(☐ , ☐)

Worked Example

493b: **Identify the equation of the vertical asymptote of a more general reciprocal function.**

The graph of $y = 4 - \dfrac{1}{5x-4}$ is shown below:

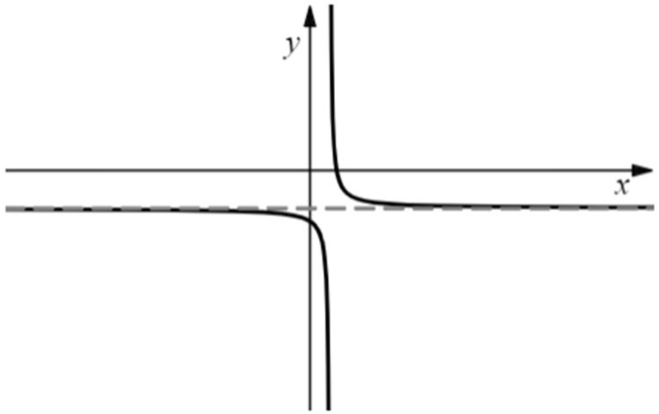


Find the equation of the vertical asymptote.

Worked Example

493c: **Identify the equation of the horizontal asymptote of a more general reciprocal function.**

The graph of $y = \dfrac{3x-1}{1-4x}$ is shown below:



Find the equation of the horizontal asymptote.

Worked Example

493g: Determine coefficients in a mo general reciprocal function that lead to a given x or y intercept.

A curve has equation

$$y = \frac{ax - b}{cx + d} \qquad x \neq -\frac{d}{c}$$

where a, b and d are positive prime numbers and c is a positive integer.

The curve crosses the x-axis at the point $\left(\frac{5}{7}, 0\right)$

Find the values of a and b.

Only on DFM

Worked Example

493h: Determine coefficients in a more general reciprocal function that lead to a given vertical or horizontal asymptote.

A curve has equation

$$y = \frac{ax+b}{c-dx} \qquad x \neq \frac{c}{d}$$

where a, c and d are positive prime numbers and b is a positive integer.

An asymptote to the curve has equation $y = -\frac{5}{2}$

Find the values of a and d.

Only on DFM

4.4) Points of intersection

Notes

Worked Example

On the same diagram sketch the curves with equations $y = -x^2(5x - a)$ and $y = -\frac{b}{x}$, where a, b are positive constants.

State, giving a reason, the number of real solutions to the equation $x^2(5x - a) + \frac{b}{x} = 0$

Worked Example

On the same diagram sketch the curves with equations $y = \frac{3}{x^2}$ and $y = x^2(x-4)$.
State, giving a reason, the number of real solutions to the equation $x^4(x-4) - 3 = 0$

T.69 4D: Qs1-2, P.29 4.4: Qs 2-3

Worked Example

On the same diagram sketch the curves with equations $y = x(x-5)$ and $y = x(x-3)^2$, and hence find the coordinates of any points of intersection.

Worked Example

Work out the range of values of a such that the graphs of $y = x^2 + a$ and $3y = x - 2$ have two points of intersection

T.69 4D: Qs3+, P.29 4.4: Qs 1, 4+

4.5) Translating graphs

This is all you need to remember when considering how transforming your function transforms your graph...

	Affects which axis?	What we expect or opposite?
Change **inside** $f(\,)$	x	Opposite
Change **outside** $f(\,)$	y	Direct

Therefore...

$y = f(x-3)$ ➡ Translation by $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$

$y = f(x)+4$ ➡ Translation by $\begin{pmatrix} 0 \\ 4 \end{pmatrix}$

Effect of transformation on specific points

$y = f(x)$	$(4, 3)$	$(1, 0)$	$(6, -4)$
$y = f(x+1)$	$(3,3)$	$(0,0)$	$(5, -4)$
$y = f(x) - 1$	$(4,2)$	$(1, -1)$	$(6, -5)$

T.74 4E: Qs 1,2, P.30 4.5: Qs 1,2

Notes

Worked Example

448h: **Determine the new equation of a function after a translation by $\begin{pmatrix} a \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 0 \\ a \end{pmatrix}$**

The curve $y = x^2 - x - 5$ is translated by $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$.

State the equation of the new curve after this transformation.

Worked Example

The point with coordinates $(-1.5, 0)$ lies on the curve with equation

$$y = (x + a)^3 + 6(x + a)^2 + 9(x + a)$$

where a is a constant. Find the two possible values of a

Worked Example

Sketch $y = x(x-3)$. On the same axes, sketch $y = (x+a)(x+a-3)$, where $a > 3$.

T.74 4E: Qs 3+, P.30 4.5: Qs 3+

4.6) Stretching graphs

This is all you need to remember when considering how transforming your function transforms your graph...

	Affects which axis?	What we expect or opposite?
Change **inside** $f(\,)$	x	Opposite
Change **outside** $f(\,)$	y	Direct

Therefore...

$y = f(5x)$ ➡ Stretch in x-direction by scale factor $\frac{1}{5}$

$y = 2f(x)$ ➡ Stretch in y-direction by scale factor 2

$\boldsymbol{y = f(x)}$	$\boldsymbol{(4,3)}$	$\boldsymbol{(1,0)}$	$\boldsymbol{(6,-4)}$
$y = f(2x)$	$(2,3)$	$(0.5,0)$	$(3,-4)$
$y = 3f(x)$	$(4,9)$	$(1,0)$	$(6,-12)$
$y = f(-x)$	$(-4,3)$	$(-1,0)$	$(-6,-4)$
$y = -f(x)$	$(4,-3)$	$(1,0)$	$(6,4)$

Notes

Notes

Worked Example

Sketch $y = x^2(x+8)$. On the same axes, sketch the graph with equation

$$y = (4x)^2(4x+8)$$

If $y = (x+2)(x-1)$, sketch $y = f(x)$ and $y = f(\frac{x}{4})$ on the same axes.

Worked Example

On the same axes, sketch:

$$y = x(x+2)(x-1) \quad y = 4x(4x+2)(4x-1) \quad y = -x(x+2)(x-1)$$

T.78 4F: Qs 1-3, P.32 4.6: Qs 1-3

Worked Example

450e: **Understand the effect on a point under the transformation $y = af(x)$**

The curve with equation $y = f(x)$ has the maximum point $P(8, -14)$.

Find the image of P on the curve with equation $y = \frac{1}{2}f(x)$

Worked Example

450f: **Understand the effect on a point under the transformation $y = f(ax)$**

The point $P\,(6, -3)$ lies on the curve with equation $y = f\,(x)$.

Find the image of P on the curve with equation $y = f\,(3x)$

Worked Example KS 450i

The graph of $y = f(x)$ is shown in Figure 1.

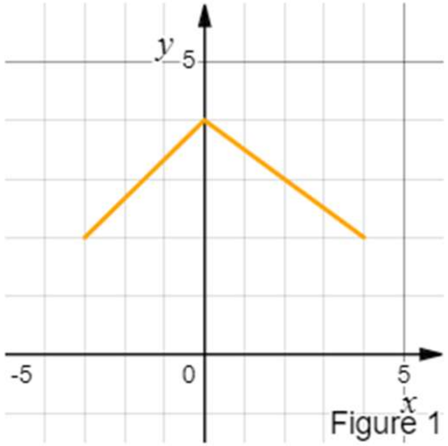


Figure 1

The graph of $y = af(x)$ is shown in Figure 2.

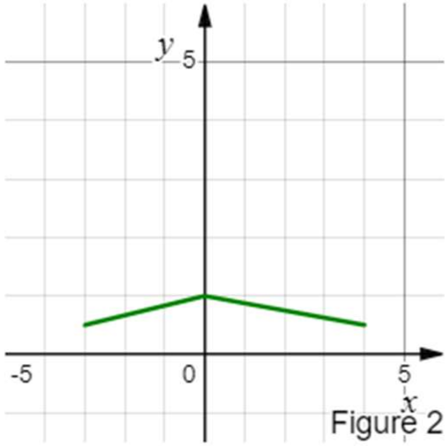


Figure 2

Determine the value of a.

T.78 4F: Qs 4+, P.32 4.6: Qs 4+

4.7) Transforming functions

Notes

Worked Example

Find the new coordinates under the transformations

$\boldsymbol{y = f(x)}$	$\boldsymbol{(-6, 4)}$	$\boldsymbol{(0, 1)}$
$y = f(x+2)$		
$y = f(x) - 2$		
$y = f(3x)$		
$y = 4f(x)$		
$y = f\left(\frac{x}{5}\right)$		
$y = 6f(x)$		
$y = -f(x)$		
$y = f(-x)$		

Worked Example

The point $A(2, 5)$ is the minimum of the curve with equation $y = f(x)$. Write the new coordinates of the new minimum of the curve:

a) $y = 2f(x) + 3$
b) $y = 3f(x) - 2$
c) $y = f(2x) + 3$
d) $y = f(3x) - 2$
e) $y = -f(x) + 3$
f) $y = -f(x) - 2$
g) $y = f(-x) + 3$
h) $y = f(-x) - 2$

Worked Example

The point $A(2, 5)$ is the minimum of the curve with equation $y = f(x)$. Write the new coordinates of the new minimum of the curve:

a) $y = -2f(x) + 3$
b) $y = -3f(x) - 2$
c) $y = 2f(-x) + 3$
d) $y = 3f(-x) - 2$
e) $y = -2f(-x) + 3$
f) $y = -3f(-x) - 2$
g) $y = 3f(2x) + 7$
h) $y = 7f(5x) - 2$
i) $y = -3f(2x) + 7$
j) $y = -7f(5x) - 2$
k) $y = -3f(-2x) + 7$
l) $y = -7f(-5x) - 2$

T.80 4G: all Qs, P.33 4.7 all Qs

AS Specimen

Graphs & Transformations

4.

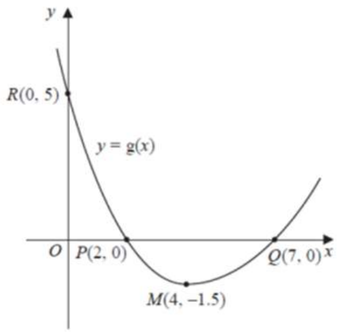


Figure 1

Figure 1 shows a sketch of the curve with equation $y = \mathrm{g}(x)$.

The curve has a single turning point, a minimum, at the point $M(4, -1.5)$.

The curve crosses the x-axis at two points, $P(2, 0)$ and $Q(7, 0)$.

The curve crosses the y-axis at a single point $R(0, 5)$.

(*a*) State the coordinates of the turning point on the curve with equation $y = 2\mathrm{g}(x)$.

(1)

(*b*) State the largest root of the equation $\mathrm{g}(x + 1) = 0$.

(1)

(*c*) State the range of values of x for which $\mathrm{g}'(x) \leq 0$.

(1)

Given that the equation $\mathrm{g}(x) + k = 0$, where k is a constant, has no real roots,

(*d*) state the range of possible values for k.

(1)

Exams

- Formula Booklet
- Past Papers
- Practice Papers
- past paper Qs by topic

Past paper practice by topic. Both new and old specification can be found via this link on hgsmaths.com

Question	Scheme	Marks	AOs
4(a)	$(4, -3)$	B1	1.2
		(1)	
(b)	$x = 6$	B1	1.1b
		(1)	
(c)	$x \leq 4$	B1	1.1b
		(1)	
(d)	$k > 1.5$	B1	2.2a
		(1)	
		(4 marks)	

Summary of Key Points

Summary of key points

1 If p is a root of the function $f(x)$, then the graph of $y = f(x)$ touches or crosses the x-axis at the point $(p, 0)$.

2 The graphs of $y = \frac{k}{x}$ and $y = \frac{k}{x^2}$, where k is a real constant, have asymptotes at $x = 0$ and $y = 0$.

3 The x-coordinate(s) at the points of intersection of the curves with equations $y = f(x)$ and $y = g(x)$ are the solution(s) to the equation $f(x) = g(x)$.

4 The graph of $y = f(x) + a$ is a translation of the graph $y = f(x)$ by the vector $\begin{pmatrix} 0 \\ a \end{pmatrix}$.

5 The graph of $y = f(x + a)$ is a translation of the graph $y = f(x)$ by the vector $\begin{pmatrix} -a \\ 0 \end{pmatrix}$.

6 When you translate a function, any asymptotes are also translated.

7 The graph of $y = af(x)$ is a stretch of the graph $y = f(x)$ by a scale factor of a in the vertical direction.

8 The graph of $y = f(ax)$ is a stretch of the graph $y = f(x)$ by a scale factor of $\frac{1}{a}$ in the horizontal direction.

9 The graph of $y = -f(x)$ is a reflection of the graph of $y = f(x)$ in the x-axis.

10 The graph of $y = f(-x)$ is a reflection of the graph of $y = f(x)$ in the y-axis.

T.82 mixed ex   P.35 BSG