KING EDWARD VI
HANDSWORTH GRAMMAR
SCHOOL FOR BOYS

KING EDWARD VI
ACADEMY TRUST
BIRMINGHAM

Year 12
Pure Mathematics
Chapter 14 pt 2 Exponentials and Logarithms

HGS Maths

Dr Frost Course

Name: ______________________________

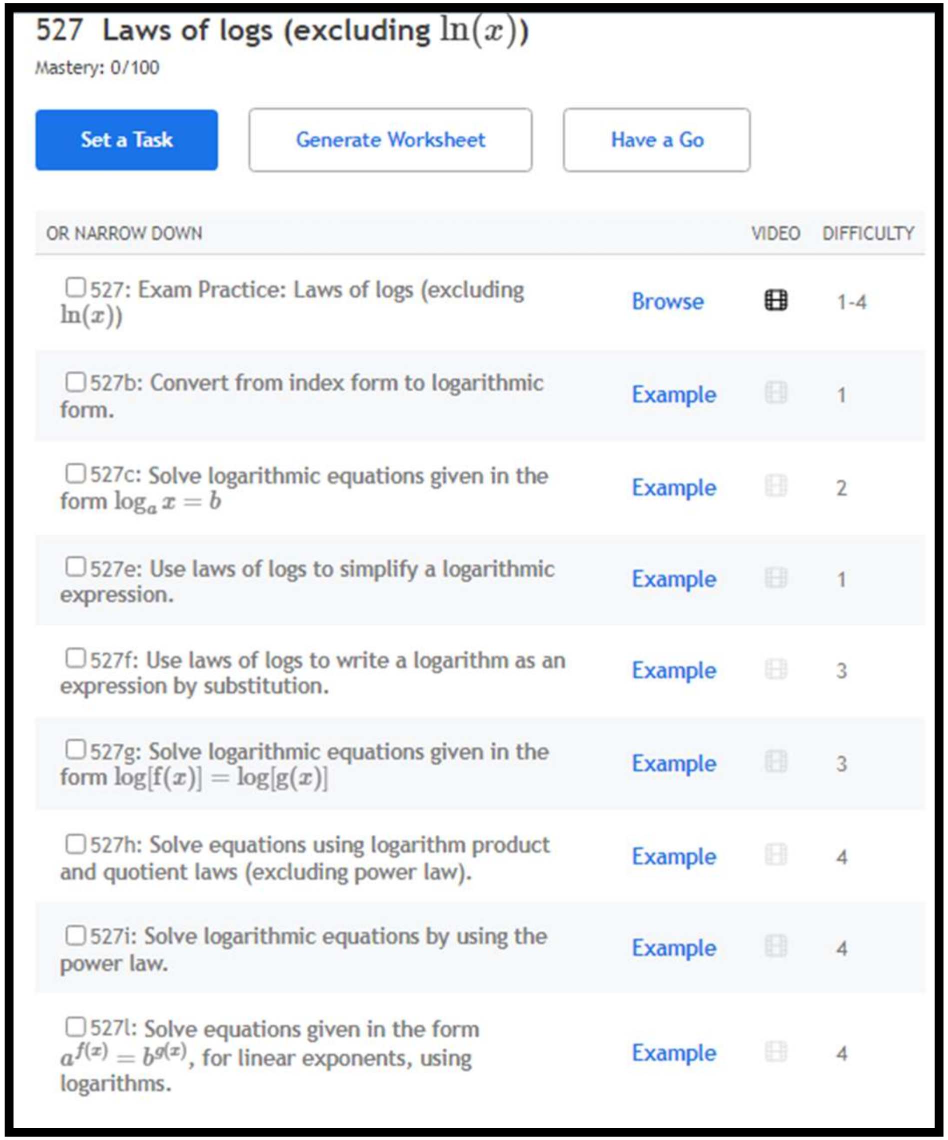
Class: ______________________________

Contents

Prior knowledge check

All key skills 527 :

[Chp14 - Exponentials & Logarithms (drfrost.org)](drfrost.org)


527 Laws of logs (excluding $\ln(x)$)

Mastery: 0/100

Set a Task | Generate Worksheet | Have a Go

OR NARROW DOWN		VIDEO	DIFFICULTY
527: Exam Practice: Laws of logs (excluding $\ln(x)$)	Browse		1-4
527b: Convert from index form to logarithmic form.	Example		1
527c: Solve logarithmic equations given in the form $\log_a x = b$	Example		2
527e: Use laws of logs to simplify a logarithmic expression.	Example		1
527f: Use laws of logs to write a logarithm as an expression by substitution.	Example		3
527g: Solve logarithmic equations given in the form $\log[f(x)] = \log[g(x)]$	Example		3
527h: Solve equations using logarithm product and quotient laws (excluding power law).	Example		4
527i: Solve logarithmic equations by using the power law.	Example		4
527l: Solve equations given in the form $a^{f(x)} = b^{g(x)}$, for linear exponents, using logarithms.	Example		4

14.1) Exponential functions

Notes

Worked Example

On the same axes, sketch $y = 4^x$, $y = 5^x$ and $y = 3.5^x$

Worked Example

On the same axes, sketch $y = 3^x$ and $y = \left(\frac{1}{3}\right)^x$

Worked Example

Sketch $y = 3^{x-2}$

T.313: 14A Qs4-5, P.113: 14.1 Q4-6

Worked Example

The graph of $y = ka^x$ passes through the points $(4, \frac{16}{3})$ and $(0, \frac{1}{3})$

Find the values of the constants k and a

T.313: 14A Qs6-7, P.113: 14.1 Q7

14.2) $y = e^x$

Notes

Worked Example

Differentiate with respect to x:

a) e^{2x}

b) e^{-3x}

c) $4e^{5x}$

d) $6e^{\frac{1}{3}x}$

T.316: 14B Qs3-5,7 ,P.114: 14.2 Qs 2, 3, 5,6,8

Worked Example

Sketch

$$y = e^{2x}\ ,\ y = e^{3x},\ y = e^{-2x}$$

On the same axis as $y = e^{x}$

Worked Example

Sketch:

$$y = 3 + 4e^{\frac{1}{2}x}$$

Worked Example

Sketch:

$$y = e^{-3x} - 2$$

T.316: 14B Qs3-5, P.114: 14.3 Qs 1, 4,7

14.3) Exponential modelling

Notes

Worked Example

Suppose the population P of a village is modelled by $P = 500e^{2t}$ where t is the numbers of years since February 2009. Find:

a) The initial population

b) The initial rate of growth

c) The population in February 2014

T.318: 14C Qs1-3

Your Turn

Suppose the population P of a village is modelled by $P = 100e^{3t}$ where t is the numbers of years since January 2010. Find:

a) The initial population

b) The initial rate of growth

c) The population in January 2014

a) 100

b) $\frac{dP}{dt} = 300$

c) 16275479

Worked Example

The density of a pesticide in a given section of field, P mg/m², can be modelled by the equation $P = 80e^{-0.003t}$ where t is the time in days since the pesticide was first applied.

a. Use this model to estimate the density of pesticide after 30 days.

b. Interpret the meaning of the value 80 in this model.

c. Show that $\frac{dP}{dt} = kP$, where k is a constant, and state the value of k.

d. Interpret the significance of the sign of your answer in part (c).

e. Sketch the graph of P against t.

T.318: 14B Qs3+, P.115: 14.3 all Qs

14.7) Working with natural logarithms

Notes

Worked Example

528b: Solve exponential equations involving a single occurrence of e

Solve for x:

$$3e^{3x+1} = 7$$

Give your solution in exact form.

Worked Example

Solve the equation:

$$3\ln x - 7 = 5$$

T.327: 14G Qs 1,2, P.119: 14.7 Qs 1,2

Worked Example

528g: **Solve equations which are quadratic in e^x**

Find the exact solution of

$$3e^{2x} + e^x = 10.$$

Input note: Give any answers involving logs in the form $\log_a b$.

Worked Example

Solve the equation:

$$e^x - 12e^{-x} = -1$$

Worked Example

528c: Solve logarithmic equations involving a single occurrence of $\ln$

Solve for x:

$$3\ln(-3x-3)+3=9$$

Give your solution in exact form.

Worked Example

Find the exact coordinates of the points where the graph with equation $y = 6 + \ln(5 - x)$ intersects the axes

T.327: 14G Qs 3,4, P.119: 14.7 Qs 3-6

Worked Example

Solve the equation:

$$3^x e^{x+4} = 2$$

Give your answer as an exact value

T.327: 14G Qs 5+ P.119: 14.7 Qs 7+

14.8) Logarithms and non-linear data

Notes

Worked Example

Use logarithms to convert the non-linear relationship into a linear form and sketch the resulting straight line.

$$y = ax^n$$

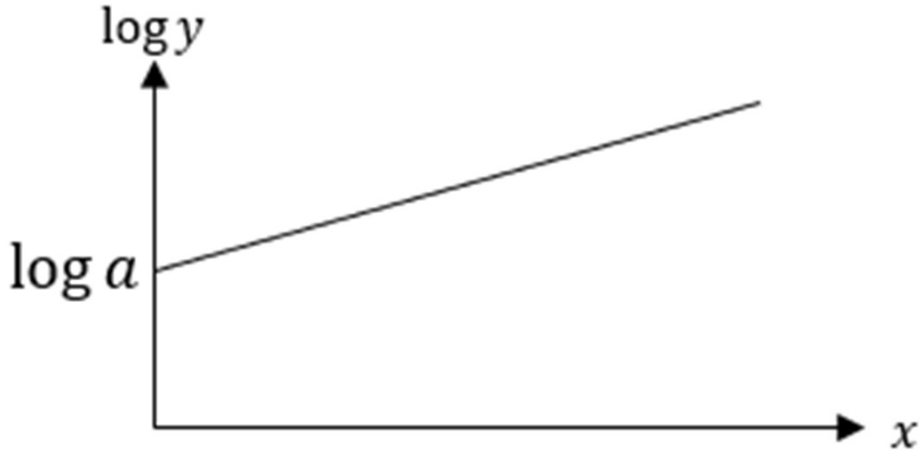
T.331: 14H Qs 1,2 P.120: 14.8 Qs 1,2

Your Turn

Use logarithms to convert the non-linear relationship into a linear form and sketch the resulting straight line.

$$y = ab^x$$

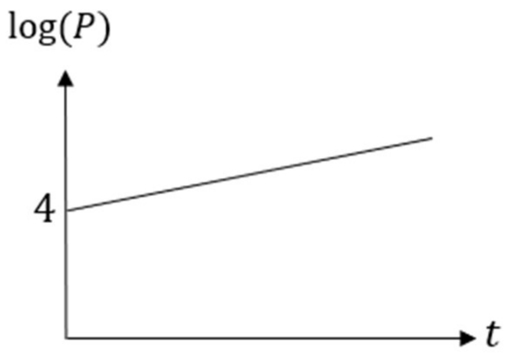
$$\log y = (\log b)\, x + \log a$$



Worked Example

The graph represents the growth of a population of bacteria, P, over t hours.
The graph has a gradient of 0.3 and meets the vertical axis at $(0,4)$.
A scientist suggest that this growth can be modelled by the equation $P = ab^t$, where a and b are constants to be found.

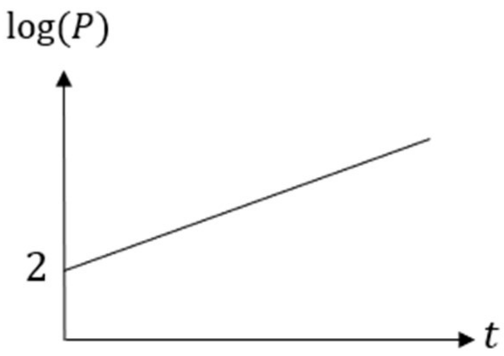
a. Write down an equation for the line.
b. Find the values of a and b, giving them to 3 sf where necessary.
c. Interpret the meaning of the constant a in this model.



Your Turn

The graph represents the growth of a population of bacteria, P, over t hours.
The graph has a gradient of 0.6 and meets the vertical axis at $(0,2)$.
A scientist suggest that this growth can be modelled by the equation $P = ab^t$, where a and b are constants to be found.

a. Write down an equation for the line.
b. Find the values of a and b, giving them to 3 sf where necessary.
c. Interpret the meaning of the constant a in this model.



a) $\log P = 0.6t + 2$
b) $a = 100, b = 3.98$ (3 sf)
c) The initial size of the bacteria population was 100

Worked Example

The table below gives the rank (by size) and population of a country's largest cities and districts (the capital city is number 1 but has been excluded as an outlier).

City	A	B	C	D	E
Rank, R	2	3	4	5	6
Population	2 000 000	1 400 000	1 200 000	1 000 000	900 000

The relationship between the rank and population can be modelled by the formula:
$P = aR^n$ where a and n are constants.

a) Draw a table giving values of $\log R$ and $\log P$ to 2dp.

b) Plot a graph of $\log R$ against $\log P$ using the values from your table and draw the line of best fit.

c) Use your graph to estimate the values of a and n to two significant figures.

Your Turn

The table below gives the rank (by size) and population of the UK's largest cities and districts in the past (London is number 1 but has been excluded as an outlier).

City	Birmingha m	Leeds	Glasgow	Sheffield	Bradford
Rank, R	2	3	4	5	6
Population	1 000 000	730 000	620 000	530 000	480 000

The relationship between the rank and population can be modelled by the formula:

$P = aR^n$ where a and n are constants.

a) Draw a table giving values of $\log R$ and $\log P$ to 2dp.

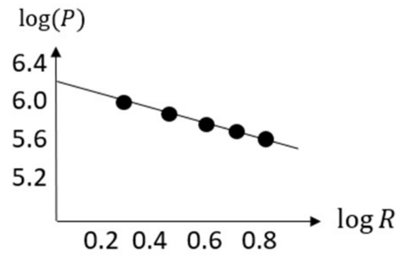
b) Plot a graph of $\log R$ against $\log P$ using the values from your table and draw the line of best fit.

c) Use your graph to estimate the values of a and n to two significant figures.

a)

$\log R$	0.30	0.48	0.60	0.70	0.78
$\log P$	6	5.86	5.79	5.72	5.68

b)



c) $a = 1600000$, $n = -0.67$ (2 sf)

Worked Example

A population is increasing exponentially according to the model $P = ab^t$, where a, b are constants to be found.

The population is recorded as follows:

Years t after 2016	1.4	2.6	4.4
Population P	4706	7346	14324

a) Draw a table giving values of t and $\log P$ (to 3dp).

b) A line of best fit is drawn for the data in your new table, and it happens to go through the first data point above (where $t = 1.4$) and last (where $t = 4.4$).
Determine the equation of this line of best fit.

c) Hence, determine the values of a and b in the model.

d) Estimate the population in 2020

T.331: 14H Qs 3+ P.120: 14.8 Qs 3+

Your Turn

A population is increasing exponentially according to the model $P = ab^t$, where a, b are constants to be found.

The population is recorded as follows:

Years t after 2015	0.7	1.3	2.2
Population P	2353	3673	7162

a) Draw a table giving values of t and $\log P$ (to 3dp).
b) A line of best fit is drawn for the data in your new table, and it happens to go through the first data point above (where $t = 0.7$) and last (where $t = 2.2$).
Determine the equation of this line of best fit
c) Hence, determine the values of a and b in the model.
d) Estimate the population in 2020

a)

t	0.7	1.3	2.2
$\log P$	3.372	3.565	3.855

b) $\log P = 0.322t + 3.147$
c) $a = 1403$, $b = 2.099$ (4 sf)
d) 57164

A2 2019 Paper 1

Exponentials and Logs

7. In a simple model, the value, £V, of a car depends on its age, t, in years.

The following information is available for car A

- its value when new is £20 000
- its value after one year is £16 000

(a) Use an exponential model to form, for car A, a possible equation linking V with t.

(4)

The value of car A is monitored over a 10-year period.
Its value after 10 years is £2 000

(b) Evaluate the reliability of your model in light of this information.

(2)

The following information is available for car B

- it has the same value, when new, as car A
- its value depreciates more slowly than that of car A

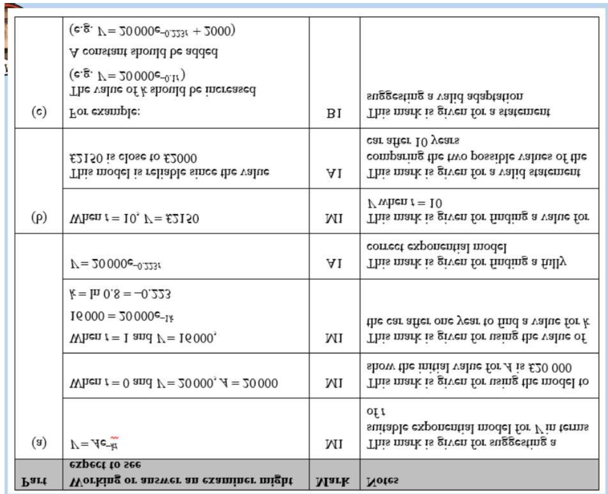
(c) Explain how you would adapt the equation found in (a) so that it could be used to model the value of car B.

(1)

Exams

- Formula Booklet
- Past Papers
- Practice Papers
- past paper Qs by topic

Past paper practice by topic. Both new and old specification can be found via this link on hgsmaths.com



Part	Working or answer an examiner might expect to see	Mark	Notes
(a)	$V = Ae^{-kt}$	M1	This mark is given for suggesting a suitable exponential model for V in terms of t
	When $t = 0$ and $V = 20\,000$, $A = 20\,000$	M1	This mark is given for using the model to show the initial value for A is £20 000
	When $t = 1$ and $V = 16\,000$, $16\,000 = 20\,000e^{-k}$, $k = \ln 0.8 = -0.223$	M1	This mark is given for using the value of the car after one year to find a value for k
	$V = 20\,000e^{-0.223t}$	A1	This mark is given for finding a fully correct exponential model
(b)	When $t = 10$, $V = £2150$	M1	This mark is given for finding a value for V when t = 10
	This model is reliable since the value £2150 is close to £2000	A1	This mark is given for a valid statement comparing the two possible values of the car after 10 years
(c)	For example: The value of k should be increased (e.g. $V = 20\,000e^{-0.1t}$) A constant should be added (e.g. $V = 20\,000e^{-0.223t} + 2000$)	B1	This mark is given for a statement suggesting a valid adaptation

Summary of Key Points

Summary of key points

1 For all real values of x:

- If $f(x) = e^x$ then $f'(x) = e^x$
- If $y = e^x$ then $\frac{dy}{dx} = e^x$

2 For all real values of x and for any constant k:

- If $f(x) = e^{kx}$ then $f'(x) = ke^{kx}$
- If $y = e^{kx}$ then $\frac{dy}{dx} = ke^{kx}$

3 $\log_a n = x$ is equivalent to $a^x = n$ $(a \neq 1)$

4 **The laws of logarithms:**

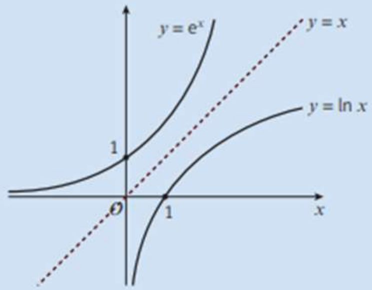
- $\log_a x + \log_a y = \log_a xy$ (the multiplication law)
- $\log_a x - \log_a y = \log_a \left(\frac{x}{y}\right)$ (the division law)
- $\log_a (x^k) = k \log_a x$ (the power law)

5 You should also learn to recognise the following special cases:

- $\log_a \left(\frac{1}{x}\right) = \log_a (x^{-1}) = -\log_a x$ (the power law when $k = -1$)
- $\log_a a = 1$ $(a > 0, a \neq 1)$
- $\log_a 1 = 0$ $(a > 0, a \neq 1)$

6 Whenever $f(x) = g(x)$, $\log_a f(x) = \log_a g(x)$

7 The graph of $y = \ln x$ is a reflection of the graph $y = e^x$ in the line $y = x$.



8 $e^{\ln x} = \ln(e^x) = x$

9 If $y = ax^n$ then the graph of $\log y$ against $\log x$ will be a straight line with gradient n and vertical intercept $\log a$.

10 If $y = ab^x$ then the graph of $\log y$ against x will be a straight line with gradient $\log b$ and vertical intercept $\log a$.

T.336 mixed ex. P.122: BSG