KING EDWARD VI
HANDSWORTH GRAMMAR
SCHOOL FOR BOYS

KING EDWARD VI
ACADEMY TRUST
BIRMINGHAM

Year 12
Pure Mathematics
P1 6 Circles
Booklet

HGS Maths

Dr Frost Course

Name: ______________________________________

Class: ______________________________________

Contents

Prior knowledge check

Prior knowledge check

1 Write each of the following in the form $(x + p)^2 + q$:

a $x^2 + 10x + 28$ **b** $x^2 - 6x + 1$

c $x^2 - 12x$ **d** $x^2 + 7x$ ← Section 2.

2 Find the equation of the line passing through each of the following pairs of points:

a $A(0, -6)$ and $B(4, 3)$

b $P(7, -5)$ and $Q(-9, 3)$

c $R(-4, -2)$ and $T(5, 10)$ ← Section 5.

3 Use the discriminant to determine whether the following have two real solutions, one real solution or no real solutions.

a $x^2 - 7x + 14 = 0$

b $x^2 + 11x + 8 = 0$

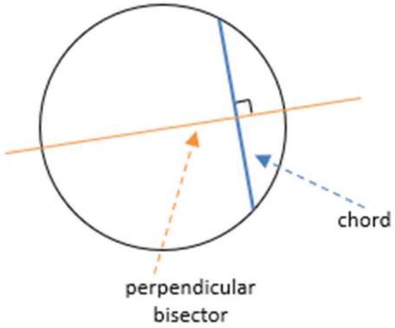
c $4x^2 + 12x + 9 = 0$ ← Section 2.

4 Find the equation of the line that passes through the point $(3, -4)$ and is perpendicula to the line with equation $6x - 5y - 1 = 0$

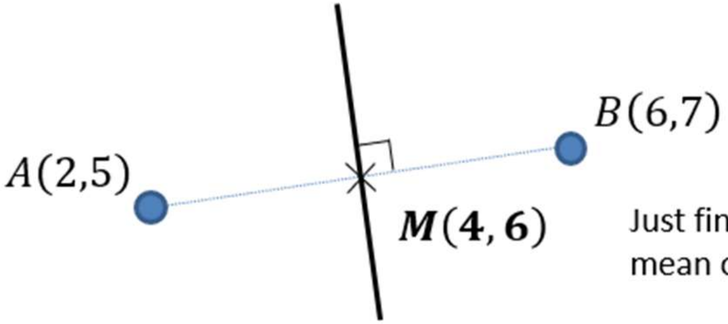
← Section 5.

6.1) Midpoints and perpendicular bisectors

Later in the chapter you will need to find the perpendicular bisector of a chord of a circle.



What two properties does a perpendicular bisector of two points A and B have?



Just find the mean of the x values and the mean of the y values.

1. It passes through the midpoint of AB.
2. It is perpendicular to AB.

Equation?

$$\boldsymbol{m_{AB} = \frac{2}{4} = \frac{1}{2}}$$
$$\boldsymbol{m_{\perp} = -2}$$
$$\boldsymbol{y - 6 = -2(x - 4)}$$

Notes

Worked Example

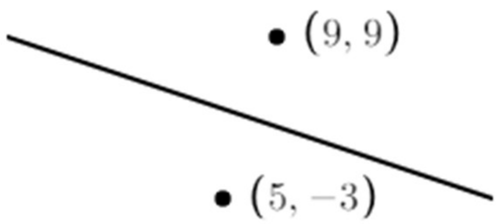
495e: Determine the perpendicular bisector of a line using $y - y_1 = m(x - x_1)$

A straight line passes through the points $A(5, -3)$ and $B(9, 9)$.

Find the equation of the perpendicular bisector of AB.

Give your answer in the form $y = mx + c$.

Simplify your answer where possible.



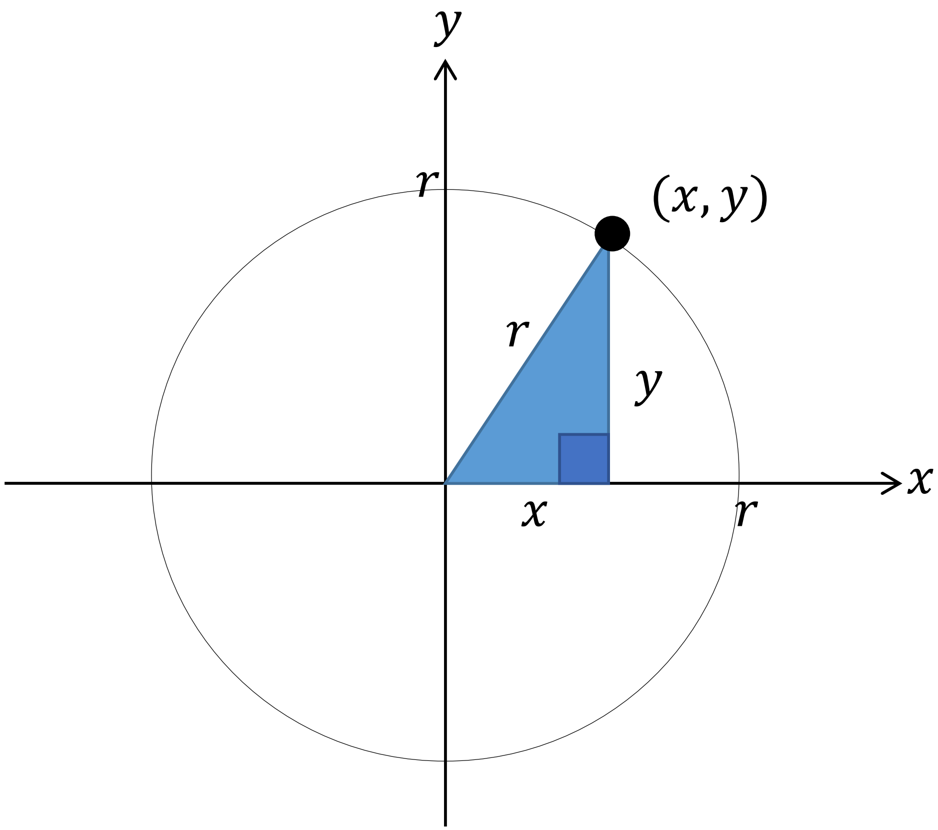
T.116: 6B Qs 1, P.46 6.1 Qs 1-5

Worked Example

A line segment AB is the diameter of a circle with centre $(4, -5)$. If A has coordinates $(2, -1)$, what are the coordinates of B?

T.116: 6B Qs 2+, P.46 6.1 Qs 6+

6.2) Equation of a circle

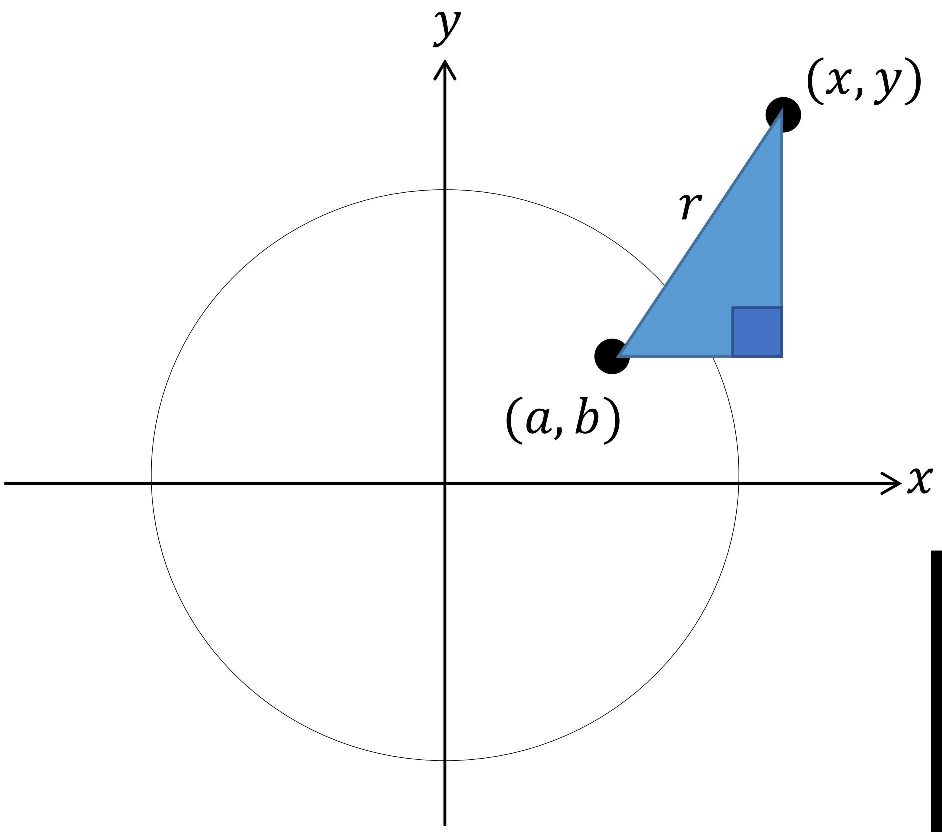


Recall that a line can be a set of points (x, y) that satisfy some equation. Suppose we have a point (x, y) on a circle centred at the origin, with radius r. What equation must (x, y) satisfy?

(Hint: draw a right-angled triangle inside your circle, with one vertex at the origin and another at the circumference)

$$x^2 + y^2 = r^2$$

Notes



Now suppose we shift the circle so it's now centred at (a, b). What's the equation now?

(Hint: What would the sides of this right-angled triangle be now?)

✎ The equation of a circle with centre (a, b) and radius r is:

$$(x-a)^2 + (y-b)^2 = r^2$$

Notes

Quickfire Questions

Centre	Radius	Equation
$(0,0)$	5	
$(1,2)$	6	
		$(x+3)^2+(y-5)^2=1$
		$(x+5)^2+(y-2)^2=49$
		$(x+6)^2+y^2=16$
		$(x-1)^2+(y+1)^2=3$
		$(x+2)^2+(y-3)^2=8$

Worked Example

496a: Determine the equation of a circle, not centred at the origin, given its centre and an integer radius.

A circle has centre $(11, 5)$ and radius 7.

Write down the equation of the circle.

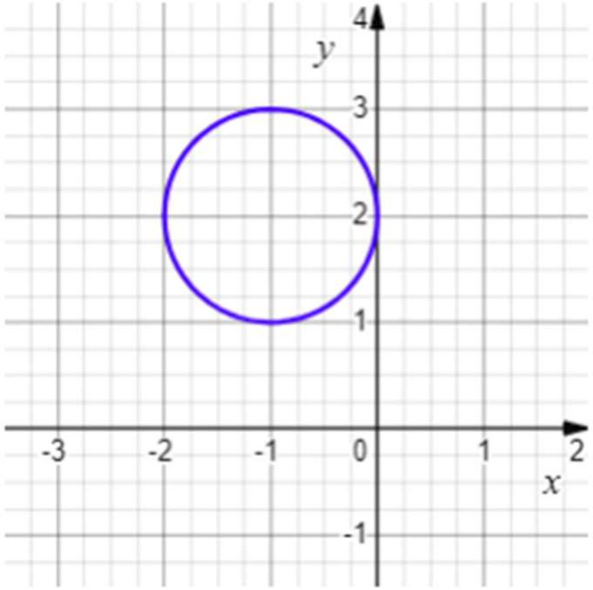
Give your answer in the form

$$(x-a)^2+(y-b)^2=r^2$$

Worked Example

496b: Determine the equation of a circle, not centred at the origin, which is drawn.

Find an equation of the circle drawn below.



Worked Example

Skill involved: 496d: Determine the centre and radius of a circle given in completed square form.

Find the centre and exact value of the radius of the circle with equation

$$(x+1)^2 + (y+5)^2 = 19$$

where the centre of the circle is (a, b).

Worked Example

Skill involved: 496e: Determine the equation of a circle, with given centre and radius/diameter, in expanded form.

Write down the equation of a circle with centre $(0, -1)$ and diameter of $6\sqrt{3}$

Give your answer in the form $x^2 + y^2 + ay + b = 0$, where a and b are constants to be found.

....

Worked Example

K328g: Find the equation of a circle using the endpoints of the diameter.

The line CD is the diameter of a circle where C and D have coordinates $(-3, 4)$ and $(9, -12)$ respectively.

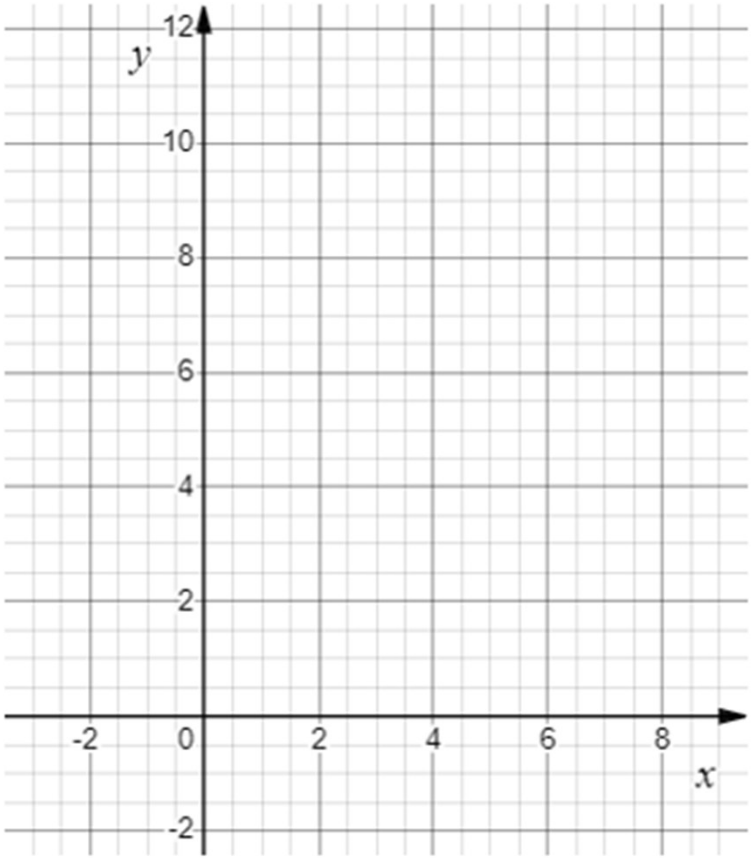
Find the equation of the circle, giving your answer in the form

$$x^2 + y^2 + ax + by + c = 0$$

Worked Example

Skill involved: 496f: Plot the graph of a circle not centred at the origin.

Draw the graph of $(x-3)^2 + (y-5)^2 = 9$



Worked Example

Skill involved: 496g: Appreciate that a point on the circumference of a circle, not centred at the origin, satisfies its equation.

Determine whether the point $(-1,7)$ is inside the circle, on the circle, or outside the circle with equation

$$(x-1)^2 + (y-2)^2 = 30$$

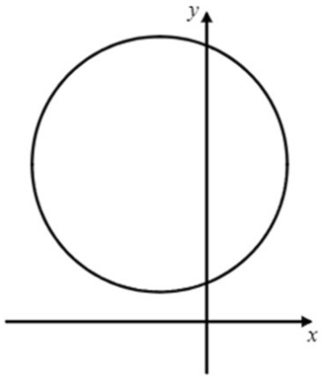
T.119: 6C Qs 1-4, P.47 6.2 Qs 1-4

Worked Example

Skill involved: 496h: Determine the intercepts of circle given its equation.

The circle C has the equation

$$(x+3)^2 + (y-10)^2 = 66$$



Find the exact y coordinates of the points where the circle C intersects the y-axis.

$y =$ ....

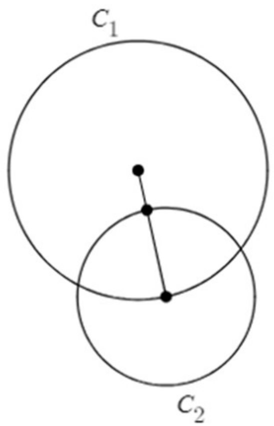
or $y =$ ....

Worked Example

Skill involved: 496i: Determine a constant in the equation of a circle given a point on the circumference.

Circle C_1 has equation $x^2 + y^2 - 4x - gy - 56 = 0$

Circle C_2 has equation $(x-4)^2 + (y+4)^2 = 40$



The centre of C_2 lies on the circumference of C_1

Find the value of g, where g is a positive constant.

Worked Example

Skill involved: 496j: Determine the equation of a circle using the endpoints of the diameter.

The line AB is the diameter of a circle where A and B have coordinates $(3,12)$ and $(19,24)$ respectively.

Find the equation of the circle, giving your answer in the form

$$x^2+y^2+ax+by+c=0$$

Worked Example

Skill involved: 496k: Complete the square to determine the centre and radius of a circle.

A circle has equation

$$x^2 + y^2 - 2y - 120 = 0$$

Find the centre and radius of the circle

Worked Example

Skill involved: 496l: Complete the square to determine the centre and radius of a circle, where the equation involves algebraic constants.

A circle C has equation

$$x^2 + y^2 + 2kx - 4ky = -20$$

where k is a constant.

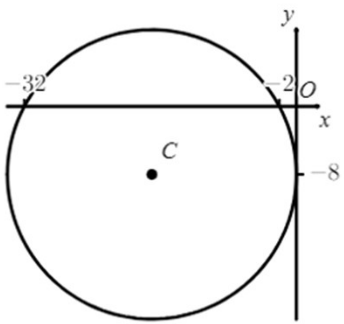
By considering the radius of C, state the range of possible values for k.

Worked Example

Skill involved: 496m: Determine the equation of a circle using simple reasoning to establish the centre or radius of the circle.

The diagram shows a circle with centre C.

The circle intersects the x-axis at $(-2,0)$ and $(0,-8)$
The circle touches the y-axis at $(-32,0)$



Work out the equation of the circle.

T.119: 6C Qs 5+, P.47 6.2 Qs 5+

6.3) Intersections of straight lines and circles

Notes

Worked Example

Skill involved: 496u: Determine whether a straight line cuts or is tangent to a circle.

A circle C has equation $(x-4)^2 + (y-3)^2 = 20$

Given that $y = -\frac{1}{2}x + k$, where k is a constant does not meet C, find the range of values for k.

....

Worked Example

The line with equation $y = 5x + 2$ meets the circle with equation $x^2 + kx + y^2 = 6$ at exactly one point.
Find the two possible values of k

Worked Example

The line with equation $y = 4x - 3$ does not intersect the circle with equation $x^2 + 2x + y^2 = k$.
Find the range of possible values of k.

T.122: 6D Qs all, P.48 6.3 Qs all

6.4) Use tangent and chord properties

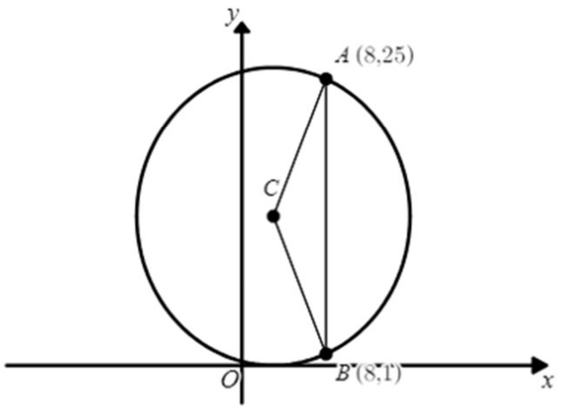
Notes

Worked Example

Skill involved: 496n: Use the coordinates of the endpoint of chord of a circle and the radius to determine the circle's centre.

The diagram shows a circle, centre C with radius 13.

The circle passes through the points $A(8,25)$ and $B(8,1)$.



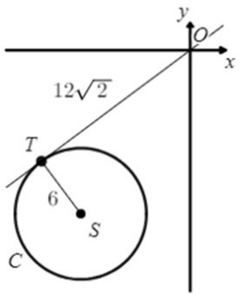
Work out the equation of the circle.

Worked Example

Skill involved: 496o: Determine the coordinate of a centre of a circle given its radius and the length of a tangent.

The diagram shows a circle C with centre S and radius T and the point T which lies on C.

The tangent to C at point T passes through the origin O and $OT = 12\sqrt{2}$



Given that the coordinates of S are $(-10, m)$, find the exact value of m.

Worked Example

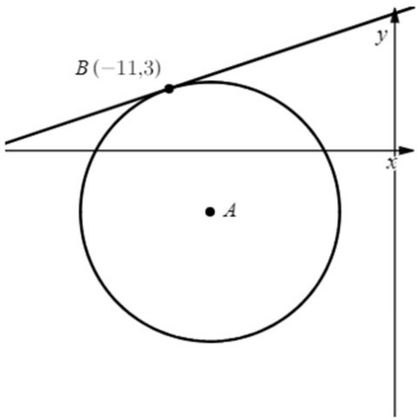
Skill involved: 496p: Determine the tangent to a circle centred at any point.

A circle has equation

$$x^2 + y^2 + 18x + 6y + 50 = 0$$

Find the equation of the tangent to the circle at the point $(-11,3)$.

Give your answer in the form $y = mx + c$, where m and c are constants to be found.



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T.126: 6E Qs 1-5, P.49 6.4 Qs 1-5

Worked Example

Skill involved: 496q: Determine the shortest distance between two circles with given equations.

Circle A has equation $x^2 + y^2 + 40x - 42y + 480 = 0$

Circle B has equation $x^2 + y^2 - 24x + 10y - 231 = 0$

Find the exact shortest distance between the two circles.

....

Worked Example

Skill involved: 496r: Determine whether circles overlap or touch.

Circle A has equation $(x-7)^2+(y+3)^2=32$

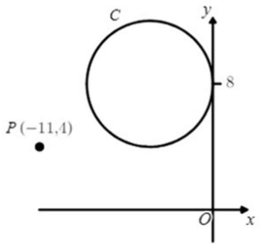
Circle B has equation $(x+16)^2+(y+10)^2=338$

Select the correct statement.

Worked Example

Skill involved: 496s: Determine the length of a tangent from a point to a circle with given equation.

A circle C has radius $P(-11,4)$ and touches the y-axis as the point $(0,8)$, as shown in the figure.



A line through the point $P(-11,4)$ is a tangent to the circle C at the point T.

Find the length of PT.

$PT =$ .... units

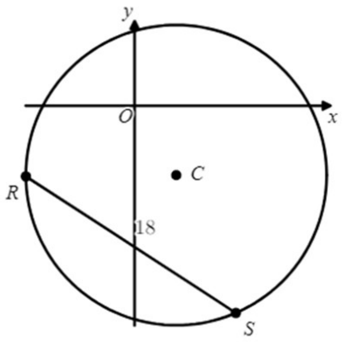
Worked Example

Skill involved: 496t: Determine the shortest distance from a chord of the circle of given length to the centre of the circle.

A circle has centre C and equation $(x-3)^2 + (y+5)^2 = 117$.

There are two points R and S which lie on the circle.

Given that the length of the chord RS is 18 units, find the length of the shortest distance from C to the chord RS.



Worked Example

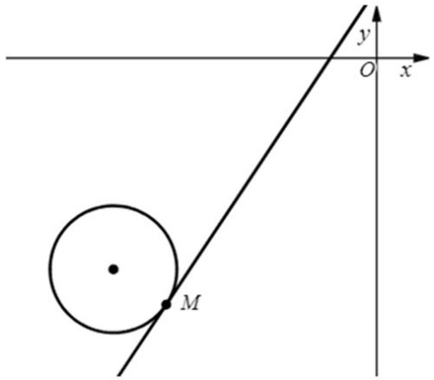
Skill involved: 496v: Determine the equation of a tangent to a circle that is parallel to another given tangent.

Given the equation of a circle $(x+15)^2+(y+12)^2=13$, and a tangent line with equation $y=\frac{3}{2}x+4$.

This tangent meets the circle at point M.

The line with equation, $y=\frac{3}{2}x+k$, where $k \neq 4$ is also a tangent to the circle.

Find the value of k.



Worked Example

Skill involved: 496u: Determine whether a straight line cuts or is tangent to a circle.

A circle C has equation $(x-4)^2 + (y-3)^2 = 20$

Given that $y = -\frac{1}{2}x + k$, where k is a constant does not meet C, find the range of values for k.

....

Worked Example

A circle C has equation

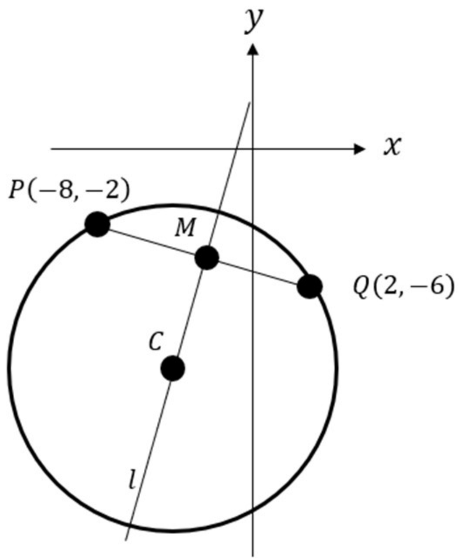
$$(x - 4)^2 + (y + 4)^2 = 10$$

The line l is a tangent to the circle and has gradient -3. Find two possible equations for l, giving your answers in the form $y = mx + c$.

Worked Example

The point P has coordinates $(-8, -2)$ and the point Q has coordinates $(2, -6)$.
M is the midpoint of the line segment PQ.

a) Find an equation for l.
b) Given that the y-coordinate of C is -9:
  i) show that the x-coordinate of C is -5.
  ii) find an equation of the circle.



Worked Example

The line with equation $4x + y - 5 = 0$ is a tangent to the circle with equation $(x - 3)^2 + (y - p)^2 = 2$.
Find the two possible values of p

Your Turn

The line with equation $4x + y - 3 = 0$ is a tangent to the circle with equation $(x-2)^2 + (y-p)^2 = 5$.
Find the two possible values of p

$p = 3 \pm \sqrt{19}$

Worked Example

A circle has centre $C(5,3)$, and passes through the point $P(2,6)$.
Find the equation of the tangent of the circle at the point P, giving your equation in the form $ax + by + c = 0$ where a, b, c are integers..

Worked Example

A circle passes through the points $A(0,0)$ and $B(2,8)$.
The centre of the circle has x value -2. Determine the equation of the circle.

T.126: 6E Qs 6+, P.49 6.4 Qs 6+

Mark Scheme for Extension Question 2

(a) $(x+4)^2+(y-7)^2=13$ and $y=mx$

$\therefore (x^2+8x+16)+(m^2x^2-14mx+49)=13$ — M1

$(1+m^2)x^2+(8-14m)x+52=0$ — (3 t-quad) A1

Touches, so "$b^2=4ac$" $\quad (8-14m)^2=4\times52\times(1+m^2)$ — M1 ($b^2=4ac$)

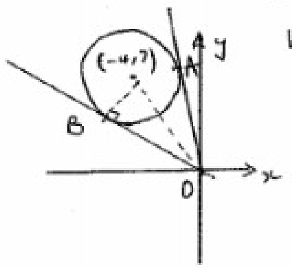
$(4-7m)^2=52+52m^2$

$\therefore \; \underline{3m^2+56m+36=0}$ * — A1 cso

(4)

(b) $(3m+2)(m+18)=0$ — M1

$\therefore m=-2/3$ or -18 — (both m) A1



Let A or B be (X,Y)

then $(X^2+Y^2)+13=4^2+7^2=65$

$X^2+Y^2=52$ — M1, A1

$Y=-\frac{2}{3}X \Rightarrow \frac{13}{9}X^2=52 \Rightarrow X=\pm6$

From the configuration $X_B=-6 \;\therefore Y_B=+4$: $\underline{B \text{ is } (-6,4)}$ — M1, A1

$Y=-18X \Rightarrow 325X^2=52 \Rightarrow X^2=\frac{4}{25}$

Again $X<0$ for A $\;\therefore X_A=-\frac{2}{5}$; $Y_A=\frac{36}{5}$

$\underline{A=\left(-\frac{2}{5},\frac{36}{5}\right)}$ — M1, A1

(8)

(c) Situation is a translation of problem in (b) by $\begin{pmatrix}4\\-7\end{pmatrix}$ — M1

So P, Q are $\begin{pmatrix}-6\\4\end{pmatrix}+\begin{pmatrix}4\\-7\end{pmatrix}$ and $\left(-\frac{2}{5},\frac{36}{5}\right)+(4,-7)$

$=\underline{(-2,-3)}$ and $\underline{\left(\frac{18}{5},\frac{1}{5}\right)}$ — A1 (either)

(2)

Mark Scheme for Extension Question 3

(i) $PA = 2PB \Rightarrow (x-5)^2 + (y-16)^2 = 4\left((x+4)^2 + (y-4)^2\right)$

$\Rightarrow x^2 + y^2 - 10x - 32y + 281 = 4x^2 + 4y^2 + 32x - 32y + 128$

$\Rightarrow 3x^2 + 3y^2 + 42x - 153 = 0$

$\Rightarrow x^2 + y^2 + 14x - 51 = 0$

$\Rightarrow (x+7)^2 - 49 + y^2 - 51 = 0$

$\Rightarrow (x+7)^2 + y^2 = 100$

which is a circle centre $(-7,\ 0)$ with radius 10.

(ii) $QC = k \times QD \Rightarrow (x-a)^2 + y^2 = k^2(x-b)^2 + k^2y^2$

$\Rightarrow x^2\left(k^2-1\right) + y^2\left(k^2-1\right) + x\left(2a - 2k^2b\right) + \left(k^2b^2 - a^2\right) = 0$

If this locus is the same as the locus of P, then the ratios of the coefficients must be the same.

$\Rightarrow \dfrac{2a - 2k^2b}{k^2-1} = 14$ and $\dfrac{k^2b^2 - a^2}{k^2-1} = -51.$

Notice that you **cannot** conclude that $k^2 - 1 = 1$.

$\Rightarrow k^2 = \dfrac{a+7}{b+7}$ and $k^2 = \dfrac{a^2+51}{b^2+51}$

$\Rightarrow \dfrac{a+7}{b+7} = \dfrac{a^2+51}{b^2+51}$

$\Rightarrow (a+7)\left(b^2+51\right) = (b+7)\left(a^2+51\right)$

$\Rightarrow ab^2 - a^2b = 7\left(a^2-b^2\right) + 51(b-a)$

$\Rightarrow ab(b-a) = 7(a-b)(a+b) + 51(b-a)$

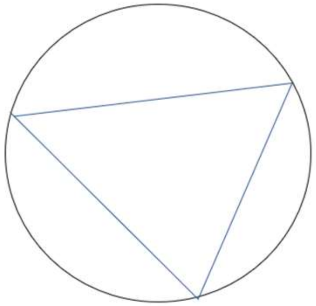
$\Rightarrow ab = 51 - 7(a+b)$ since $a \neq b \Rightarrow a - b \neq 0$

$\Rightarrow ab + 7(a+b) = 51$

$\Rightarrow ab + 7(a+b) + 49 = 51 + 49$

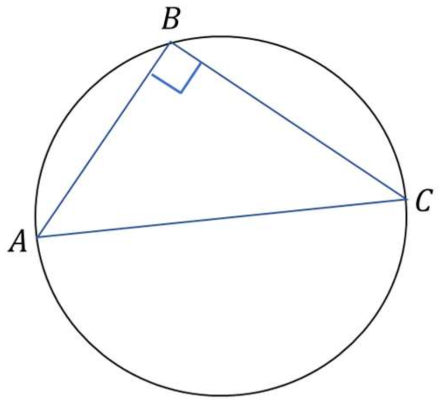
$\Rightarrow (a+7)(b+7) = 100$

6.5) Circles and triangles



We'd say:

- The triangle **inscribes** the circle.
  (A shape inscribes another if it is inside and its boundaries touch but do not intersect the outer shape)
- The circle **circumscribes** the triangle.
- If the circumscribing shape is a circle, it is known as the **circumcircle** of the triangle.
- The centre of a circumcircle is known as the **circumcentre**.

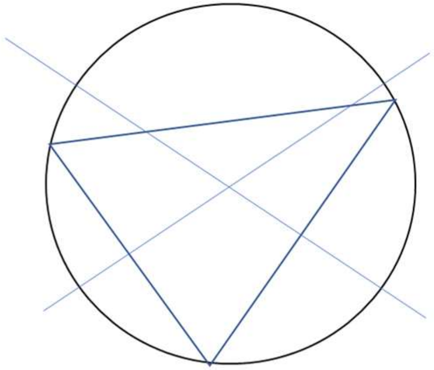


If $\angle ABC = 90^\circ$ then:

- **AC is the diameter of the circumcircle of triangle ABC.**

Similarly if AC is the diameter of a circle:

- **$\angle ABC = 90^\circ$ therefore AB is perpendicular to BC.**
- **$AB^2 + BC^2 = AC^2$**



Given three points/a triangle we can find the centre of the circumcircle by:

- **Finding the equation of the perpendicular bisectors of two different sides.**
- **Find the point of intersection of the two bisectors.**

Notes

Worked Example

Skill involved: 496w: Determine the equation of a circle given the coordinates of 3 points.

The coordinates $(-7,19)$, $(1,15)$ and $(-23,7)$ lie on the circle C.

Find the equation of C.

....

T.131: 6F Qs all, P.51 6.5 Qs all

9. A circle with centre $A(3,-1)$ passes through the point $P(-9,8)$ and the point $Q(15,-10)$

(a) Show that PQ is a diameter of the circle.

(2)

(b) Find an equation for the circle.

(3)

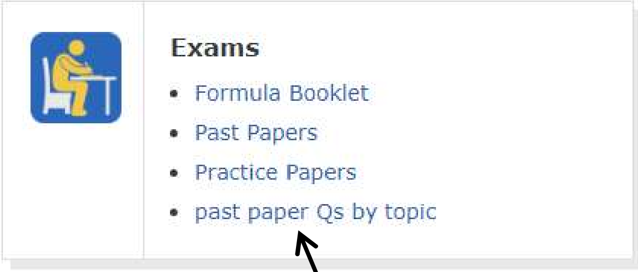
A point R also lies on the circle.
Given that the length of the chord PR is 20 units,

(c) find the length of the shortest distance from A to the chord PR.
Give your answer as a surd in its simplest form.

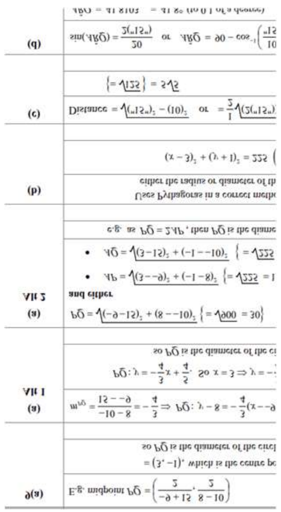
(2)

(d) Find the size of angle ARQ, giving your answer to the nearest 0.1 of a degree.

(2)



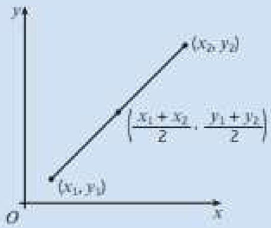
Past paper practice by topic. Both new and old specification can be found via this link on hgsmaths.com



Summary of Key Points

Summary of key points

1 The midpoint of a line segment with endpoints (x_1, y_1) and (x_2, y_2) is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.



2 The perpendicular bisector of a line segment AB is the straight line that is perpendicular to AB and passes through the midpoint of AB.

If the gradient of AB is m then the gradient of its perpendicular bisector, l, will be $-\frac{1}{m}$.

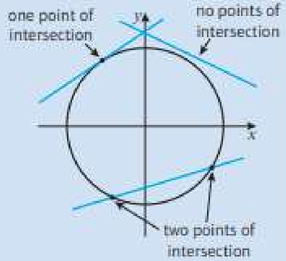
3 The equation of a circle with centre (0, 0) and radius r is $x^2 + y^2 = r^2$.

4 The equation of the circle with centre (a, b) and radius r is $(x - a)^2 + (y - b)^2 = r^2$.

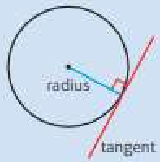
5 The equation of a circle can be given in the form: $x^2 + y^2 + 2fx + 2gy + c = 0$

This circle has centre $(-f, -g)$ and radius $\sqrt{f^2 + g^2 - c}$

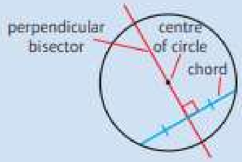
6 A straight line can intersect a circle once, by just touching the circle, or twice. Not all straight lines will intersect a given circle.



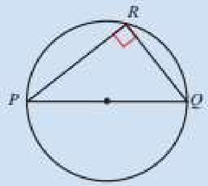
7 A tangent to a circle is perpendicular to the radius of the circle at the point of intersection.



8 The perpendicular bisector of a chord will go through the centre of a circle.



9
- If $\angle PRQ = 90°$ then R lies on the circle with diameter PQ.
- The angle in a semicircle is always a right angle.



10 To find the centre of a circle given any three points:
- Find the equations of the perpendicular bisectors of two different chords.
- Find the coordinates of intersection of the perpendicular bisectors.