Exam-style practice: AS level

1 a A four digit number, $a_3a_2a_1a_0$ (with this notation we mean that for 1485 we have $a_3=1,\ a_2=4,\ a_1=8$ and $a_0=5$) can be written as $10^3a_3+10^2a_2+10a_1+a_0$.
Using modular arithmetic we can say that if $n=10^3a_3+10^2a_2+10a_1+a_0$ is divisible by 11 then n (mod 11) must be 0.
This is equivalent to saying that $(990+10)a_3+(99+1)a_2+(11-1)a_1+a_0$ (mod 11) must be 0.
Since we know that 990, 99 and 11 are all clearly divisible by 11, we can reduce the statement to say that $-a_3+a_2-a_1+a_0\equiv 0$ (mod 11) (note that we have used that $10\equiv -1$ (mod 11)).
So with our example $-1+4-8+5=0\equiv 0$ (mod 11), thus we have 1485 divisible by 11.

b We first apply the Euclidean algorithm to 1485 and 143 in order to find their greatest common divisor.

$$1485=10\times143+55$$
$$143=2\times55+33$$
$$55=1\times33+22$$
$$33=1\times22+11$$
$$22=2\times11+0$$

so $\gcd(1485,143)=11$.

We now work backwards through the Euclidean algorithm in order to find solutions to the equation $1485p+143q=11$.

$$\begin{aligned}11&=33-1(22)\\&=33-(55-1(33))\\&=2(33)-1(55)\\&=2(143-2(55))-1(55)\\&=2(143)-5(55)\\&=2(143)-5(1485-10(143))\\&=52(143)-5(1485)\end{aligned}$$

Hence $p=-5$ and $q=52$.

1 c We found the values $p=-5$ and $q=52$ give us $-5(1485)+52(143)=11$ and so we just need to double both sides of this expression in order to see $-10(1485)+104(143)=22$.
Thus we have $a=-10$ and $b=104$.

2 a

$\times_{20}$	1	3	7	9	11	13	17	19
1	1	3	7	9	11	13	17	19
3	3	9	1	7	13	19	11	9
7	7	1	9	3	17	11	19	13
9	9	7	3	1	19	17	13	11
11	11	13	17	19	1	3	7	9
13	13	19	11	17	3	9	1	7
17	17	11	19	13	7	1	9	3
19	19	17	13	11	9	7	3	1

b By Lagrange's theorem, for a subgroup H, of a finite group G, the order of H should divide the order of G. But the group in question has order 8 and 3 does not divide 8. Thus there cannot be a subgroup of order 3 for this group.

c One way to find a subgroup of order 2 in this group is to find numbers which square to 1, since the only operations valid in this hypothetical subgroup are

$1\times1=1$

$1\times k=k$

$k\times1=k$

$k\times k=k^2\equiv1$

which are clearly only two elements.
Looking along the diagonal of the Cayley table gives us values of $k=9,\ k=11$ and $k=19$.
So three subgroups of $(G,\ \times_{20})$ of order two are $\{1,9\},\ \{1,11\}$ and $\{1,19\}$.

3 a The locus L defined by $|z-3| = \sqrt{2}|z+9|$ is a circle. We transform the equation into Cartesian form, substituting $z = x + iy$.

$|z-3| = \sqrt{2}|z+9|$

$\Rightarrow |x+iy-3| = \sqrt{2}|x+iy+9|$

$\Rightarrow |(x-3)+iy| = \sqrt{2}|(x+9)+iy|$

$\Rightarrow \sqrt{(x-3)^2+y^2} = \sqrt{2}\sqrt{(x+9)^2+y^2}$

$\Rightarrow (x-3)^2+y^2 = 2(x+9)^2+2y^2$

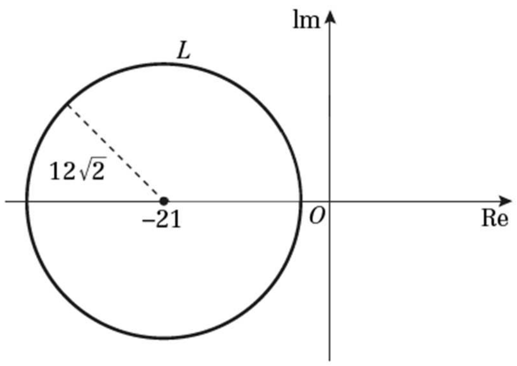
$\Rightarrow x^2-6x+9+y^2 = 2x^2+36x+162+2y^2$

$\Rightarrow x^2+42x+153+y^2 = 0$

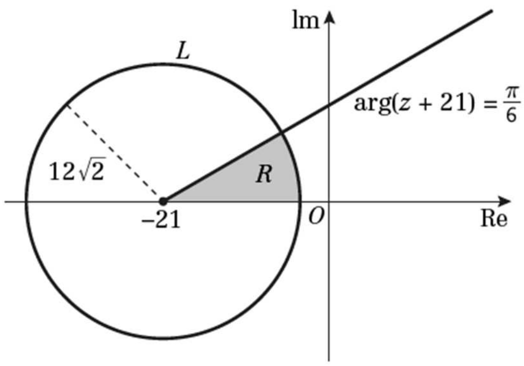
$\Rightarrow (x+21)^2-441+153+y^2 = 0$

$\Rightarrow (x+21)^2+y^2 = 288$

This is a circle centred at (−21, 0) with radius $\sqrt{288} = 12\sqrt{2}$.



b i The condition $0 \leqslant \arg(z+21) \leqslant \frac{\pi}{6}$ is a segment of the plane, between two half lines starting at (−21, 0) and with argument between 0 and $\frac{\pi}{6}$.



3 b ii The region R is the sector of a circle, radius $r = 12\sqrt{2}$, internal angle $\theta = \frac{\pi}{6}$

The area is therefore

$$\frac{r^2\theta}{2} = \frac{288 \times \frac{\pi}{6}}{2} = \frac{144\pi}{6} = 24\pi$$

4 a $\mathbf{M} - \lambda\mathbf{I} = \begin{pmatrix} 6 & 1-p \\ -3 & 4 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} 6-\lambda & 1-p \\ -3 & 4-\lambda \end{pmatrix}$

Therefore

$|\mathbf{M} - \lambda\mathbf{I}| = (6-\lambda)(4-\lambda)+3(1-p)$

$= 24-10\lambda+\lambda^2+3-3p$

$= \lambda^2-10\lambda-3p+27.$

Since there are two real solutions, we deduce that 'b^2-4ac' > 0

Therefore

$10^2-4\times1\times(27-3p) > 0$

$100-108+12p > 0$

$12p > 8$

Therefore $p > \frac{2}{3}$.

b i Let $f(\lambda) = \lambda^2-10\lambda-3p+27$

If one of the eigenvalues is 1, then $f(1) = 0$.

$f(1) = 1-10-3p+27 = 0$

$\Rightarrow 3p = 18$

$\Rightarrow p = 6$

ii Substituting $p = 6$:

$f(\lambda) = \lambda^2-10\lambda-18+27$

$= \lambda^2-10\lambda+9$

$= (\lambda-1)(\lambda-9)$

Therefore

$|\mathbf{M} - \lambda\mathbf{I}| = 0 \Rightarrow (\lambda-1)(\lambda-9) = 0$

So $\lambda = 1$ or $\lambda = 9$

4 c We find the eigenvectors using the equation

$$\begin{pmatrix} 6 & -5 \\ -3 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda\begin{pmatrix} x \\ y \end{pmatrix}.$$

For $\lambda = 1$:

$6x - 5y = x$

$\Rightarrow x = y.$

So, a corresponding eigenvector is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

For $\lambda = 9$:

$6x - 5y = 9x,$

$\Rightarrow y = -\dfrac{3x}{5}.$

Choosing $x = 5$ gives $y = -3$

Therefore, a corresponding eigenvector is

$\begin{pmatrix} 5 \\ -3 \end{pmatrix}$

So $\mathbf{P} = \begin{pmatrix} 1 & 5 \\ 1 & -3 \end{pmatrix}$ and $\mathbf{D} = \begin{pmatrix} 1 & 0 \\ 0 & 9 \end{pmatrix}$.

5 a Since we have an initial condition of 500g, we set $c = 500$.

Since we have a constant value of 50g added to the dishwasher periodically, we set $b = 50$

Finally, we know that the dishwasher uses its salt at a rate of 30% per week. Therefore there is 70% left of the salt from the week before at the start of the new week.

So we set $a = 0.7$.

b Substituting the values above into the recurrence relation gives us

$s_n = 0.7s_{n-1} + 50,$

$s_0 = 500.$

First we solve the homogeneous recurrence relation $s_n = 0.7s_{n-1}$ in order to find the complementary function

$s_n = A(0.7)^n.$

Now we try the particular solution $s_n = \lambda$;

$s_n = 0.7s_{n-1} + 50$

$\lambda = 0.7\lambda + 50$

$\lambda = \dfrac{50}{0.3} = \dfrac{500}{3}.$

So we have a general solution of

$s_n = A(0.7)^n + \dfrac{500}{3}$

Using the boundary condition:

$s_0 = 500 = A(0.7)^0 + \dfrac{500}{3}$

$\Rightarrow A = \dfrac{1000}{3}.$

So the closed form is

$$s_n = \frac{1000}{3}(0.7)^n + \frac{500}{3}$$
$$= \frac{500}{3}\left(2(0.7)^n + 1\right).$$

c The value of the salt in grams at the end of the eleventh week was

$s_{11} - 50 = \dfrac{500}{3}\left(2(0.7)^{11} + 1\right) - 50 \approx 123.26$

and at the very end of the twelfth week before the top up,

$s_{12} - 50 = \dfrac{500}{3}\left(2(0.7)^{12} + 1\right) - 50 \approx 121.28.$

This means the value of x can at most be 123.2 since if it were 123.3 then the light would have turned on during the eleventh week. The lowest value x can be is 121.3 since if it were 121.2 then it would not have yet turned on.

Thus, $121.3 \leqslant x \leqslant 123.2.$