Exam-style Practice

1 a A general point on line l_1 is

$$\begin{pmatrix}-3\\0\\5\end{pmatrix}+\lambda\begin{pmatrix}5\\-1\\1\end{pmatrix}=\begin{pmatrix}-3+5\lambda\\-\lambda\\5+\lambda\end{pmatrix}$$

A general point on line l_2 is

$$\begin{pmatrix}10\\-1\\15\end{pmatrix}+\mu\begin{pmatrix}6\\-2\\4\end{pmatrix}=\begin{pmatrix}10+6\mu\\-1-2\mu\\15+4\mu\end{pmatrix}$$

If l_1 and l_2 intersect, then there are unique values of λ and μ such that

$$-3+5\lambda=10+6\mu \qquad \textbf{(1)}$$
$$-\lambda=-1-2\mu \qquad \textbf{(2)}$$
$$5+\lambda=15+4\mu \qquad \textbf{(3)}$$

Substituting **(2)** into **(3)** gives

$$5+(1+2\mu)=15+4\mu$$
$$-9=2\mu$$
$$\mu=-\frac{2}{9}$$

Substituting $\mu=-\frac{2}{9}$ back into **(2)** gives

$$\lambda=1+2\left(-\frac{2}{9}\right)=\frac{5}{9}$$

Determine if $\mu=-\frac{2}{9}$ and $\lambda=\frac{5}{9}$ are consistent with **(1)**:

$$\text{LHS}=-3+5\left(\frac{5}{9}\right)=-\frac{2}{9}$$
$$\text{RHS}=10+6\left(-\frac{2}{9}\right)=\frac{26}{3}$$

$-\frac{2}{9}\neq\frac{26}{3}$ so the system of equations is not consistent, and hence l_1 and l_2 do not meet.

2 a

$$\det\begin{vmatrix}2&k&3\\1&-3&1\\3&-1&2\end{vmatrix}=2\begin{vmatrix}-3&1\\-1&2\end{vmatrix}-k\begin{vmatrix}1&1\\3&2\end{vmatrix}+3\begin{vmatrix}1&-3\\3&-1\end{vmatrix}$$
$$=2(-6+1)-k(2-3)+3(-1+9)$$
$$=-10+k+24=k+14$$

b The three planes do not meet at a single point, so

$$\det\begin{vmatrix}2&k&3\\1&-3&1\\3&-1&2\end{vmatrix}=0$$
$$k+14=0$$
$$k=-14$$

The three equations then become:

$$2x-14y+3z=1 \qquad \textbf{(1)}$$
$$x-3y+z=-2 \qquad \textbf{(2)}$$
$$3x-y+2z=3 \qquad \textbf{(3)}$$

Determine whether this system has any solutions:

$\textbf{(3)}-2\times\textbf{(2)}$: $x+5y=7$ **(4)**

$3\times(\textbf{2})-1$: $x+5y=-7$ **(5)**

Equations **(4)** and **(5)** are inconsistent so the system is inconsistent and has no solutions. The planes form a prism.

3 Let $w=2x-1 \Rightarrow x=\frac{w+1}{2}$

Substituting $x=\frac{w+1}{2}$ in the cubic equation:

$$2\left(\frac{w+1}{2}\right)^3-3\left(\frac{w+1}{2}\right)^2-7\left(\frac{w+1}{2}\right)-1=0$$
$$2\left(\frac{w^3+3w^2+3w+1}{8}\right)-3\left(\frac{w^2+2w+1}{4}\right)-7\left(\frac{w+1}{2}\right)-1=0$$
$$w^3-17w-20=0$$
$$p=1,\ q=0,\ r=-17,\ s=-20$$

4 a $n=1$: LHS $= \sum_{r=1}^{1} r^3 = 1^3 = 1$

RHS $= \frac{1}{4}(1)^2((1)+1)^2 = \frac{4}{4} = 1$

As LHS = RHS, the summation formula is true for $n = 1$.

Assume that the summation formula is true for $n = k$:

$$\sum_{r=1}^{k} r^3 = \frac{1}{4}(k)^2(k+1)^2$$

With $n = k + 1$ terms the summation formula becomes:

$$\sum_{r=1}^{k+1} r^3 = \sum_{r=1}^{k} r^3 + (k+1)^3$$
$$= \frac{1}{4}k^2(k+1)^2 + (k+1)^3$$
$$= \frac{1}{4}(k+1)^2(k^2 + 4(k+1))$$
$$= \frac{1}{4}(k+1)^2(k^2+4k+4)$$
$$= \frac{1}{4}(k+1)^2(k+2)^2$$
$$= \frac{1}{4}(k+1)^2\left[((k+1)+1)^2\right]$$

Therefore, the summation formula is true when $n = k+1$. If the summation formula is true for $n = k$ then it is shown to be true for $n = k + 1$. As the result is true for $n = 1$, it is now also true for all $n \in \mathbb{Z}^+$ by mathematical induction.

b $\sum_{r=1}^{n} 2r(r+1) = 2\sum_{r=1}^{n} r^2 + 2\sum_{r=1}^{n} r$

$$= 2 \times \frac{1}{6}n(n+1)(2n+1) + 2 \times \frac{1}{2}n(n+1)$$
$$= 2n(n+1)\left(\frac{1}{6}(2n+1) + \frac{1}{2}\right)$$
$$= 2n(n+1)\left(\frac{n+2}{3}\right)$$
$$= \frac{2}{3}n(n+1)(n+2)$$

c $\sum_{r=1}^{n} 2r(r+1) = 4\sum_{r=1}^{n} r^3$

$\Rightarrow \frac{2}{3}n(n+1)(n+2) = n^2(n+1)^2$ for positive integers n, by parts **a** and **b**

$\Rightarrow 3n^4 + 4n^3 - 3n^2 - 4n = 0$

$\Rightarrow n(n-1)(n+1)(3n+4) = 0$

So $n = 0, 1, -1$ or $-\frac{4}{3}$

However, the result only holds for positive integers, so $n = 1$

5 a $(3+2\text{i})$ and $(3-2\text{i})$ are complex conjugate roots of $\text{f}(z) = 0$

So $(z-(3+2\text{i}))(z-(3-2\text{i}))$ is a factor of $\text{f}(z)$.

$(z-(3+2\text{i}))(z-(3-2\text{i}))$

$= z^2 - (3+2\text{i}+3-2\text{i})z + (3+2\text{i})(3-2\text{i})$

$= z^2 - 6z + 3^2 + 6\text{i} - 6\text{i} - (2i)^2$

$= z^2 - 6z + 9 - (-4)$

$= z^2 - 6z + 13$

So $(z^2 - 6z + 13)$ is a factor of $\text{f}(z)$.

5 b $\left(z^2-6z+13\right)\left(az^2+bz+c\right)$

$$=z^4-14z^3+78z^2+kz+221$$

Equate coefficients of z^4 :

$a=1$

Equate coefficients of z^3 :

The z^3 terms on the LHS are $z^2\times bz$ and $-6z\times az^2$, so

$$bz^3-6az^3=-14z^3$$
$$b-6a=-14$$
$$b-6\times(1)=-14$$
$$b=-8$$

Equate constant terms:

$$13c=221$$
$$c=17$$

To find k, equate coefficients of z :

The z terms on the LHS are $-6z\times c$ and $13\times bz$,

so $\quad -6cz+13bz=kz$

$$-6c+13b=k$$
$$-6\times(17)+13\times(-8)=k$$
$$k=-206$$

c $\left(z^2-6z+13\right)\left(z^2-8z+17\right)$

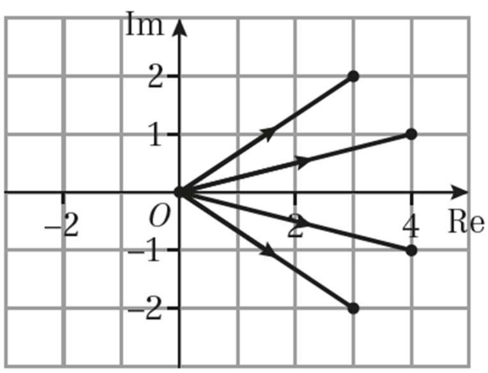
$$=z^4-14z^3+78z^2-206z+221$$

Solving $\left(z^2-8z+17\right)=0$

$$(z-4)^2+1=0$$
$$(z-4)^2=-1$$
$$z-4=\pm i$$
$$z=4\pm i$$

So the roots of $f(z)=0$ are

$3+2i,\ 3-2i,\ 4+i$ and $4-i$



6 a $\sqrt{1^2+(-1)^2+2^2}=\sqrt{6}$

$$\hat{\mathbf{n}}=\begin{pmatrix}\frac{1}{\sqrt{6}}\\-\frac{1}{\sqrt{6}}\\\frac{2}{\sqrt{6}}\end{pmatrix}$$

b $\mathbf{r}=\begin{pmatrix}0\\3\\-1\end{pmatrix}+\lambda\begin{pmatrix}2\\4\\-3\end{pmatrix}$

The cartesian coordinates of any point on this line are

$$x=2\lambda$$
$$y=3+4\lambda$$
$$z=-1-3\lambda$$

Sub these coordiantes into the equation of Π :

$$1(2\lambda)-1(3+4\lambda)+2(-1-3\lambda)=3$$
$$-8\lambda-8=0$$
$$-8\lambda=8\Rightarrow\lambda=-1$$

Now use $\lambda=-1$ to find coordiantes of P :

$$x=2(-1)\Rightarrow x=-2$$
$$y=3+4(-1)\Rightarrow y=-1$$
$$z=-1-3(-1)\Rightarrow z=1$$

Coordinates of P: $(-2,-1,\ 2)$

Now let θ be the angle between the line l and the *normal* to the plane.

$$\cos\theta=\frac{\begin{pmatrix}1\\-1\\2\end{pmatrix}\cdot\begin{pmatrix}2\\4\\-3\end{pmatrix}}{\sqrt{1^2+1^2+2^2}\sqrt{2^2+4^2+3^2}}$$

$$=\frac{|2-4-6|}{\sqrt{6}\sqrt{29}}=0.606$$

$$\theta=0.92$$

Then the angle between l and Π is

$$\frac{\pi}{2}-\theta=\frac{\pi}{2}-0.92=0.65$$

7 a $4x - x^2 = 0$

$x(4 - x) = 0$

$x = 0$ or $x = 4$

$$V = \pi\int_0^4 \left(4x - x^2\right)^2 \, dx$$

$$= \pi\int_0^4 x^4 - 8x^3 + 16x^2 \, dx$$

$$= \pi\left[\frac{x^5}{5} - \frac{8x^4}{4} + \frac{16x^3}{3}\right]_0^4$$

$$= \pi\left[\frac{x^5}{5} - 2x^4 + \frac{16x^3}{3}\right]_0^4$$

$$= \pi\left[\left(\frac{(4)^5}{5} - 2(4)^4 + \frac{16(4)^3}{3}\right) - 0\right]$$

$$= \frac{512\pi}{15} \text{ mm}^3$$

Cost of silver for 500 beads:

$$£0.05 \times 500 \times \frac{512\pi}{15} = £2680.83$$

b The model does not account for a hole through the middle of the bead.

8 a
$$\det\mathbf{M} = \begin{vmatrix} -\frac{5}{\sqrt{2}} & -\frac{5}{\sqrt{2}} \\ \frac{5}{\sqrt{2}} & -\frac{5}{\sqrt{2}} \end{vmatrix}$$

$$= -\frac{5}{\sqrt{2}} \times -\frac{5}{\sqrt{2}} - \left(-\frac{5}{\sqrt{2}}\right) \times \frac{5}{\sqrt{2}}$$

$$= \frac{25}{2} + \frac{25}{2} = 25$$

Area scale factor $= 25$

Linear scale factor of enlargement $= \sqrt{25} = 5$

b
$$\begin{pmatrix} -\frac{5}{\sqrt{2}} & -\frac{5}{\sqrt{2}} \\ \frac{5}{\sqrt{2}} & -\frac{5}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 5\cos\theta & -5\sin\theta \\ 5\sin\theta & 5\cos\theta \end{pmatrix}$$

$$5\sin\theta = \frac{5}{\sqrt{2}}$$

$$\sin\theta = \frac{5}{5\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$\theta = 45^\circ \Rightarrow 180^\circ - 45^\circ = 135^\circ$

Check using the lower-right element:

$5\cos 135^\circ = -\frac{5}{\sqrt{2}}$ so $\theta = 135^\circ$

So $\mathbf{M}$ is a rotation anticlockwise throught 135°.

c Let the coordinates of P be (x, y). Then

$$\begin{pmatrix} -\frac{5}{\sqrt{2}} & -\frac{5}{\sqrt{2}} \\ \frac{5}{\sqrt{2}} & -\frac{5}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{M}^{-1}\begin{pmatrix} a \\ b \end{pmatrix}$$

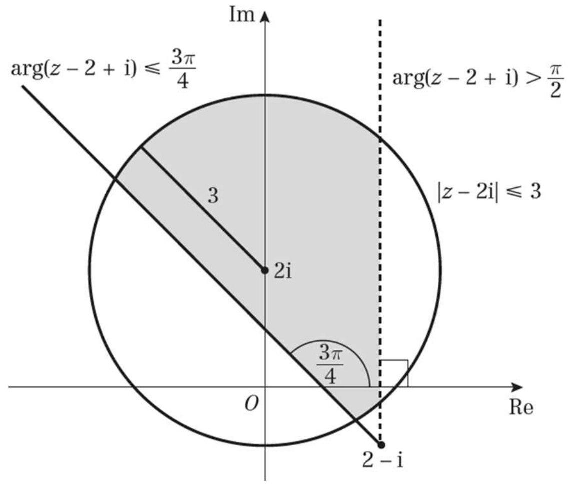
$$= \frac{1}{25}\begin{pmatrix} -\frac{5}{\sqrt{2}} & \frac{5}{\sqrt{2}} \\ -\frac{5}{\sqrt{2}} & -\frac{5}{\sqrt{2}} \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{5\sqrt{2}} & \frac{1}{5\sqrt{2}} \\ -\frac{1}{5\sqrt{2}} & -\frac{1}{5\sqrt{2}} \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix}$$

$$= \begin{pmatrix} \frac{-a+b}{5\sqrt{2}} \\ \frac{-a-b}{5\sqrt{2}} \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

The coordinates of P are $\left(\frac{-a+b}{5\sqrt{2}}, \frac{-a-b}{5\sqrt{2}}\right)$

9



$|z-2\mathrm{i}|=3$ represents a circle centred (0, 2) with radius 3.

$|z-2\mathrm{i}|\le 3$ is the area inside this circle

$\arg(z-2+\mathrm{i})=\frac{3\pi}{4}$ is the half-line from the point $2-\mathrm{i}$ which makes an angle $\frac{3\pi}{4}$ with the positive real axis.

$\arg(z-2+\mathrm{i})=\frac{\pi}{2}$ is is the half-line from the point $2-\mathrm{i}$ which makes an angle $\frac{\pi}{2}$ with the positive real axis.

$$\{z\in\mathbb{C}:|z-2\mathrm{i}|\leqslant 3\}\cap\left\{z\in\mathbb{C}:\frac{\pi}{2}<\arg(z-2+\mathrm{i})\leqslant\frac{3\pi}{4}\right\}$$

is the area inside the circle $|z-2\mathrm{i}|\leqslant 3$ which lies between the two half-lines $\arg(z-2+\mathrm{i})=\frac{\pi}{2}$ and $\arg(z-2+\mathrm{i})=\frac{3\pi}{4}$

The half line $\arg(z-2+\mathrm{i})=\frac{\pi}{2}$ is not included in the region, so is shown dotted on the Argand diagram.

10 a The stolen car is travelling along the line

$$\overrightarrow{AB}=\begin{pmatrix}5\\3\\-1\end{pmatrix}+t\begin{pmatrix}2\\-8\\9\end{pmatrix}$$

So a general point on the line $\overrightarrow{AB}$ has position vector $\begin{pmatrix}5+2t\\3-8t\\-1+9t\end{pmatrix}$.

The stinger can intercept the car if the perpendicular distance from the origin to $\overrightarrow{AB}$ is less than 6 m.

The perpendicular distance occurs when

$$\begin{pmatrix}5+2t\\3-8t\\-1+9t\end{pmatrix}\cdot\begin{pmatrix}2\\-8\\9\end{pmatrix}=0$$

$$10+4t-24+64t-9+81t=0$$
$$149t-23=0$$
$$149t=23$$
$$t=\frac{23}{149}$$

Substitute $t=\frac{23}{149}$ into the expression for a general point on the line $\overrightarrow{AB}$:

$$5+2\left(\frac{23}{149}\right)=\frac{791}{149}$$
$$3-8\left(\frac{23}{149}\right)=\frac{263}{149}$$
$$-1+9\left(\frac{23}{149}\right)=\frac{58}{149}$$

$$\left|\overrightarrow{AB}\right|_{\min}=\sqrt{\frac{791^2+263^2+58^2}{149^2}}$$
$$=5.6<6$$

The stolen car passes 5.6 m from the origin. The police can stop the car.

b Limitation of the model – the car is unlikely to drive exactly in a straight line, as roads are not straight.