YEAR 12 & 13 FURTHER MATHEMATICS

TMUA Problems

Test of Mathematics for University Admission past questions, arranged by our A level units

359 questions · 13 chapters · full worked solutions

How to use this book

Work on these when you have finished your Your Turns and problems.
Each chapter is one of our units: only try a chapter once you have been taught that unit.
Within a chapter the questions get harder as you go.
Every question is multiple choice with one correct option. Show your working in your exercise book, and do not use a calculator — you cannot in the real test.
In the TMUA you get 75 minutes for 20 questions: about 3½ minutes each.
Check your answer against the key, then read the worked solution, even when you got it right: the solutions often show a quicker way.

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CHAPTER 1 · UNIT P1

Algebra and Functions 1

40 questions

1.1

TMUA Specimen · Paper 1 · Q1

The sum of the two values of x that satisfy the simultaneous equations

$x - 3y + 1 = 0$ and $3x^2 - 7xy = 5$ is

A -8.5

B -7.5

C -1.5

D 3.5

E 4.5

F 5

1.2

TMUA 2019 · Paper 1 · Q1

$\mathrm{f}(x)$ is a quadratic function in x.

The graph of $y = \mathrm{f}(x)$ passes through the point $(1, -1)$ and has a turning point at $(-1, 3)$.

Find an expression for $\mathrm{f}(x)$.

A $-x^2 - 2x + 2$

B $-x^2 + 2x + 3$

C $x^2 - 2x$

D $x^2 + 2x - 4$

E $2x^2 + 4x + 1$

F $-2x^2 - 4x + 5$

1.3

TMUA 2020 · Paper 2 · Q1

Find the complete set of values of k for which the line $y = x - 2$ crosses or touches the curve $y = x^2 + kx + 2$

A $-1 \leq k \leq 3$

B $-3 \leq k \leq 5$

C $-4 \leq k \leq 4$

D $k \leq -1$ or $k \geq 3$

E $k \leq -3$ or $k \geq 5$

F $k \leq -4$ or $k \geq 4$

1.4

TMUA 2023 · Paper 2 · Q1

Given that

$$\frac{1}{\sqrt{x}-6}-\frac{1}{\sqrt{x}+6}=\frac{3}{11}$$

what is the value of x?

A $2\sqrt{15}$
B $4\sqrt{5}$
C $5\sqrt{2}$
D $\sqrt{58}$
E 50
F 58
G 60
H 80

1.5

TMUA 2016 · Paper 1 · Q2

The expression $3x^3 + 13x^2 + 8x + a$, where a is a constant, has $(x + 2)$ as a factor.

Which one of the following is a complete factorisation of the expression?

A $(x+2)(x-1)(3x-2)$
B $(x+2)(x+1)(3x-2)$
C $(x+2)(x+1)(3x+2)$
D $(x+2)(x-3)(3x+2)$
E $(x+2)(x+3)(3x-2)$
F $(x+2)(x+3)(3x+2)$

1.6

TMUA 2019 · Paper 1 · Q2

Find the complete set of values of the real constant k for which the expression

$$x^2 + kx + 2x + 1 - 2k$$

is positive for all real values of x.

A $-12 < k < 0$
B $k < -12 \quad \text{or} \quad k > 0$
C $-\sqrt{6} - 3 < k < \sqrt{6} - 3$
D $k < -\sqrt{6} - 3 \quad \text{or} \quad k > \sqrt{6} - 3$
E $-2 < k < \frac{1}{2}$
F $k < -2 \quad \text{or} \quad k > \frac{1}{2}$
G $0 < k < 4$
H $k < 0 \quad \text{or} \quad k > 4$

1.7

TMUA 2023 · Paper 1 · Q2

The graphs of $y = x^2 + 5x + 6$ and $y = mx - 3$, where m is a constant, are plotted on the same set of axes.

Given that the graphs do not meet, what is the complete range of possible values of m?

A $-1 < m < 11$

B $m < -1,\ m > 11$

C $-\sqrt{11} < m < \sqrt{11}$

D $m < -\sqrt{11},\ m > \sqrt{11}$

E $-11 < m < 1$

F $m < -11,\ m > 1$

1.8

TMUA 2019 · Paper 2 · Q2 (also 2020 · Paper 1 · Q2)

$(2x + 1)$ and $(x - 2)$ are factors of $2x^3 + px^2 + q$.

What is the value of $2p + q$?

A -10

B $-\frac{38}{5}$

C $-\frac{22}{3}$

D $\frac{22}{3}$

E $\frac{38}{5}$

F 10

1.9

TMUA 2017 · Paper 1 · Q4

When $(3x^2 + 8x - 3)$ is multiplied by $(px - 1)$ and the resulting product is divided by $(x + 1)$, the remainder is 24.

What is the value of p?

A -4

B 2

C 4

D $\frac{8}{7}$

E $\frac{11}{4}$

1.10

TMUA 2018 · Paper 1 · Q4

Consider the simultaneous equations

$$3x^2 + 2xy = 4$$
$$x + y = a$$

where a is a real constant.

Find the complete set of values of a for which the equations have two distinct real solutions for x.

A There are no values of a.

B $-2 < a < 2$

C $-1 < a < 1$

D $a = 0$

E $a < -1$ or $a > 1$

F $a < -2$ or $a > 2$

G All real values of a

1.11

TMUA 2017 · Paper 1 · Q5

S is the complete set of values of x which satisfy **both** the inequalities

$$x^2 - 8x + 12 < 0 \text{ and } 2x + 1 > 9$$

The set S can also be represented as a single inequality.

Which one of the following single inequalities represents the set S ?

A $(x^2 - 8x + 12)(2x + 1) < 0$

B $(x^2 - 8x + 12)(2x + 1) > 0$

C $x^2 - 10x + 24 < 0$

D $x^2 - 10x + 24 > 0$

E $x^2 - 6x + 8 < 0$

F $x^2 - 6x + 8 > 0$

G $x < 2$

H $x > 6$

1.12

TMUA 2018 · Paper 1 · Q5

The function f is defined by $f(x) = x^3 + ax^2 + bx + c$.

a, b and c take the values 1, 2 and 3 with no two of them being equal and not necessarily in this order.

The remainder when $f(x)$ is divided by $(x + 2)$ is R.

The remainder when $f(x)$ is divided by $(x + 3)$ is S.

What is the largest possible value of $R - S$?

A -26
B 5
C 7
D 17
E 29

1.13

TMUA 2020 · Paper 1 · Q5

The curve S has equation

$$y = px^2 + 6x - q$$

where p and q are constants.

S has a line of symmetry at $x = -\frac{1}{4}$ and touches the x-axis at exactly one point.

What is the value of $p + 8q$?

A 6
B 18
C 21
D 25
E 38

1.14

TMUA 2021 · Paper 1 · Q5

The function f is such that

$$f(mn) = \begin{cases} f(m)f(n) & \text{if } mn \text{ is a multiple of 3} \\ mn & \text{if } mn \text{ is not a multiple of 3} \end{cases}$$

for all positive integers m and n.

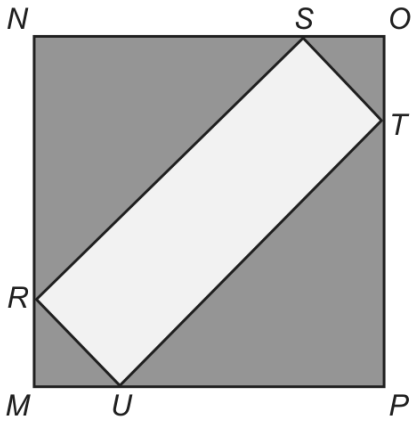
Given that $f(9) + f(16) - f(24) = 0$, what is the value of $f(3)$?

A $\frac{8}{3}$
B $2\sqrt{2}$
C 3
D $\frac{16}{5}$
E $3\sqrt{2}$
F 4

1.15

TMUA 2023 · Paper 1 · Q5

The following shape has two lines of reflectional symmetry.



[diagram not to scale]

MNOP is a square of perimeter 40 cm.

The vertices of rectangle *RSTU* lie on the edge of square *MNOP*.

MR has length x cm.

What is the largest possible value of x such that *RSTU* has area 20 cm^2?

A $\sqrt{2}$

B $\sqrt{10}$

C $2\sqrt{15}$

D $10\sqrt{2}$

E $5+\sqrt{5}$

F $5+\sqrt{15}$

1.16

TMUA Specimen · Paper 1 · Q6

It is given that $x+2$ is a factor of $x^3+4cx^2+x(c+1)^2-6$.

The sum of the possible values of c is

A -10

B -6

C 0

D 6

E 10

1.17

TMUA 2016 · Paper 1 · Q6

P, Q, and R are each mixtures of red and white paint.

The percentage by volume of red paint in P is 30%.

The percentage by volume of red paint in Q is 20%.

The mixtures P, Q, and R are combined in the proportion 12 : 5 : 3 respectively.

If the resulting mixture contains 25% by volume of red paint, what percentage by volume of mixture R is red paint?

A 25%

B 23%

C $13\frac{1}{3}\%$

D $19\frac{1}{2}\%$

E $9\frac{3}{4}\%$

F It is impossible to achieve this result.

1.18

TMUA Specimen · Paper 2 · Q6

The area of a rectangle is measured to be 5600 cm^2 correct to 2 significant figures.

The width of the rectangle is measured to be 80 cm correct to the nearest centimetre.

Which one of the following expressions gives the greatest possible height of the rectangle?

A 70.5 cm

B 75 cm

C $\frac{5650}{85}$ cm

D $\frac{5650}{80.5}$ cm

E $\frac{5650}{75}$ cm

F $\frac{5650}{79.5}$ cm

1.19

TMUA 2020 · Paper 1 · Q8

The function f is defined for all real x as

$$f(x) = (p - x)(x + 2)$$

Find the complete set of values of p for which the maximum value of $f(x)$ is less than 4.

A $-2 - 4\sqrt{2} < p < -2 + 4\sqrt{2}$

B $-2 - 2\sqrt{2} < p < -2 + 2\sqrt{2}$

C $-2\sqrt{5} < p < 2\sqrt{5}$

D $-6 < p < 2$

E $-4 < p < 0$

F $-2 < p < 2$

1.20

TMUA 2021 · Paper 1 · Q8

The line $y = 2x + 3$ meets the curve $y = x^2 + bx + c$ at exactly one point.

The line $y = 4x - 2$ also meets the curve $y = x^2 + bx + c$ at exactly one point.

What is the value of $b - c$?

A -9

B -5.5

C -1

D 5

E 6

F 14

1.21

TMUA Specimen · Paper 1 · Q9

The roots of the equation $2x^2 - 11x + c = 0$ differ by 2. The value of c is

A $\frac{105}{8}$

B $\frac{113}{8}$

C $\frac{117}{8}$

D $\frac{119}{8}$

1.22

TMUA 2020 · Paper 1 · Q9

The quadratic expression $x^2 - 14x + 9$ factorises as $(x - \alpha)(x - \beta)$, where α and β are positive real numbers.

Which quadratic expression can be factorised as $(x - \sqrt{\alpha})(x - \sqrt{\beta})$?

A $x^2 - \sqrt{10}x + 3$

B $x^2 - \sqrt{14}x + 3$

C $x^2 - \sqrt{20}x + 3$

D $x^2 - 178x + 81$

E $x^2 - 176x + 81$

F $x^2 + 196x + 81$

1.23

TMUA Specimen · Paper 1 · Q12

The cross-section of a triangular prism is an equilateral triangle with side $2x$ cm.
The length of the prism is d cm.

Let the total surface area of the prism be T cm^2. Given that the volume of the prism is T cm^3, which one of the following is an expression for d in terms of x?

A $\frac{x}{2x-3}$

B $\frac{3x}{3x-2\sqrt{3}}$

C $\frac{2x}{x-4\sqrt{3}}$

D $\frac{2x}{x-2\sqrt{3}}$

E $\frac{2x}{x-\sqrt{3}}$

1.24

TMUA 2021 · Paper 1 · Q12

The minimum value of the function $x^4 - p^2x^2$ is -9
p is a real number.

Find the minimum value of the function $x^2 - px + 6$

A -3

B $6 - \frac{3\sqrt{2}}{2}$

C $\frac{3}{2}$

D 3

E $\frac{9}{2}$

F $6 + \frac{3\sqrt{2}}{2}$

1.25

TMUA 2022 · Paper 1 · Q12

A family of quadratic curves is given by

$$y_k = 2\left(x - \frac{k}{2}\right)^2 + \frac{k^2}{2} + 4k + 3$$

where k is any real number and y_k is a function of x.

All these curves are sketched, and the point with the lowest y-coordinate among all the curves y_k is (a, b).

Find the value of $a + b$

A –1
B –3
C –5
D –7
E –9

1.26

TMUA Specimen · Paper 2 · Q12

A polynomial $p(x)$ has the property that $p(1) = 2$.

Which one of the following can be deduced from this?

A $p(x) = (x - 1)q(x) + 2$ for some polynomial $q(x)$.
B $p(x) = (x + 1)q(x) + 2$ for some polynomial $q(x)$.
C $p(x) = (x - 1)q(x) - 2$ for some polynomial $q(x)$.
D $p(x) = (x + 1)q(x) - 2$ for some polynomial $q(x)$.
E $p(x) = (x - 2)q(x) + 1$ for some polynomial $q(x)$.
F $p(x) = (x + 2)q(x) + 1$ for some polynomial $q(x)$.
G $p(x) = (x - 2)q(x) - 1$ for some polynomial $q(x)$.
H $p(x) = (x + 2)q(x) - 1$ for some polynomial $q(x)$.

1.27

TMUA Specimen · Paper 1 · Q13

How many real roots does the equation $x^4 - 4x^3 + 4x^2 - 10 = 0$ have?

A 0
B 1
C 2
D 3
E 4

Given that

$$\left(a^3+\frac{2}{b^3}\right)\left(\frac{2}{a^3}-b^3\right)=\sqrt{2}$$

where a and b are real numbers, what is the least value of ab?

A $-\sqrt{2}$

B $\sqrt{2}$

C $-2\sqrt{2}$

D $2\sqrt{2}$

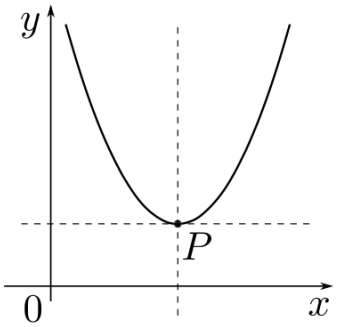
E $-\frac{\sqrt{2}}{2}$

F $\frac{\sqrt{2}}{2}$

G $-2^{\frac{1}{6}}$

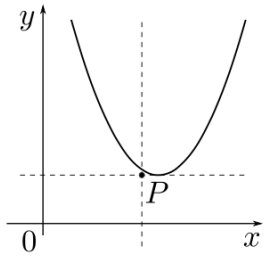
H $2^{\frac{1}{6}}$

The diagram below shows the graph of $y = x^2 - 2bx + c$. The vertex of this graph is at the point P.

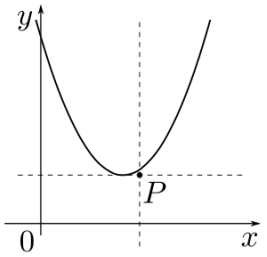


Which one of the following could be the graph of $y = x^2 - 2Bx + c$, where $B > b$?

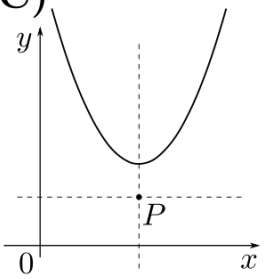
(A)



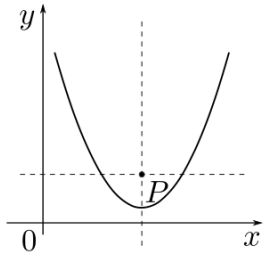
(B)



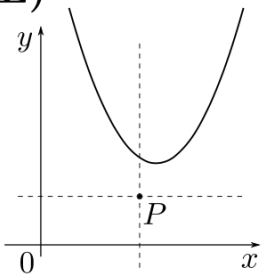
(C)



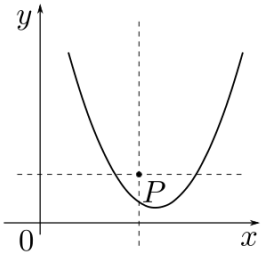
(D)



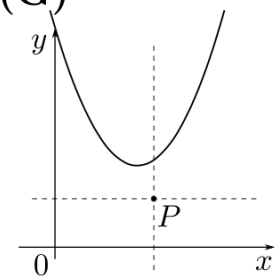
(E)



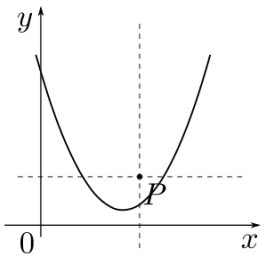
(F)



(G)



(H)



1.30

TMUA 2016 · Paper 2 · Q15

Consider the quadratic $f(x) = x^2 - 2px + q$ and the statement:

(*) $f(x) = 0$ has two real roots whose difference is greater than 2 and less than 4.

Which one of the following statements is true **if and only if** (*) is true?

A $q < p^2 < q + 4$

B $\sqrt{q+1} < p < \sqrt{q+4}$

C $q - 3 \leq p^2 - 4 \leq q$

D $q < p^2 - 1 < q + 3$

E $q - 2 < p^2 - 3 < q + 2$

1.31

TMUA Specimen · Paper 1 · Q16

Given that c and d are non-zero integers, the expression $\frac{10^{c-2d} \times 20^{2c+d}}{8^c \times 125^{c+d}}$ is an integer if

A $c < 0$

B $d < 0$

C $c < 0$ and $d < 0$

D $c < 0$ and $d > 0$

E $c > 0$ and $d < 0$

F $c > 0$ and $d > 0$

G $d > 0$

H $c > 0$

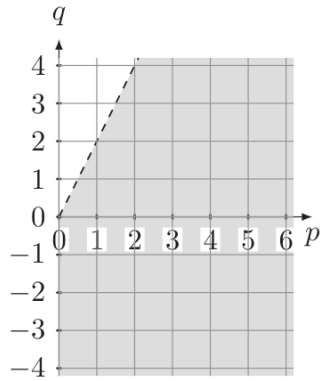
The graph of the quadratic

$$y = px^2 + qx + p$$

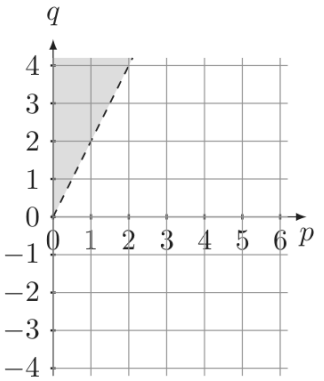
where $p > 0$, intersects the x-axis at two distinct points.

In which one of the following graphs does the **shaded** region show the complete set of possible values that p and q could take?

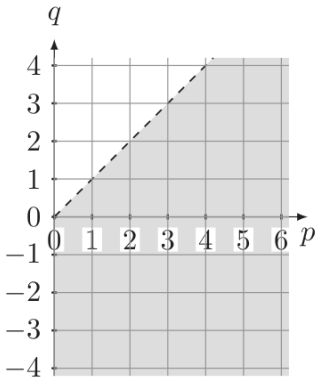
A



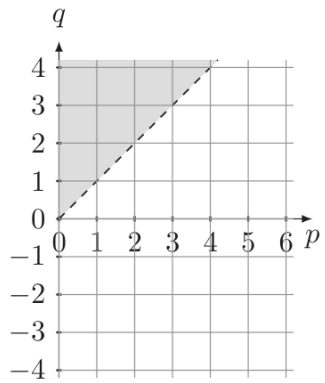
B



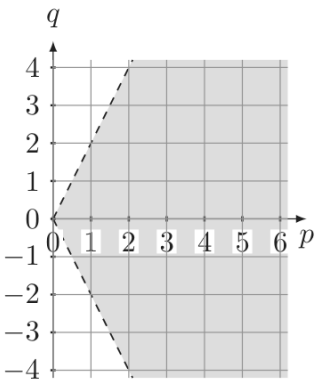
C



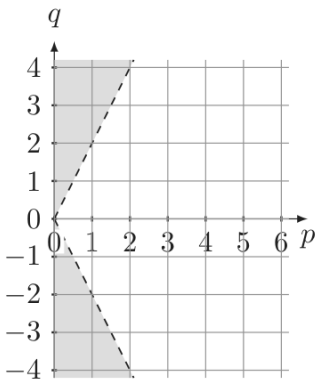
D



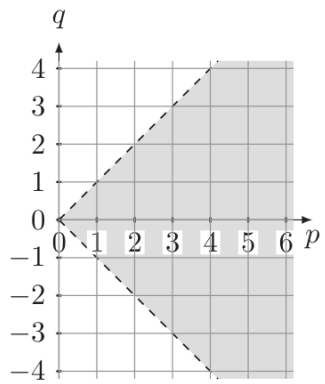
E



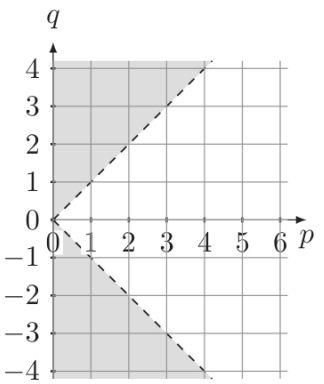
F



G



H



1.33

TMUA Specimen · Paper 1 · Q17

For what values of the non-zero real number a does the quadratic equation $ax^2 + (a-2)x = 2$ have real distinct roots?

A All values of a

B $a = -2$

C $a > -2$

D $a \neq -2$

E No values of a

1.34

TMUA 2020 · Paper 1 · Q17

Find the complete set of values of m in terms of c such that the graphs of $y = mx + c$ and $y = \sqrt{x}$ have two points of intersection.

A $0 < m < \frac{1}{4c}$

B $0 < m < 4c^2$

C $m > \frac{1}{4c}$

D $m < \frac{1}{4c}$

E $m > 4c^2$

F $m < 4c^2$

1.35

TMUA 2023 · Paper 2 · Q18

The equation $x^4 + bx^2 + c = 0$ has four distinct real roots **if and only if** which of the following conditions is satisfied?

A $b^2 > 4c$

B $b^2 < 4c$

C $c > 0$ **and** $b > 2\sqrt{c}$

D $c > 0$ **and** $b < -2\sqrt{c}$

E $c < 0$ **and** $b < 0$

F $c < 0$ **and** $b > 0$

1.36

TMUA 2017 · Paper 1 · Q19

The set of solutions to the inequality $x^2 + bx + c < 0$ is the interval $p < x < q$ where b, c, p and q are real constants with $c < 0$.

In terms of p, q and c, what is the set of solutions to the inequality $x^2 + bcx + c^3 < 0$?

A $\frac{p}{c} < x < \frac{q}{c}$

B $\frac{q}{c} < x < \frac{p}{c}$

C $pc < x < qc$

D $qc < x < pc$

E $pc^2 < x < qc^2$

F $qc^2 < x < pc^2$

1.37

TMUA 2020 · Paper 1 · Q19

Find the lowest positive integer for which $x^2 - 52x - 52$ is positive.

A 26

B 27

C 51

D 52

E 53

F 54

1.38

TMUA 2019 · Paper 2 · Q19

Find the value of the expression

$$\sqrt{8 - 4\sqrt{2} + 1} + \sqrt{9 - 12\sqrt{2} + 8}$$

A $\sqrt{26 - 16\sqrt{2}}$

B $4\sqrt{2} - 4$

C -2

D $4 - 4\sqrt{2}$

E 2

F $\sqrt{26} - 4\sqrt{2}$

G 1

1.39

TMUA 2020 · Paper 1 · Q20

For how many values of a is the equation

$$(x-a)(x^2-x+a)=0$$

satisfied by exactly two distinct values of x?

A 0
B 1
C 2
D 3
E 4
F more than 4

1.40

TMUA 2018 · Paper 2 · Q20

It is given that the equation $\sqrt{x+p}+\sqrt{x}=p$ has at least one real solution for x, where p is a real constant.

What is the complete set of possible values for p?

A $p=0$ or $p=1$
B $p=0$ or $p\geq 1$
C $p\geq -x$
D $p\geq \sqrt{x}$
E $p\geq 0$
F $p\geq 1$

CHAPTER 2 · UNIT P2

Coordinate Geometry 1

21 questions

2.1

TMUA 2021 · Paper 1 · Q1

Two circles have the same radius.

The centre of one circle is $(-2, 1)$.

The centre of the other circle is $(3, -2)$.

The circles intersect at two distinct points.

What is the equation of the straight line through the two points at which the circles intersect?

A $3x - 5y = 4$

B $3x + 5y = -1$

C $5x - 3y = -4$

D $5x - 3y = -1$

E $5x - 3y = 1$

F $5x - 3y = 4$

G $5x + 3y = 1$

2.2

TMUA Specimen · Paper 2 · Q1

The radius of the circle $2x^2 + 2y^2 - 8x + 12y + 15 = 0$ is

A $\sqrt{\frac{5}{2}}$

B $\sqrt{\frac{11}{2}}$

C $\sqrt{\frac{41}{2}}$

D $\sqrt{37}$

E $\sqrt{67}$

2.3 TMUA 2022 · Paper 1 · Q2

Find the complete set of values of p for which the equation

$$x^2 - 2px + y^2 - 6y - p^2 + 8p + 9 = 0$$

describes a circle in the xy-plane.

A $p < -\frac{9}{4}$

B $0 < p < 4$

C $-1 < p < 9$

D $p < 0$ or $p > 4$

E $p < -1$ or $p > 9$

F all real values of p

2.4 TMUA 2017 · Paper 2 · Q2

$PQRS$ is a rectangle.

The coordinates of P and Q are $(0, 6)$ and $(1, 8)$ respectively.

The perpendicular to PQ at Q meets the x-axis at R.

What is the area of $PQRS$?

A $\frac{5}{2}$

B $4\sqrt{10}$

C 20

D $8\sqrt{10}$

E 40

2.5 TMUA 2021 · Paper 2 · Q2

$A(0, 2)$ and $C(4, 0)$ are opposite vertices of the square $ABCD$.

What is the equation of the straight line through B and D?

A $y = -2x + 5$

B $y = -\frac{1}{2}x - 3$

C $y = -\frac{1}{2}x + 2$

D $y = x$

E $y = 2x - 3$

F $y = 2x + 2$

2.6

TMUA Specimen · Paper 1 · Q3

The perpendicular bisector of the line segment joining the points $(2, -6)$ and $(5, 4)$ cuts the x-axis at the point with x-coordinate

A $\frac{1}{20}$

B $\frac{1}{6}$

C $\frac{1}{3}$

D $\frac{19}{5}$

E $\frac{41}{6}$

2.7

TMUA 2017 · Paper 1 · Q3

A line l has equation $y = 6 - 2x$

A second line is perpendicular to l and passes through the point $(-6, 0)$.

Find the area of the region enclosed by the two lines and the x-axis.

A $16\frac{1}{5}$

B 18

C $21\frac{3}{5}$

D 27

E $40\frac{1}{2}$

2.8

TMUA 2018 · Paper 1 · Q3

Find the shortest distance between the two circles with equations:

$$(x+2)^2 + (y-3)^2 = 18$$
$$(x-7)^2 + (y+6)^2 = 2$$

A 0

B 4

C 16

D $2\sqrt{2}$

E $5\sqrt{2}$

2.9

TMUA 2022 · Paper 2 · Q4

The point P has coordinates (p, q), and the equation of a circle is

$$x^2 + 2fx + y^2 + 2gy + h = 0$$

where f, g, h, p and q are all real constants.

Let L be the distance between the centre of the circle and the point P.

Which one of the following is **sufficient** on its own to be able to calculate L?

A the values of f, g and h

B the values of f, g, p and q

C the values of f, h, p and q

D the values of g, h, p and q

E none of the options **A**-**D** is sufficient on its own

2.10

TMUA 2017 · Paper 1 · Q6

A tangent to the circle $x^2 + y^2 = 144$ passes through the point $(20, 0)$ and crosses the positive y-axis.

What is the value of y at the point where the tangent meets the y-axis?

A 12

B 15

C $\frac{49}{3}$

D 20

E $\frac{64}{3}$

F $\frac{80}{3}$

2.11

TMUA 2019 · Paper 1 · Q6

The circles with equations

$$(x + 4)^2 + (y + 1)^2 = 64 \quad \text{and}$$
$$(x - 8)^2 + (y - 4)^2 = r^2 \quad \text{where } r > 0$$

have exactly one point in common.

Find the difference between the two possible values of r.

A 4

B 10

C 16

D 26

E 50

2.12

TMUA 2021 · Paper 2 · Q7

A circle has equation $(x-9)^2+(y+2)^2=4$

A square has vertices at $(1,0)$, $(1,2)$, $(-1,2)$ and $(-1,0)$.

A straight line bisects both the area of the circle and the area of the square.

What is the x-coordinate of the point where this straight line meets the x-axis?

A 2
B 3
C 4
D 4.5
E 5
F 6
G The straight line is not uniquely determined by the information given, so there is more than one possible point of intersection.
H There is no straight line that bisects both the area of the circle and the area of the square.

2.13

TMUA 2023 · Paper 2 · Q7

The graph of the line $ax+by=c$ is drawn, where a, b and c are real non-zero constants.

Which one of the following is a **necessary** but **not sufficient** condition for the line to have a positive gradient **and** a positive y-intercept?

A $\frac{c}{b}>0$ and $\frac{a}{b}<0$
B $\frac{c}{b}<0$ and $\frac{a}{b}>0$
C $a>b>c$
D $a<b<c$
E a and c have opposite signs
F a and c have the same sign

2.14

TMUA 2016 · Paper 1 · Q9

The line segment joining the points (3, 3) and (7, 5) is a diameter of a circle.

This circle is translated by 3 units in the negative x-direction, then reflected in the x-axis, and then enlarged by a scale factor of 4 about the centre of the resulting circle.

The equation of the final circle is

A $(x-2)^2+(y-4)^2=320$
B $(x-2)^2+(y+4)^2=320$
C $(x-2)^2+(y-4)^2=80$
D $(x-2)^2+(y+4)^2=80$
E $(x-2)^2+(y-4)^2=20$
F $(x-2)^2+(y+4)^2=20$

2.15

TMUA 2017 · Paper 1 · Q9

A circle has equation $x^2 + y^2 - 18x - 22y + 178 = 0$

A regular hexagon is drawn inside this circle so that the vertices of the hexagon touch the circle.

What is the area of the hexagon?

A 6

B $6\sqrt{3}$

C 18

D $18\sqrt{3}$

E 36

F $36\sqrt{3}$

G 48

H $48\sqrt{3}$

2.16

TMUA 2023 · Paper 1 · Q13

Point P lies on the circle with equation $(x-2)^2 + (y-1)^2 = 16$

Point Q lies on the circle with equation $(x-4)^2 + (y+5)^2 = 16$

What is the maximum possible length of PQ?

A 10

B 14

C 16

D $2\sqrt{34}$

E $10\sqrt{2}$

F $8 + 2\sqrt{10}$

G $16 + 2\sqrt{6}$

2.17

TMUA 2023 · Paper 2 · Q14

Three lines are given by the equations:

$$ax + by + c = 0$$

$$bx + cy + a = 0$$

$$cx + ay + b = 0$$

where a, b and c are non-zero real numbers.

Which one of the following is correct?

A **If** two of the lines are parallel, **then** all three are parallel.

B **If** two of the lines are parallel, **then** the third is perpendicular to the other two.

C **If** two of the lines are parallel, **then** the third is parallel to $y = x$.

D **If** two of the lines are parallel, **then** the third is perpendicular to $y = x$.

E **If** two of the lines are perpendicular, **then** all three meet at a point.

F **If** two of the lines are perpendicular, **then** the third is parallel to $y = x$.

G **If** two of the lines are perpendicular, **then** the third is perpendicular to $y = x$.

2.18

TMUA 2021 · Paper 2 · Q15

A circle has equation

$$x^2 + ax + y^2 + by + c = 0$$

where a, b and c are non-zero real constants.

Which one of the following is a **necessary and sufficient** condition for the circle to be tangent to the y-axis?

A $a^2 = 4c$

B $b^2 = 4c$

C $\dfrac{a}{2} = \sqrt{\dfrac{a^2 + b^2}{4} - c}$

D $\dfrac{b}{2} = \sqrt{\dfrac{a^2 + b^2}{4} - c}$

E $-\dfrac{a}{2} = \sqrt{\dfrac{a^2 + b^2}{4} - c}$

F $-\dfrac{b}{2} = \sqrt{\dfrac{a^2 + b^2}{4} - c}$

2.19

TMUA 2020 · Paper 1 · Q16

The circle C_1 has equation $(x+2)^2+(y-1)^2=3$

The circle C_2 has equation $(x-4)^2+(y-1)^2=3$

The straight line l is a tangent to both C_1 and C_2 and has positive gradient.

The acute angle between l and the x-axis is θ

Find the value of $\tan\theta$

A $\frac{1}{2}$

B 2

C $\frac{\sqrt{2}}{2}$

D $\sqrt{2}$

E $\frac{\sqrt{6}}{2}$

F $\frac{\sqrt{6}}{3}$

G $\frac{\sqrt{3}}{3}$

H $\sqrt{3}$

2.20

TMUA 2023 · Paper 1 · Q16

A right-angled triangle has vertices at (2, 3), (9, –1) and (5, k).

Find the sum of all the possible values of k.

A –8

B –6

C 0.25

D 2

E 2.25

F 8.25

G 10.25

Circle C_1 is defined as $x^2 + y^2 = 25$

A second circle C_2 has radius 4 and centre (a, b) where

$$-2 \leq a \leq 2 \quad \text{and} \quad -3 \leq b \leq 3$$

If the centre of C_2 is equally likely to be located anywhere within the given range, what is the probability that C_2 intersects C_1?

A $\frac{1}{25}$

B $\frac{9}{25}$

C $\frac{16}{25}$

D $\frac{6 - \pi}{6}$

E $\frac{16 - \pi}{24}$

F $\frac{24 - \pi}{24}$

CHAPTER 3 · UNIT P3

Exponentials and Logarithms

29 questions

3.1

TMUA 2021 · Paper 1 · Q4

Find the minimum value of the function

$$2^{2x} - 2^{x+3} + 4$$

A -16
B -12
C -8
D 0
E 4
F 20

3.2

TMUA Specimen · Paper 1 · Q5

Given that $y = -\log_{10}(1-x)$ for $x < 1$, find x in terms of y.

A $x = -\frac{1}{\log_{10}(1-y)}$
B $x = 1 + \log_{10} y$
C $x = 1 - \log_{10} y$
D $x = 1 - 10^{-y}$
E $x = 10^{-y} - 1$
F $x = 10^{1-y}$

3.3

TMUA Specimen · Paper 2 · Q5

Using the observation that $2^5 \approx 3^3$, it is possible to deduce that $\log_3 2$ is approximately

A $\frac{3}{5}$
B $\frac{2}{3}$
C $\frac{3}{2}$
D $\frac{5}{3}$
E $\frac{1}{2}$
F 2

3.4

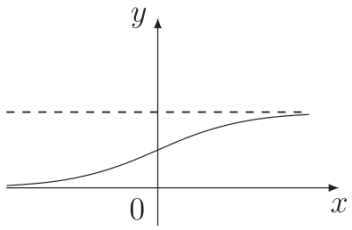
TMUA 2020 · Paper 2 · Q5

Which one of the following shows the graph of

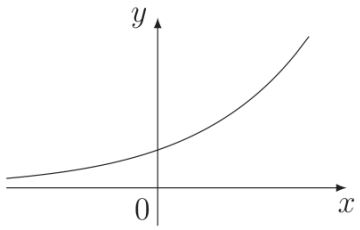
$$y = \frac{2^x}{1 + 2^x}$$

(Dotted lines indicate asymptotes.)

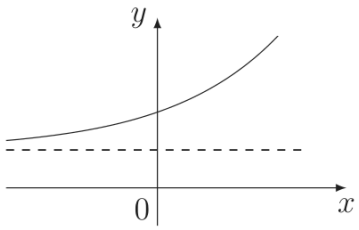
A



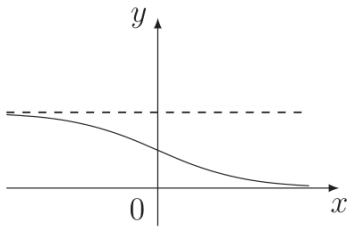
B



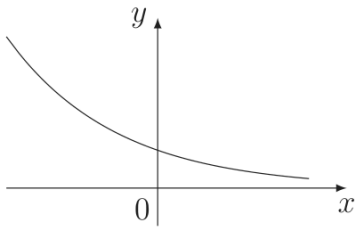
C



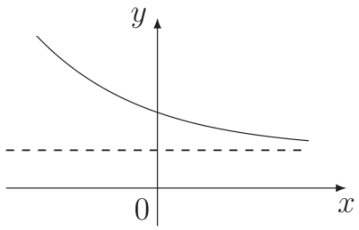
D



E



F



3.5

TMUA 2020 · Paper 1 · Q6

Find the maximum value of the function

$$\mathrm{f}(x) = \frac{1}{5^{2x} - 4(5^x) + 7}$$

A $\frac{1}{7}$

B $\frac{1}{4}$

C $\frac{1}{3}$

D 3

E 4

F 7

3.6

TMUA 2020 · Paper 1 · Q7

Given that

$$2^{3x} = 8^{(y+3)}$$

and

$$4^{(x+1)} = \frac{16^{(y+1)}}{8^{(y+3)}}$$

what is the value of $x + y$?

A -23
B -22
C -15
D -14
E -11
F -10

3.7

TMUA 2023 · Paper 1 · Q7

P(x) and Q(x) are defined as follows:

$$\mathrm{P}(x) = 2^x + 4$$

$$\mathrm{Q}(x) = 2^{(2x-2)} - 2^{(x+2)} + 16$$

Find the largest value of x such that P(x) and Q(x) are in the ratio 4 : 1, respectively.

A 5
B 12
C 32
D $\log_2 3$
E $\log_2 5$
F $\log_2 12$
G $\log_2 20$

The four real numbers a, b, c, and d are all greater than 1.

Suppose that they satisfy the equation $\log_c d = (\log_a b)^2$.

Use some of the lines given to construct a proof that, in this case, it follows that

$$(*) \quad \log_b d = (\log_a b)(\log_a c).$$

(1) Let $x = \log_a b$ and $y = \log_a c$

(2) $d = (c^x)^2$

(3) $d = c^{(x^2)}$

(4) $d = b^{xy}$

(5) $d = (a^y)^{(x^2)}$

(6) $d = ((a^y)^x)^2$

(7) $d = (a^x)^{xy}$

(8) $d = a^{(y^{2x})}$

(9) $d = a^{(x^2 y)}$

A (1). Then (2), so (6), so (8), so (7), and therefore (4), hence (*) as required.

B (1). Then (2), so (7), so (8), so (6), and therefore (4), hence (*) as required.

C (1). Then (3), so (5), so (9), so (7), and therefore (4), hence (*) as required.

D (1). Then (3), so (7), so (9), so (5), and therefore (4), hence (*) as required.

E (1). Then (4), so (5), so (9), so (7), and therefore (3), hence (*) as required.

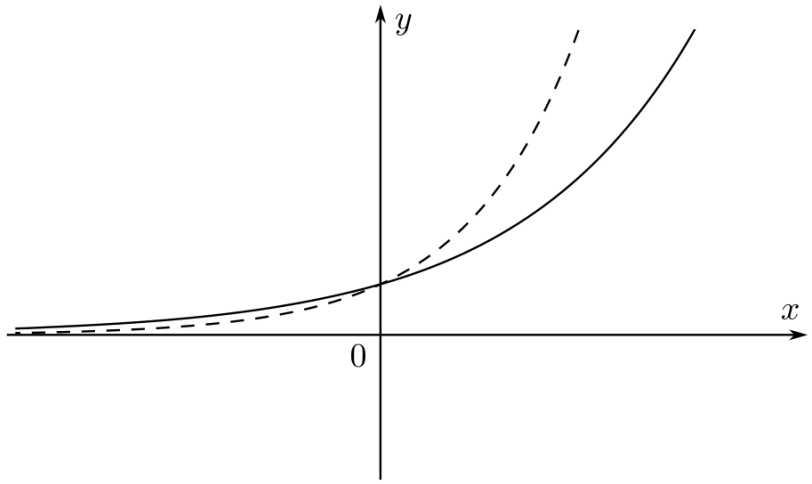
F (1). Then (4), so (6), so (8), so (7), and therefore (2), hence (*) as required.

G (1). Then (4), so (7), so (8), so (6), and therefore (2), hence (*) as required.

H (1). Then (4), so (7), so (9), so (5), and therefore (3), hence (*) as required.

The graphs of two functions are shown here:

- $y = a^x$ is shown with a solid line, where a is a positive real number
- $y = f(x)$ is shown with a dashed line



Which of the following statements (**1**, **2**, **3**, **4**) could be true?

1 $f(x) = b^x$ for some $b > a$

2 $f(x) = b^x$ for some $b < a$

3 $f(x) = a^{kx}$ for some $k > 1$

4 $f(x) = a^{kx}$ for some $k < 1$

A **1** only

B **2** only

C **3** only

D **4** only

E **1** and **3** only

F **1** and **4** only

G **2** and **3** only

H **2** and **4** only

Given that $a^x b^{2x} c^{3x} = 2$, where a, b, and c are positive real numbers, then $x =$

A $\log_{10}\left(\frac{2}{a+2b+3c}\right)$

B $\frac{\log_{10} 2}{\log_{10}(a+2b+3c)}$

C $\frac{2}{\log_{10}(a+2b+3c)}$

D $\frac{2}{a+2b+3c}$

E $\log_{10}\left(\frac{2}{ab^2c^3}\right)$

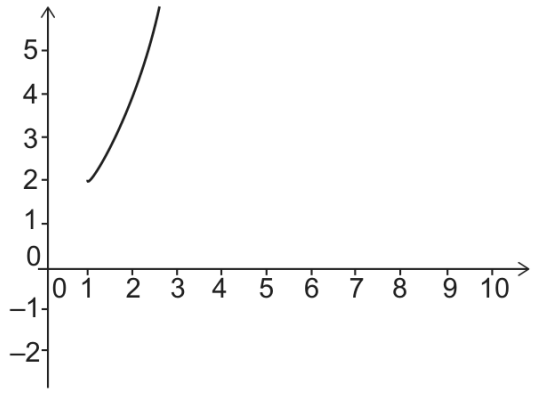
F $\frac{\log_{10} 2}{\log_{10}(ab^2c^3)}$

G $\frac{2}{\log_{10}(ab^2c^3)}$

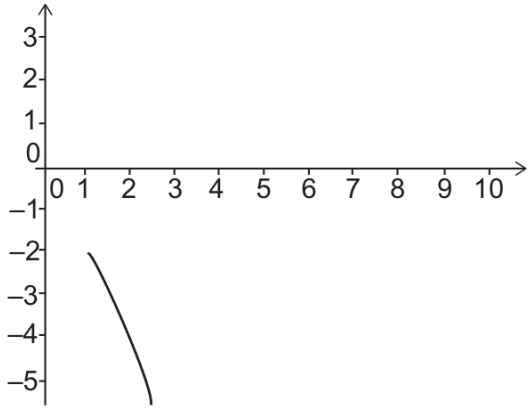
H $\frac{2}{ab^2c^3}$

Which one of the following is a sketch of the graph of $y = \log_x 2$ for $x > 1$?

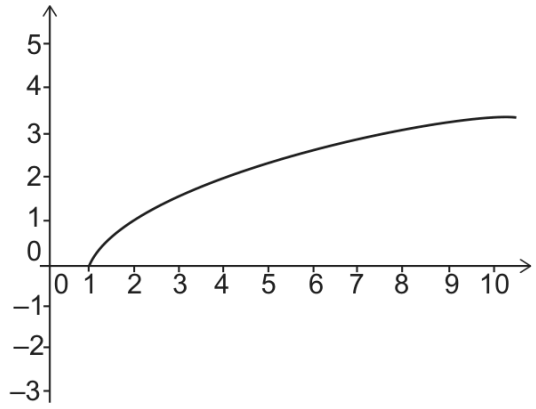
A



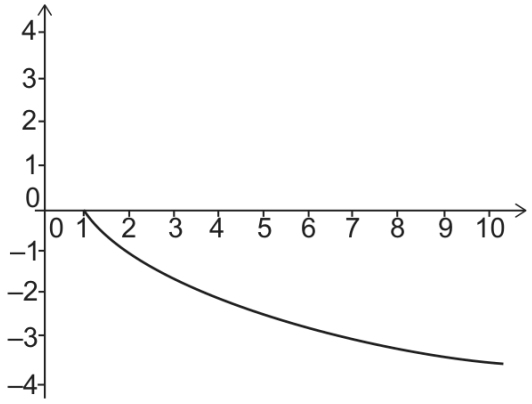
B



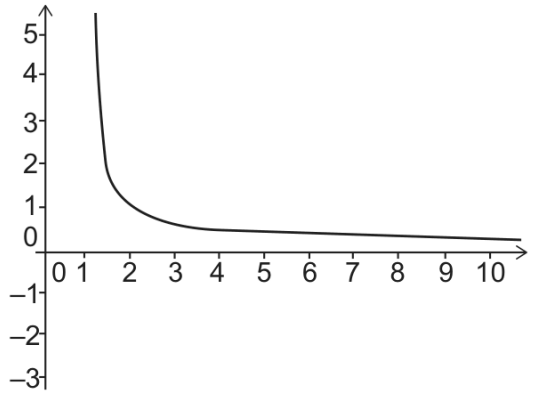
C



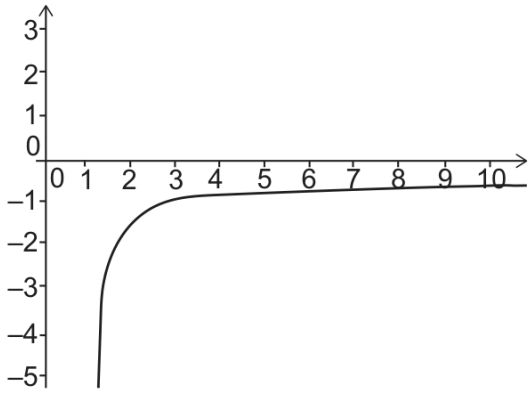
D



E



F



3.12

TMUA Specimen · Paper 1 · Q11

The sum of the roots of the equation $2^{2x} - 8 \times 2^x + 15 = 0$ is

A 3

B 8

C $2\log_{10} 2$

D $\log_{10}\left(\frac{15}{4}\right)$

E $\frac{\log_{10} 15}{\log_{10} 2}$

3.13

TMUA 2016 · Paper 1 · Q11

The real roots of the equation $4^{2x} + 12 = 2^{2x+3}$ are p and q, where $p > q$.

The value of $p - q$ can be expressed as

A $\frac{3}{4}$

B 1

C 4

D $-\frac{1}{2} + \log_{10}\frac{3}{2}$

E $\frac{\log_{10} 3}{\log_{10} 4}$

F $\frac{\log_{10} 3}{\log_{10} 2}$

3.14

TMUA 2019 · Paper 1 · Q11

Find the sum of the real values of x that satisfy the simultaneous equations:

$$\log_3(xy^2) = 1$$
$$(\log_3 x)(\log_3 y) = -3$$

A $\frac{1}{3}$

B 1

C 3

D $3\frac{1}{9}$

E $9\frac{1}{27}$

F $9\frac{1}{3}$

G 27

H $27\frac{1}{9}$

3.15

TMUA 2018 · Paper 2 · Q11

Consider the equation $2^x = mx + c$, where m and c are real constants.

Which of the following statements is/are true?

I The equation has a negative real solution **only if** $c > 1$.

II The equation has two distinct real solutions **if** $c > 1$.

III The equation has two distinct positive real solutions **if and only if** $c \leq 1$.

A none of them
B I only
C II only
D III only
E I and II only
F I and III only
G II and III only
H I, II and III

3.16

TMUA Specimen · Paper 1 · Q14

a, b, x, and y are real and positive.

a and b are constants.

x and y are related.

A graph of $\log y$ against $\log x$ is drawn.

For which one of the following relationships will this graph be a straight line?

A $y^b = a^x$
B $y = ab^x$
C $y^2 = a + x^b$
D $y = ax^b$
E $y^x = a^b$

3.17

TMUA 2017 · Paper 1 · Q14

The solution of the simultaneous equations

$$2^x + 3 \times 2^y = 3$$
$$2^{2x} - 9 \times 2^{2y} = 6$$

is $x = p,\ y = q$.

Find the value of $p - q$

A $\frac{5}{12}$

B $\frac{7}{3}$

C $\log_2 \frac{5}{12}$

D $\log_2 \frac{7}{3}$

E $\log_2 9$

F $\log_2 15$

3.18

TMUA 2018 · Paper 1 · Q14

The line $y = mx + 4$ passes through the points $(3, \log_2 p)$ and $(\log_2 p, 4)$.

What are the possible values of p?

A $p = 1$ and $p = 4$

B $p = 1$ and $p = 16$

C $p = \frac{1}{4}$ and $p = 4$

D $p = \frac{1}{4}$ and $p = 64$

E $p = \frac{1}{64}$ and $p = 4$

F $p = \frac{1}{64}$ and $p = 16$

3.19

TMUA 2021 · Paper 2 · Q14

Consider the following simultaneous equations, where p is a real number:

$$p2^x + \log_2 y = 2$$
$$2^x + \log_2 y = 1$$

What is the complete range of p for which these simultaneous equations have a real solution (x, y)?

A $p < 1$

B $p \neq 1$

C $p > 1$

D $p < 1$ or $p > 2$

E $p \neq 1$ and $p < 2$

F $p > 1$ and $p < 2$

G $p > 2$

H All real values of p

3.20

TMUA 2018 · Paper 1 · Q15

Find the sum of the real solutions of the equation:

$$3^x - (\sqrt{3})^{x+4} + 20 = 0$$

A 1

B 4

C 9

D $\log_3 20$

E $2\log_3 20$

F $4\log_3 20$

3.21

TMUA 2019 · Paper 1 · Q15

Find the real non-zero solution to the equation

$$\frac{2^{(9^x)}}{8^{(3^x)}} = \frac{1}{4}$$

A $\log_3 2$

B $2\log_3 2$

C 1

D 2

E $\log_2 3$

F $2\log_2 3$

3.22

TMUA 2020 · Paper 1 · Q15

Find the positive difference between the two real values of x for which

$$(\log_2 x)^4 + 12\left(\log_2\left(\frac{1}{x}\right)\right)^2 - 2^6 = 0$$

A 4

B 16

C $\frac{15}{4}$

D $\frac{17}{4}$

E $\frac{255}{16}$

F $\frac{257}{16}$

3.23

TMUA 2019 · Paper 2 · Q15

The numbers a, b and c are each greater than 1.

The following logarithms are all to the same base:

$$\log(ab^2c) = 7$$
$$\log(a^2bc^2) = 11$$
$$\log(a^2b^2c^3) = 15$$

What is this base?

A a

B b

C c

D It is possible to determine the base, but the base is not a, b or c.

E There is insufficient information given to determine the base.

3.24

TMUA 2022 · Paper 2 · Q15

The real numbers x, y and z are all greater than 1, and satisfy the equations

$$\log_x y = z \quad \text{and} \quad \log_y z = x$$

Which one of the following equations for $\log_z x$ **must** be true?

A $\log_z x = y$

B $\log_z x = \frac{1}{y}$

C $\log_z x = xy$

D $\log_z x = \frac{1}{xy}$

E $\log_z x = xz$

F $\log_z x = \frac{1}{xz}$

G $\log_z x = yz$

H $\log_z x = \frac{1}{yz}$

3.25

TMUA 2016 · Paper 1 · Q16

Given the simultaneous equations

$$\log_{10} 2 + \log_{10}(y-1) = 2\log_{10} x$$
$$\log_{10}(y+3-3x) = 0$$

the values of y are

A $\frac{5}{2} \pm \frac{3\sqrt{5}}{2}$

B $3 \pm \sqrt{3}$

C $7 \pm 3\sqrt{3}$

D $3, 9$

E $1, 13$

Consider the following functions defined for $x > 1$:

$$\mathrm{f}(x) = \log_2\left(\log_2 \sqrt{x}\,\right)$$

$$\mathrm{g}(x) = \log_2\left(\sqrt{\log_2 x}\,\right)$$

Which one of the following is true for **all** values of $x > 1$?

A $0 \leq \mathrm{f}(x) \leq \mathrm{g}(x)$ **or** $\mathrm{g}(x) \leq \mathrm{f}(x) \leq 0$

B $0 \leq \mathrm{g}(x) \leq \mathrm{f}(x)$ **or** $\mathrm{f}(x) \leq \mathrm{g}(x) \leq 0$

C $\frac{1}{2} \leq \mathrm{f}(x) \leq \mathrm{g}(x)$ **or** $\mathrm{g}(x) \leq \mathrm{f}(x) \leq \frac{1}{2}$

D $\frac{1}{2} \leq \mathrm{g}(x) \leq \mathrm{f}(x)$ **or** $\mathrm{f}(x) \leq \mathrm{g}(x) \leq \frac{1}{2}$

E $1 \leq \mathrm{f}(x) \leq \mathrm{g}(x)$ **or** $\mathrm{g}(x) \leq \mathrm{f}(x) \leq 1$

F $1 \leq \mathrm{g}(x) \leq \mathrm{f}(x)$ **or** $\mathrm{f}(x) \leq \mathrm{g}(x) \leq 1$

Consider the following problem:

Solve the inequality $\left(\frac{1}{4}\right)^n < \left(\frac{1}{32}\right)^{10}$, where n is a positive integer.

A student produces the following argument:

$$\left(\frac{1}{4}\right)^n < \left(\frac{1}{32}\right)^{10}$$

$\Big\downarrow$ (I)

$$\log_{\frac{1}{2}}\left(\frac{1}{4}\right)^n < \log_{\frac{1}{2}}\left(\frac{1}{32}\right)^{10}$$

$\Big\downarrow$ (II)

$$n\log_{\frac{1}{2}}\left(\frac{1}{4}\right) < 10\log_{\frac{1}{2}}\left(\frac{1}{32}\right)$$

$\Big\downarrow$ (III)

$$n < \frac{10\log_{\frac{1}{2}}\left(\frac{1}{32}\right)}{\log_{\frac{1}{2}}\left(\frac{1}{4}\right)}$$

$\Big\downarrow$ (IV)

$$n < \frac{10 \times 5}{2} = 25$$

$\Big\downarrow$ (V)

$$1 \leqslant n \leqslant 24$$

Which step (if any) in the argument is invalid?

A There are no invalid steps; the argument is correct

B Only step (I) is invalid; the rest are correct

C Only step (II) is invalid; the rest are correct

D Only step (III) is invalid; the rest are correct

E Only step (IV) is invalid; the rest are correct

F Only step (V) is invalid; the rest are correct

3.28

TMUA 2018 · Paper 2 · Q19

Three **real** numbers x, y and z satisfy $x > y > z > 1$.

Which one of the following statements **must** be true?

A $\dfrac{2^{z+1}}{2^x} > \dfrac{2^x + 2^z}{2^y}$

B $2 > \dfrac{3^x + 3^z}{3^y}$

C $\dfrac{2 \times 5^x}{5^z} > \dfrac{5^x + 5^z}{5^y}$

D $2 < \dfrac{7^x + 7^z}{7^y}$

3.29

TMUA 2021 · Paper 1 · Q20

Find the length of the curve with equation

$$2\log_{10}(x - y) = \log_{10}(2 - 2x) + \log_{10}(y + 5)$$

A 5
B 10
C 15
D 3π
E 9π
F 12π

CHAPTER 4 · UNIT P4

Trigonometry 1

39 questions

4.1

TMUA 2022 · Paper 1 · Q1

How many real solutions are there to the equation

$$2\cos^4\theta - 5\cos^2\theta + 3 = 0$$

in the interval $0 \leq \theta \leq 2\pi$?

A 1
B 2
C 3
D 4
E 5
F 6
G 7
H 8

4.2

TMUA Specimen · Paper 1 · Q2

The number of solutions in the interval $0 \leq \theta \leq 4\pi$ of the equation

$\sin^2\theta + 3\cos\theta = 3$ is

A 0
B 1
C 2
D 3
E 4
F 5
G 6

4.3 TMUA 2020 · Paper 2 · Q2

Given that $\tan\theta = 2$ and $180° < \theta < 360°$, find the value of $\cos\theta$

A $\sqrt{3}$

B $-\sqrt{3}$

C $\frac{\sqrt{3}}{2}$

D $-\frac{\sqrt{3}}{2}$

E $\frac{\sqrt{5}}{5}$

F $-\frac{\sqrt{5}}{5}$

G $\frac{2\sqrt{5}}{5}$

H $-\frac{2\sqrt{5}}{5}$

4.4 TMUA 2016 · Paper 2 · Q3

What is the value, in radians, of the largest angle x in the range $0 \leq x \leq 2\pi$ that satisfies the equation $8\sin^2 x + 4\cos^2 x = 7$?

A $\frac{2\pi}{3}$

B $\frac{5\pi}{6}$

C $\frac{4\pi}{3}$

D $\frac{5\pi}{3}$

E $\frac{7\pi}{4}$

F $\frac{11\pi}{6}$

4.5

TMUA 2017 · Paper 2 · Q4

The following question appeared in an examination:

Given that $\tan x = \sqrt{3}$, find the possible values of $\sin 2x$.

A student gave the following answer:

$\tan x = \sqrt{3}$ so $x = 60°$ and $2x = 120°$,

therefore $\sin 2x = \dfrac{\sqrt{3}}{2}$.

Which one of the following statements is correct?

A $\dfrac{\sqrt{3}}{2}$ is the only possible value, and this is fully supported by the reasoning given in the student's answer.

B $\dfrac{\sqrt{3}}{2}$ is the only possible value, but the reasoning given should consider other possible values of x for which $\tan x = \sqrt{3}$.

C $\dfrac{\sqrt{3}}{2}$ is the only possible value, but the reasoning given should consider other possible values of x for which $\sin 2x = \dfrac{\sqrt{3}}{2}$.

D $\dfrac{\sqrt{3}}{2}$ is **not** the only possible value because the reasoning given should have considered other possible values of x for which $\tan x = \sqrt{3}$.

E $\dfrac{\sqrt{3}}{2}$ is **not** the only possible value because the reasoning given should have considered other possible values of x for which $\sin 2x = \dfrac{\sqrt{3}}{2}$.

4.6

TMUA 2018 · Paper 2 · Q4

The non-zero real number c is such that the equation $\cos x = c$ has two solutions for $0 < x < \frac{3}{2}\pi$.

How many solutions of the equation $\cos^2 2x = c^2$ are there in the range $0 < x < \frac{3}{2}\pi$?

A 2
B 3
C 4
D 6
E 7
F 8

4.7

TMUA 2021 · Paper 2 · Q5

On which line is the first error in the following argument?

A $\sin^2 x + \cos^2 x = 1$ for all values of x.

B Therefore $\cos x = \sqrt{1 - \sin^2 x}$ for all values of x.

C Hence $1 + \cos x = 1 + \sqrt{1 - \sin^2 x}$ for all values of x.

D Thus $(1 + \cos x)^2 = \left(1 + \sqrt{1 - \sin^2 x}\right)^2$ for all values of x.

E Substituting $x = \pi$ gives $0 = 4$.

4.8

TMUA 2021 · Paper 1 · Q6

The function f is given by

$$\mathrm{f}(x) = \frac{\cos x + 3}{7 + 5\cos x - \sin^2 x}$$

Find the positive difference between the maximum and the minimum values of $\mathrm{f}(x)$.

A 0

B $\frac{1}{3}$

C $\frac{1}{2}$

D $\frac{2}{3}$

E 1

F 2

A student attempts to solve the equation

$$\cos x + \sin x \tan x = 2\sin x - 1$$

in the range $0 \leq x \leq 2\pi$.

The student's attempt is as follows:

$\cos x + \sin x \tan x = 2\sin x - 1$

So $\cos x - \sin x + \sin x \tan x - \sin x = -1$ (I)

So $(\sin x - \cos x)(\tan x - 1) = -1$ (II)

So $\sin x - \cos x = -1$ or $\tan x - 1 = -1$ (III)

So $(\sin x - \cos x)^2 = 1$ or $\tan x = 0$ (IV)

So $2\sin x \cos x = 0$ or $\tan x = 0$ (V)

So $x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$ (VI)

Which of the following best describes this attempt?

A It is completely correct

B It is incorrect, and the first error occurs on line (I)

C It is incorrect, and the first error occurs on line (II)

D It is incorrect, and the first error occurs on line (III)

E It is incorrect, and the first error is that extra solutions were introduced on line (IV)

F It is incorrect, and the first error is that extra solutions were introduced on line (V)

Find the maximum angle x in the range $0° \leq x \leq 360°$ which satisfies the equation

$$\cos^2(2x) + \sqrt{3}\sin(2x) - \frac{7}{4} = 0$$

A 30°

B 60°

C 120°

D 150°

E 210°

F 240°

4.11

TMUA 2017 · Paper 1 · Q8

Find the complete set of values of x, with $0 \leq x \leq \pi$, for which

$$(1 - 2\sin x)\cos x \geq 0$$

A $0 \leq x \leq \frac{\pi}{6}, \quad \frac{\pi}{2} \leq x \leq \frac{5\pi}{6}$

B $0 \leq x \leq \frac{\pi}{6}, \quad \frac{5\pi}{6} \leq x \leq \pi$

C $\frac{\pi}{6} \leq x \leq \frac{\pi}{2}, \quad \frac{5\pi}{6} \leq x \leq \pi$

D $\frac{\pi}{6} \leq x \leq \frac{5\pi}{6}$

4.12

TMUA 2023 · Paper 1 · Q8

A triangle *XYZ* is called *fun* if it has the following properties:

angle $YXZ = 30°$

$XY = \sqrt{3}\,a$

$YZ = a$

where a is a constant.

For a given value of a, there are two distinct *fun* triangles *S* and *T*, where the area of *S* is greater than the area of *T*.

Find the ratio

area of *S* : area of *T*

A 1 : 1

B 2 : 1

C $2 : \sqrt{3}$

D $\sqrt{3} : 1$

E 3 : 1

4.13

TMUA 2017 · Paper 2 · Q8

Which one of the following numbers is smallest in value?

A $\log_2 7$

B $(2^{-3} + 2^{-2})^{-1}$

C $2^{(\pi/3)}$

D $\dfrac{1}{4\left(\sqrt{2}-1\right)^3}$

E $4\sin^2\left(\dfrac{\pi}{4}\right)$

4.14

TMUA 2022 · Paper 1 · Q9

This question is about pairs of functions f and g that satisfy

$$f(x) - g(x) = 2\sin x$$

$$f(x)\,g(x) = \cos^2 x$$

for all real numbers x.

Across all solutions for $f(x)$, what is the minimum value that $f(x)$ attains for any x?

A $1 - \sqrt{2}$

B $-1 - \sqrt{2}$

C 0

D -1

E -2

F -3

G -4

4.15

TMUA 2023 · Paper 1 · Q9

How many solutions are there to

$$(1 + 3\cos 3\theta)^2 = 4$$

in the interval $0° \leq \theta \leq 180°$?

A 1

B 2

C 3

D 4

E 5

F 6

4.16

TMUA Specimen · Paper 2 · Q11

Which one of the following numbers is largest in value?
(All angles are given in radians.)

A $\tan\left(\frac{3\pi}{4}\right)$

B $\log_{10} 100$

C $\sin^{10}\left(\frac{\pi}{2}\right)$

D $\log_2 10$

E $\left(\sqrt{2} - 1\right)^{10}$

4.17

How many solutions are there to the equation

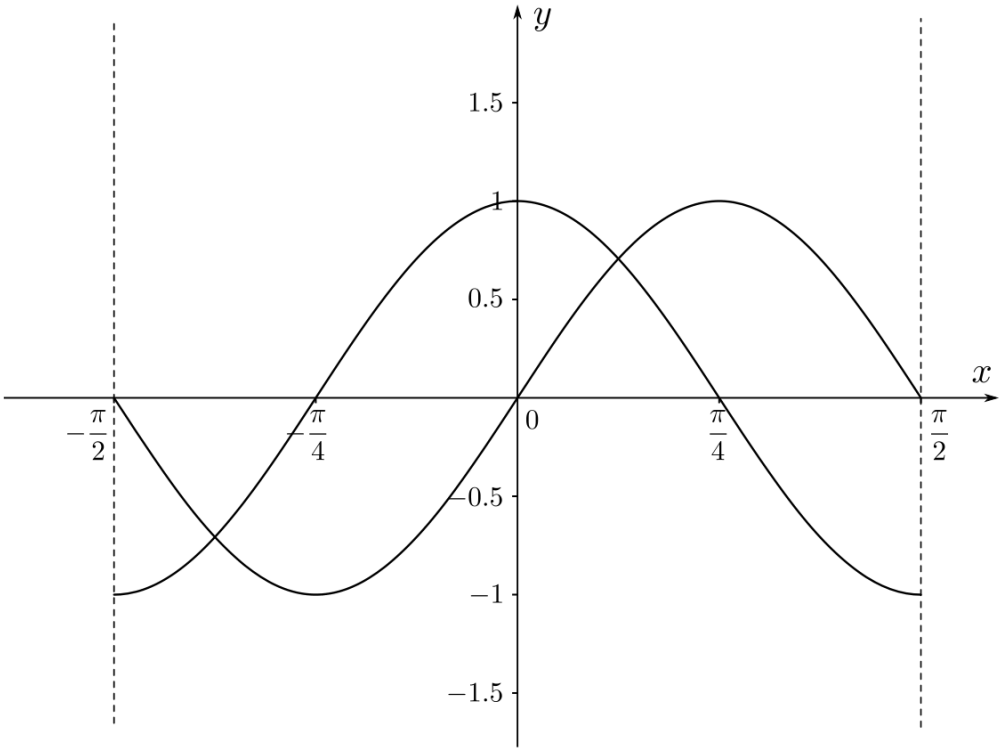
$$\frac{2^{\tan^2 x}}{4^{\sin^2 x}} = 1$$

in the range $0 \leq x \leq 2\pi$?

A 2
B 3
C 4
D 5
E 6
F 7
G 8

4.18

The diagram shows the graphs of $y = \sin 2x$ and $y = \cos 2x$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$



Which one of the following is **not** true?

A $\cos 2x < \sin 2x < \tan x$ for some real number x with $-\frac{\pi}{2} < x < \frac{\pi}{2}$
B $\cos 2x < \tan x < \sin 2x$ for some real number x with $-\frac{\pi}{2} < x < \frac{\pi}{2}$
C $\sin 2x < \cos 2x < \tan x$ for some real number x with $-\frac{\pi}{2} < x < \frac{\pi}{2}$
D $\sin 2x < \tan x < \cos 2x$ for some real number x with $-\frac{\pi}{2} < x < \frac{\pi}{2}$
E $\tan x < \sin 2x < \cos 2x$ for some real number x with $-\frac{\pi}{2} < x < \frac{\pi}{2}$
F $\tan x < \cos 2x < \sin 2x$ for some real number x with $-\frac{\pi}{2} < x < \frac{\pi}{2}$

4.19

TMUA 2023 · Paper 2 · Q12

In this question, p is a real constant.

The equation $\sin x \cos^2 x = p^2 \sin x$ has n distinct solutions in the range $0 \leq x \leq 2\pi$

Which of the following statements is/are true?

I $n = 3$ is **sufficient** for $p > 1$

II $n = 7$ **only if** $-1 < p < 1$

A none of them

B I only

C II only

D I and II

4.20

TMUA 2019 · Paper 1 · Q13

Find the maximum value of

$$4^{\sin x} - 4 \times 2^{\sin x} + \tfrac{17}{4}$$

for real x.

A $\frac{1}{4}$

B $\frac{5}{2}$

C $\frac{13}{2}$

D $\frac{21}{2}$

E $\frac{65}{4}$

F There is no maximum value.

4.21

TMUA 2018 · Paper 2 · Q13

The following is an attempted proof of the conjecture:

$$\textbf{if } \tan\theta > 0, \textbf{ then } \sin\theta + \cos\theta > 1.$$

Suppose $\tan\theta > 0$, so in particular $\cos\theta \neq 0$.

Since $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$, then $\sin\theta\cos\theta = \tan\theta\cos^2\theta > 0$. (I)

It follows that $1 + 2\sin\theta\cos\theta > 1$. (II)

Therefore $\sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta > 1$, (III)

which factorises to give $(\sin\theta + \cos\theta)^2 > 1$. (IV)

Therefore $\sin\theta + \cos\theta > 1$. (V)

Which one of the following is the case?

A The proof is correct.

B The proof is incorrect, and the first error occurs in line (I).

C The proof is incorrect, and the first error occurs in line (II).

D The proof is incorrect, and the first error occurs in line (III).

E The proof is incorrect, and the first error occurs in line (IV).

F The proof is incorrect, and the first error occurs in line (V).

4.22

TMUA 2019 · Paper 1 · Q14

x satisfies the simultaneous equations

$$\sin 2x + \sqrt{3}\cos 2x = -1$$

and

$$\sqrt{3}\sin 2x - \cos 2x = \sqrt{3}$$

where $0° \leq x \leq 360°$.

Find the sum of the possible values of x.

A $210°$

B $330°$

C $390°$

D $660°$

E $780°$

F $930°$

4.23

TMUA 2022 · Paper 1 · Q14

A circle has centre O and radius 6.

P, Q and R are points on the circumference with angle $POQ \geq \frac{\pi}{2}$

The area of the triangle POQ is $9\sqrt{3}$

What is the greatest possible area of triangle PRQ?

A $18 + 9\sqrt{3}$

B $18\sqrt{3}$

C $27 + 9\sqrt{3}$

D $27\sqrt{3}$

E $36 + 9\sqrt{3}$

F $36\sqrt{3}$

4.24

TMUA 2018 · Paper 2 · Q14

In the triangle PQR, $PR = 2$, $QR = p$ and $\angle RPQ = 30°$.

What is the set of **all** the values of p for which this information uniquely determines the length of PQ?

A $p = 1$

B $p = \sqrt{3}$

C $1 \leq p < 2$

D $\sqrt{3} \leq p < 2$

E $p = 1$ or $p \geq 2$

F $p = \sqrt{3}$ or $p \geq 2$

G $p < 2$

H $p \geq 2$

4.25

TMUA 2023 · Paper 1 · Q15

The difference between the maximum and minimum values of the function $f(x) = a^{\cos x}$, where $a > 0$ and x is real, is 3.

Find the sum of the possible values of a.

A 0

B $\frac{3}{2}$

C $\frac{5}{2}$

D 3

E $\sqrt{10}$

F $\sqrt{13}$

4.26

TMUA 2022 · Paper 1 · Q16

The solutions to $7x^4 - 6x^2 + 1 = 0$ are $\pm\cos\theta$ and $\pm\cos\beta$.

Which one of the following equations has solutions $\pm\sin\theta$ and $\pm\sin\beta$?

A $7x^4 - 8x^2 - 5 = 0$

B $7x^4 - 8x^2 + 2 = 0$

C $7x^4 - 6x^2 - 2 = 0$

D $7x^4 - 6x^2 + 1 = 0$

E $7x^4 + 6x^2 - 1 = 0$

F $7x^4 + 6x^2 + 5 = 0$

4.27

TMUA 2016 · Paper 1 · Q17

It is given that

$$y = (1 + 2\cos x)\cos 2x \quad \text{for } 0 < x < \pi$$

The complete set of values of x for which y is negative is

A $0 < x < \frac{\pi}{4}, \ \frac{2\pi}{3} < x < \frac{3\pi}{4}$

B $0 < x < \frac{\pi}{4}, \ \frac{3\pi}{4} < x < \pi$

C $0 < x < \frac{2\pi}{3}, \ \frac{3\pi}{4} < x < \pi$

D $\frac{\pi}{4} < x < \frac{2\pi}{3}, \ \frac{3\pi}{4} < x < \pi$

E $\frac{\pi}{4} < x < \frac{2\pi}{3}$

F $\frac{\pi}{4} < x < \frac{3\pi}{4}$

4.28

TMUA 2019 · Paper 1 · Q17

Find the fraction of the interval $0 \leq \theta \leq \pi$ for which the inequality

$$\left(\sin(2\theta) - \tfrac{1}{2}\right)\left(\sin\theta - \cos\theta\right) \geq 0$$

is satisfied.

A $\frac{1}{12}$

B $\frac{1}{6}$

C $\frac{1}{4}$

D $\frac{5}{12}$

E $\frac{7}{12}$

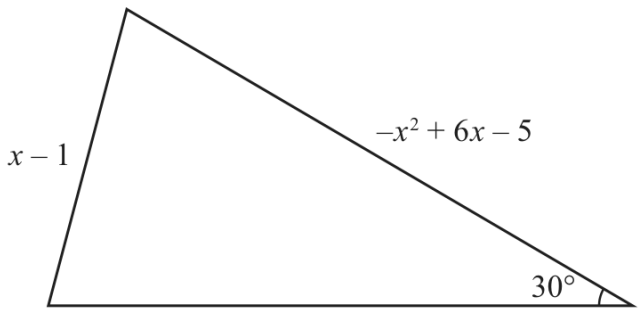
F $\frac{3}{4}$

G $\frac{5}{6}$

H $\frac{11}{12}$

4.29

TMUA 2022 · Paper 1 · Q17



Find the complete set of values of x for which there are two non-congruent triangles with the side lengths and angle as shown in the diagram.

A $1 < x < 3$

B $1 < x < 4$

C $1 < x < 5$

D $3 < x < 4$

E $3 < x < 5$

F $4 < x < 5$

4.30

TMUA 2018 · Paper 1 · Q18

What is the smallest positive value of a for which the line $x = a$ is a line of symmetry of the graph of $y = \sin\left(2x - \frac{4\pi}{3}\right)$?

A $\frac{\pi}{12}$

B $\frac{5\pi}{12}$

C $\frac{7\pi}{12}$

D $\frac{11\pi}{12}$

E $\frac{19\pi}{12}$

4.31

TMUA 2021 · Paper 2 · Q18

A student chooses two distinct real numbers x and y with $0 < x < y < 1$.

The student then attempts to draw a triangle ABC with:

$$AB = 1$$
$$\sin A = x$$
$$\sin B = y$$

Which of the following statements is/are correct?

I For some choice of x and y, there is exactly **one** triangle the student could draw.

II For some choice of x and y, there are exactly **two** different triangles the student could draw.

III For some choice of x and y, there are exactly **three** different triangles the student could draw.

(Note that congruent triangles are considered to be the same.)

A none of them

B I only

C II only

D III only

E I and II only

F I and III only

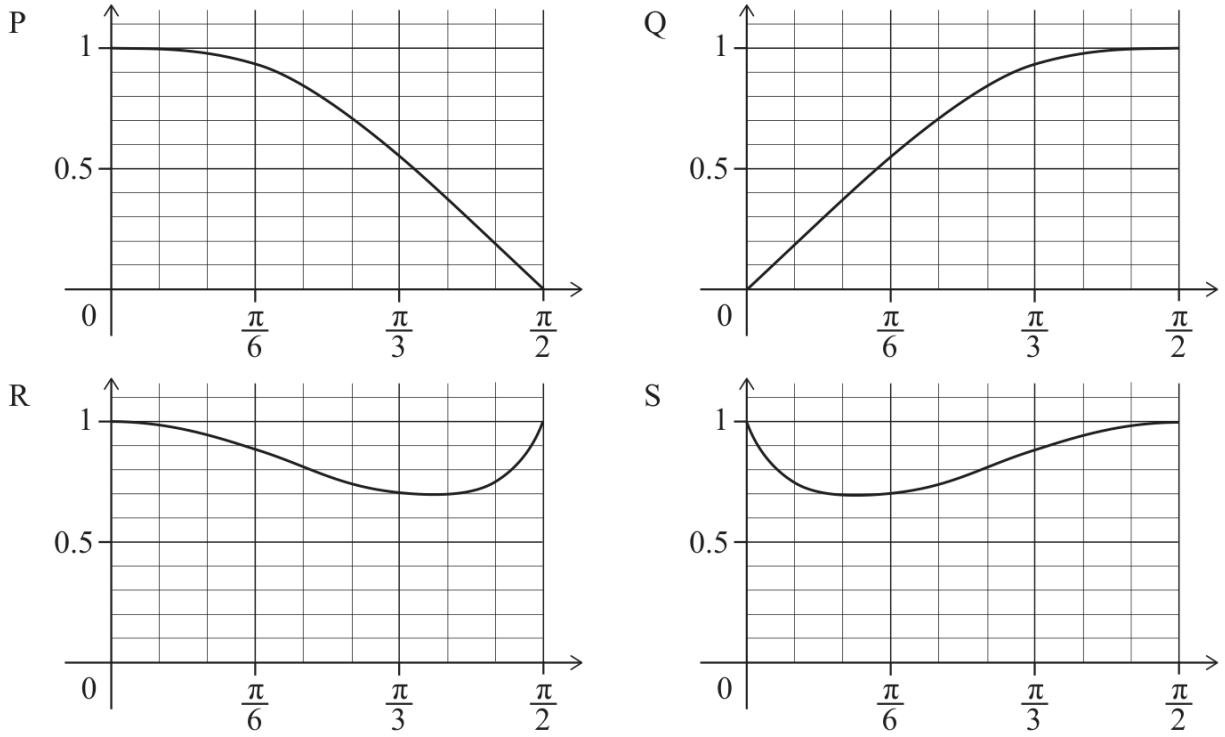
G II and III only

H I, II and III

P, Q, R and S show the graphs of

$$y = (\cos x)^{\cos x},\ y = (\sin x)^{\sin x},\ y = (\cos x)^{\sin x} \text{ and } y = (\sin x)^{\cos x}$$

for $0 < x < \frac{\pi}{2}$ in some order.



Which row in the following table correctly identifies the graphs?

	$y = (\cos x)^{\cos x}$	$y = (\sin x)^{\sin x}$	$y = (\cos x)^{\sin x}$	$y = (\sin x)^{\cos x}$
A	P	Q	R	S
B	P	Q	S	R
C	Q	P	R	S
D	Q	P	S	R
E	R	S	P	Q
F	R	S	Q	P
G	S	R	P	Q
H	S	R	Q	P

4.33

TMUA 2018 · Paper 1 · Q19

A triangle ABC is to be drawn with $AB = 10\text{cm}$, $BC = 7\text{cm}$ and the angle at A equal to θ, where θ is a certain specified angle.

Of the two possible triangles that could be drawn, the larger triangle has three times the area of the smaller one.

What is the value of $\cos\theta$?

A $\frac{5}{7}$

B $\frac{151}{200}$

C $\frac{2\sqrt{2}}{5}$

D $\frac{\sqrt{17}}{5}$

E $\frac{\sqrt{51}}{8}$

F $\frac{\sqrt{34}}{8}$

4.34

TMUA 2021 · Paper 1 · Q19

The equation

$$\sin^2\left(4^{\cos\theta} \times 60^\circ\right) = \frac{3}{4}$$

has exactly three solutions in the range $0^\circ \leq \theta \leq x^\circ$

What is the range of all possible values of x?

A $90 \leq x < 120$

B $90 \leq x < 270$

C $120 \leq x < 240$

D $270 \leq x < 300$

E $300 \leq x < 360$

F $450 \leq x < 630$

4.35

TMUA 2021 · Paper 2 · Q19

The angle θ can take any of the values 1°, 2°, 3°, $\ldots$, 359°, 360°.

For how many of these values of θ is it true that

$$\sin\theta\sqrt{1+\sin\theta}\sqrt{1-\sin\theta} + \cos\theta\sqrt{1+\cos\theta}\sqrt{1-\cos\theta} = 0$$

A 0

B 1

C 2

D 4

E 93

F 182

G 271

H 360

4.36

TMUA 2022 · Paper 2 · Q19

A polygon has n vertices, where $n \geq 3$. It has the following properties:

- Every vertex of the polygon lies on the circumference of a circle C.
- The centre of the circle C is inside the polygon.
- The radii from the centre of the circle C to the vertices of the polygon cut the polygon into n triangles of equal area.

For which values of n are these properties **sufficient** to deduce that the polygon is regular?

A no values of n

B $n = 3$ only

C $n = 3$ and $n = 4$ only

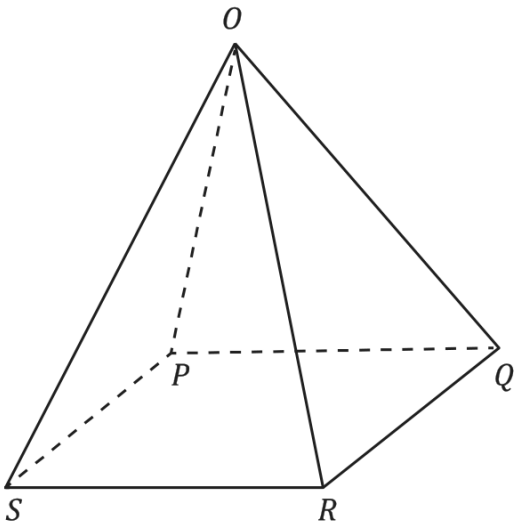
D $n = 3$ and $n \geq 5$ only

E all values of n

4.37

TMUA 2016 · Paper 1 · Q20

The diagram shows a square-based pyramid with base $PQRS$ and vertex O. All the edges of the pyramid are of length 20 metres.



[diagram not to scale]

Find the shortest distance, in metres, along the outer surface of the pyramid from P to the midpoint of OR.

A $10\sqrt{5-2\sqrt{3}}$

B $10\sqrt{3}$

C $10\sqrt{5}$

D $10\sqrt{7}$

E $10\sqrt{5+2\sqrt{3}}$

4.38

TMUA 2017 · Paper 1 · Q20

The lengths of the sides QR, RP and PQ in triangle PQR are a, $a + d$ and $a + 2d$ respectively, where a and d are positive and such that $3d > 2a$.

What is the full range, in degrees, of possible values for angle PRQ?

A $0 < \text{angle } PRQ < 60$

B $0 < \text{angle } PRQ < 120$

C $60 < \text{angle } PRQ < 120$

D $60 < \text{angle } PRQ < 180$

E $120 < \text{angle } PRQ < 180$

4.39

TMUA 2018 · Paper 1 · Q20

Find the value of

$$\sin^2 0° + \sin^2 1° + \sin^2 2° + \sin^2 3° + \cdots + \sin^2 87° + \sin^2 88° + \sin^2 89° + \sin^2 90°$$

A 0.5

B 1

C 1.5

D 45

E 45.5

F 46

CHAPTER 5 · UNIT P5

Algebra and Functions 2

25 questions

5.1

TMUA 2020 · Paper 1 · Q3

Find the complete set of values of x for which

$$(x+4)(x+3)(1-x) > 0 \quad \text{and} \quad (x+2)(x-2) < 0$$

A $1 < x < 2$

B $-2 < x < 1$

C $-2 < x < 2$

D $x < -2$ or $x > 1$

E $x < -4$ or $x > 2$

F $x < -4$ or $-3 < x < 1$

G $-4 < x < -2$ or $x > 1$

5.2

TMUA Specimen · Paper 1 · Q4

The complete set of values of x for which $(x^2-1)(x-2) > 0$ is

A $x < -1,\ 1 < x < 2$

B $x < -1,\ x > 2$

C $-1 < x < 2$

D $x < 1,\ x > 2$

E $-1 < x < 1,\ x > 2$

5.3

TMUA 2023 · Paper 2 · Q6

Consider the following equation where a is a real number and $a > 1$:

$$(*) \quad a^x = x$$

Which of the following equations must have the same number of real solutions as $(*)$?

I $\log_a x = x$

II $a^{2x} = x^2$

III $a^{2x} = 2x$

A none of them

B I only

C II only

D III only

E I and II only

F I and III only

G II and III only

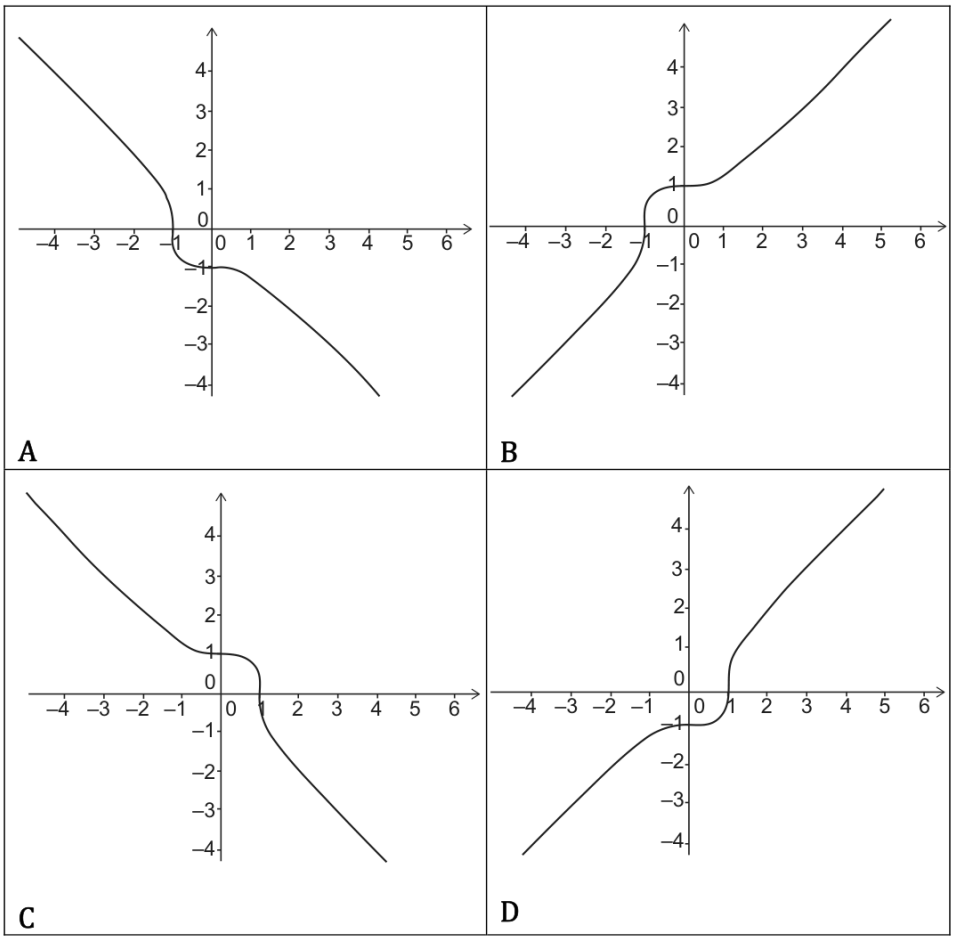
H I, II and III

5.4

TMUA Specimen · Paper 2 · Q7

Which one of the following is a sketch of the graph

$$(x+y)(x^2-xy+y^2)=1\,?$$



5.5

TMUA 2020 · Paper 2 · Q8

A student is asked to prove whether the following statement $(*)$ is true or false:

$(*)$ For all real numbers a and b, $|a+b| < |a| + |b|$

The student's proof is as follows:

Statement $(*)$ is **false**. A counterexample is $a = 3$, $b = 4$, as $|3+4| = 7$ and $|3| + |4| = 7$, but $7 < 7$ is false.

Which of the following best describes the student's proof?

A The statement $(*)$ is true, and the student's proof is not correct.

B The statement $(*)$ is false, but the student's proof is not correct: the counterexample is not valid.

C The statement $(*)$ is false, but the student's proof is not correct: the student needs to give *all* the values of a and b where $|a+b| < |a| + |b|$ is false.

D The statement $(*)$ is false, but the student's proof is not correct: the student should have instead stated that for all real numbers a and b, $|a+b| \leq |a| + |b|$.

E The statement $(*)$ is false, and the student's proof is fully correct.

5.6

TMUA 2021 · Paper 1 · Q9

Find the area enclosed by the graph of

$$|x| + |y| = 1$$

A $\frac{1}{2}$

B 1

C 2

D 4

E $\frac{1}{2}\sqrt{2}$

F $\sqrt{2}$

G $2\sqrt{2}$

5.7

TMUA Specimen · Paper 1 · Q10

The curve $y = \cos x$ is reflected in the line $y = 1$ and the resulting curve is then translated by $\frac{\pi}{4}$ units in the positive x-direction. The equation of this new curve is

A $y = 2 + \cos\left(x + \frac{\pi}{4}\right)$

B $y = 2 + \cos\left(x - \frac{\pi}{4}\right)$

C $y = 2 - \cos\left(x + \frac{\pi}{4}\right)$

D $y = 2 - \cos\left(x - \frac{\pi}{4}\right)$

.

5.8

TMUA 2018 · Paper 1 · Q10

x and y satisfy $|2 - x| \leq 6$ and $|y + 2| \leq 4$.

What is the greatest possible value of $|xy|$?

A 16

B 24

C 32

D 40

E 48

F There is no greatest possible value.

5.9

TMUA 2020 · Paper 1 · Q10

The following sequence of transformations is applied to the curve $y = 4x^2$

1. Translation by $\begin{pmatrix} 3 \\ -5 \end{pmatrix}$
2. Reflection in the x-axis
3. Stretch parallel to the x-axis with scale factor 2

What is the equation of the resulting curve?

A $y = -x^2 + 12x - 31$

B $y = -x^2 + 12x - 41$

C $y = x^2 + 12x + 31$

D $y = x^2 + 12x + 41$

E $y = -16x^2 + 48x - 31$

F $y = -16x^2 + 48x - 41$

G $y = 16x^2 - 48x + 31$

H $y = 16x^2 - 48x + 41$

5.10

TMUA 2022 · Paper 1 · Q10

A sequence of translations is applied to the graph of $y = x^3$

Which of the following graphs could be the result of this sequence of translations?

I $y = x^3 - 3x^2 + 9x - 27$

II $y = x^3 - 9x^2 + 27x - 3$

III $y = 27x^3 - 9x^2 + x - 3$

A none of them

B I only

C II only

D III only

E I and II only

F I and III only

G II and III only

H I, II and III

5.11

TMUA 2018 · Paper 2 · Q10

The function $\mathrm{f}(x)$ is defined for all real numbers.

Consider the following three conditions, where a is a real constant:

I $\quad \mathrm{f}(a-x)=\mathrm{f}(a+x)$ for all real x.

II $\quad \mathrm{f}(2a-x)=\mathrm{f}(x)$ for all real x.

III $\quad \mathrm{f}(a-x)=\mathrm{f}(x)$ for all real x.

Which of these conditions is/are **necessary and sufficient** for the graph of $y=\mathrm{f}(x)$ to have reflection symmetry in the line $x=a$?

	Condition I is necessary and sufficient	**Condition II is necessary and sufficient**	**Condition III is necessary and sufficient**
A	yes	yes	yes
B	yes	yes	no
C	yes	no	yes
D	yes	no	no
E	no	yes	yes
F	no	yes	no
G	no	no	yes
H	no	no	no

5.12

TMUA 2023 · Paper 1 · Q11

It is given that $f(x)=x^2-6x$

The curves $y=f(kx)$ and $y=f(x-c)$ have the same minimum point, where $k>0$ and $c>0$

Which of the following is a correct expression for k in terms of c?

A $\quad k=\dfrac{3-c}{3}$

B $\quad k=\dfrac{3}{c+3}$

C $\quad k=\dfrac{c-6}{6}$

D $\quad k=\dfrac{6}{6-c}$

E $\quad k=\dfrac{c+9}{9}$

F $\quad k=\dfrac{9}{c-9}$

5.13

TMUA 2016 · Paper 2 · Q14

$f(x) = ax^4 + bx^3 + cx^2 + dx + e$, where a, b, c, d, and e are real numbers.

Suppose $f(x) = 1$ has p distinct real solutions, $f(x) = 2$ has q distinct real solutions, $f(x) = 3$ has r distinct real solutions, and $f(x) = 4$ has s distinct real solutions.

Which one of the following is **not** possible?

A $p = 1, q = 2, r = 4$ and $s = 3$

B $p = 1, q = 3, r = 2$ and $s = 4$

C $p = 1, q = 4, r = 3$ and $s = 2$

D $p = 2, q = 4, r = 3$ and $s = 1$

E $p = 4, q = 3, r = 2$ and $s = 1$

5.14

TMUA 2022 · Paper 2 · Q14

Consider the two inequalities:

$$|x+5| < |x+11|$$
$$|x+11| < |x+1|$$

Which one of the following is correct?

A There is no real number for which both inequalities are true.

B There is exactly one real number for which both inequalities are true.

C The real numbers for which both inequalities are true form an interval of length 1.

D The real numbers for which both inequalities are true form an interval of length 2.

E The real numbers for which both inequalities are true form an interval of length 3.

F The real numbers for which both inequalities are true form an interval of length 4.

G The real numbers for which both inequalities are true form an interval of length 5.

5.15

TMUA 2021 · Paper 2 · Q16

p and q are real numbers, and the equation

$$x\,|x| = px + q$$

has exactly k distinct real solutions for x.

Which one of the following is the complete list of possible values for k?

A 0, 1, 2

B 0, 1, 2, 3

C 0, 1, 2, 3, 4

D 0, 2, 4

E 1, 2, 3

F 1, 2, 3, 4

5.16

TMUA 2017 · Paper 1 · Q18

The graph of $y = \log_{10} x$ is translated in the positive y-direction by 2 units.

This translation is equivalent to a stretch of factor k parallel to the x-axis.

What is the value of k?

A 0.01

B $\log_{10} 2$

C 0.5

D 2

E $\log_2 10$

F 100

5.17

TMUA 2020 · Paper 1 · Q18

Find the number of solutions and the sum of the solutions of the equation

$$1 - 2\cos^2 x = |\cos x|$$

where $0 \leq x \leq 180°$

A Number of solutions $= 2$ Sum of solutions $= 180°$

B Number of solutions $= 2$ Sum of solutions $= 240°$

C Number of solutions $= 3$ Sum of solutions $= 180°$

D Number of solutions $= 3$ Sum of solutions $= 360°$

E Number of solutions $= 4$ Sum of solutions $= 240°$

F Number of solutions $= 4$ Sum of solutions $= 360°$

5.18

TMUA 2021 · Paper 1 · Q18

The curve with equation

$$x = y^2 - 6y + 11$$

is rotated 90° clockwise about the point P to give the curve C.

P has x-coordinate -2 and y-coordinate 3.

What is the equation of C ?

A $y = -x^2 - 4x - 3$

B $y = -x^2 - 4x - 5$

C $y = -x^2 - 6x - 7$

D $y = -x^2 - 6x - 11$

E $y = x^2 - 4x + 5$

F $y = x^2 + 4x + 3$

G $y = x^2 - 6x + 11$

H $y = x^2 + 6x + 7$

5.19

TMUA 2022 · Paper 1 · Q18

It is given that

$$f(x) = x^2(x-1)^2(x-2)$$
$$g(x) = -p(x-q)^2(x-r)^2$$

where p, q and r are positive and $q < r$

Find the set of values of q and r that guarantees the greatest number of distinct real solutions of the equation $f(x) = g(x)$ for all p.

A $q < 1$ and $r < 1$

B $q < 1$ and $1 < r < 2$

C $q < 1$ and $r > 2$

D $1 < q < 2$ and $1 < r < 2$

E $1 < q < 2$ and $r > 2$

F $q > 2$ and $r > 2$

5.20

TMUA 2019 · Paper 2 · Q18

Consider the following inequality:

$$(*): \quad a|x| + 1 \leq |x - 2|$$

where a is a real constant.

Which one of the following describes the complete set of values of a such that $(*)$ is **true for all real** x?

A $a \leq \frac{3}{2}$

B $a \leq 1$

C $a \leq \frac{1}{2}$

D $a \leq 0$

E $a \leq -\frac{1}{2}$

F $a \leq -1$

G $a \leq -\frac{3}{2}$

H There are no such values of a.

5.21

TMUA Specimen · Paper 2 · Q19

The positive real numbers a, b, and c are such that the equation

$$x^3 + ax^2 = bx + c$$

has three real roots, one positive and two negative.

Which one of the following correctly describes the real roots of the equation

$$x^3 + c = ax^2 + bx\ ?$$

A It has three real roots, one positive and two negative.

B It has three real roots, two positive and one negative.

C It has three real roots, but their signs differ depending on a, b, and c.

D It has exactly one real root, which is positive.

E It has exactly one real root, which is negative.

F It has exactly one real root, whose sign differs depending on a, b, and c.

G The number of real roots can be one or three, but the number of roots differs depending on a, b, and c.

5.22

TMUA 2022 · Paper 1 · Q20

n is the number of points of intersection of the graphs

$$y = |x^2 - a^2| \text{ and } y = a^2|x - 1|$$

where a is a real number.

What is the smallest value of n that is **not** possible?

A $n = 1$

B $n = 2$

C $n = 3$

D $n = 4$

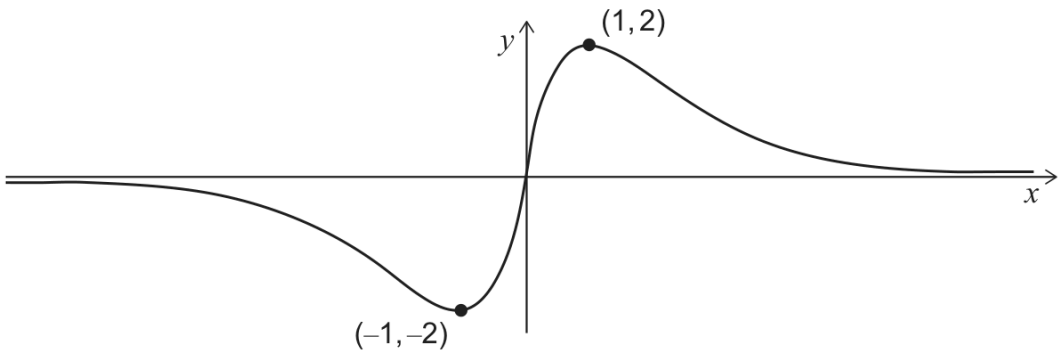
E $n = 5$

5.23

TMUA 2023 · Paper 1 · Q20

The diagram shows the graph of $y = f(x)$

The function f attains its maximum value of 2 at $x = 1$, and its minimum value of -2 at $x = -1$



Find the difference between the maximum and minimum values of $(f(x))^2 - f(x)$

A 2

B $\frac{9}{4}$

C 4

D $\frac{17}{4}$

E 6

F $\frac{25}{4}$

G 8

H $\frac{33}{4}$

5.24

TMUA 2019 · Paper 2 · Q20

When the graph of the function $y = \mathrm{f}(x)$, defined on the real numbers, is reflected in the y-axis and then translated by 2 units in the negative x-direction, the result is the graph of the function $y = \mathrm{g}(x)$.

When the graph of the same function $y = \mathrm{f}(x)$ is translated by 2 units in the negative x-direction and then reflected in the y-axis, the result is the graph of the function $y = \mathrm{h}(x)$.

Which one of the following conditions on $y = \mathrm{f}(x)$ is **necessary and sufficient** for the functions $\mathrm{g}(x)$ and $\mathrm{h}(x)$ to be identical?

A $\mathrm{f}(x) = \mathrm{f}(x + 2)$ for all x

B $\mathrm{f}(x) = \mathrm{f}(x + 4)$ for all x

C $\mathrm{f}(x) = \mathrm{f}(x + 8)$ for all x

D $\mathrm{f}(x) = \mathrm{f}(-x)$ for all x

E $\mathrm{f}(x) = \mathrm{f}(2 - x)$ for all x

F $\mathrm{f}(x) = \mathrm{f}(4 - x)$ for all x

G $\mathrm{f}(x) = \mathrm{f}(8 - x)$ for all x

The functions f_1 to f_5 are defined on the real numbers by

$$f_1(x) = \cos x$$
$$f_2(x) = \sin(\cos x)$$
$$f_3(x) = \cos(\sin(\cos x))$$
$$f_4(x) = \sin(\cos(\sin(\cos x)))$$
$$f_5(x) = \cos(\sin(\cos(\sin(\cos x))))$$

where all numbers are taken to be in radians.

These functions have maximum values m_1, m_2, m_3, m_4 and m_5, respectively.

Which one of the following statements is true?

A m_1, m_2, m_3, m_4 and m_5 are all equal to 1

B $0 < m_5 < m_4 < m_3 < m_2 < m_1 = 1$

C $m_1 = m_3 = m_5 = 1$ and $0 < m_2 = m_4 < 1$

D $m_1 = m_3 = m_5 = 1$ and $0 < m_4 < m_2 < 1$

E $m_1 = m_3 = 1$ and $0 < m_2 = m_4 < 1$ and $0 < m_5 < 1$

F $m_1 = m_3 = 1$ and $0 < m_4 < m_2 < 1$ and $0 < m_5 < 1$

CHAPTER 6 · UNIT S1

Probability

4 questions

6.1

TMUA 2021 · Paper 2 · Q3

A student is chosen at random from a class. Each student is equally likely to be chosen.

Which of the following conditions is/are **necessary** for the probability that the student wears glasses to equal $\frac{4}{15}$?

I Exactly 11 students in the class do not wear glasses.

II The number of students in the class is divisible by 3.

III The class contains 30 students, and 8 of them wear glasses.

A none of them
B I only
C II only
D III only
E I and II only
F I and III only
G II and III only
H I, II and III

6.2

TMUA Specimen · Paper 1 · Q7

A bag contains n red balls, n yellow balls, and n blue balls.

One ball is selected at random and not replaced.

A second ball is then selected at random and not replaced.

Each ball is equally likely to be chosen.

The probability that the two balls are **not** the same colour is

A $\frac{n-1}{3n-1}$

B $\frac{2n-2}{3n-1}$

C $\frac{2n}{3n-1}$

D $\frac{(n-1)^3}{27(3n-1)^3}$

E $\frac{3(n-1)}{3n-1}$

F $\frac{n^3}{27(3n-1)^3}$

6.3

TMUA 2016 · Paper 1 · Q7

60% of a sports club's members are women and the remainder are men.

This sports club offers the opportunity to play tennis or cricket. Every member plays exactly one of the two sports.

$\frac{2}{5}$ of the male members of the club play cricket;

$\frac{2}{3}$ of the cricketing members of the club are women.

What is the probability that a member of the club, chosen at random, is a woman who plays tennis?

A $\frac{1}{5}$

B $\frac{7}{25}$

C $\frac{1}{3}$

D $\frac{11}{25}$

E $\frac{3}{5}$

6.4

TMUA 2019 · Paper 2 · Q12

Most students in a large college study Mathematics. A teacher chooses three different students at random, one after the other.

Consider these three probabilities:

$$R = \text{P(At least one of the students chosen studies Mathematics)}$$
$$S = \text{P(The second student chosen studies Mathematics)}$$
$$T = \text{P(All three of the students chosen study Mathematics)}$$

Which of the following is true?

A $R < S < T$

B $R < T < S$

C $S < R < T$

D $S < T < R$

E $T < R < S$

F $T < S < R$

CHAPTER 7 · UNIT P6

Calculus 1

68 questions

7.1

TMUA 2017 · Paper 1 · Q1

Given that

$$\frac{dy}{dx} = 3x^2 - \frac{2-3x}{x^3}, \quad x \neq 0$$

and $y = 5$ when $x = 1$, find y in terms of x.

A $y = \frac{1}{3}x^3 + x^{-2} - 3x^{-1} + 6\frac{2}{3}$

B $y = x^3 + \frac{1}{2}x^{-2} - 3x^{-1} + 6\frac{1}{2}$

C $y = x^3 + x^{-2} - 3x^{-1} + 6$

D $y = x^3 + x^{-2} - x^{-1} + 4$

E $y = x^3 + 2x^{-2} - x^{-1} + 3$

F $y = 3x^3 + x^{-2} - x^{-1} + 2$

7.2

TMUA 2018 · Paper 1 · Q1

Find the value of

$$\int_1^4 \frac{3-2x}{x\sqrt{x}}\,\mathrm{d}x$$

A $-\frac{13}{2}$

B $-\frac{85}{16}$

C $-\frac{13}{8}$

D -1

E $-\frac{1}{4}$

F $\frac{7}{4}$

G 7

7.3

TMUA 2020 · Paper 1 · Q1

Which of the following is an expression for the first derivative with respect to x of

$$\frac{x^3 - 5x^2}{2x\sqrt{x}}$$

A $-\frac{\sqrt{x}}{2}$

B $\frac{\sqrt{x}}{4}$

C $\frac{3x - 5}{4\sqrt{x}}$

D $\frac{3\sqrt{x} - 5}{4\sqrt{x}}$

E $\frac{3\sqrt{x} - 10}{3\sqrt{x}}$

F $\frac{3x^2 - 10x}{3\sqrt{x}}$

7.4

TMUA 2023 · Paper 1 · Q1

Given that

$$\int_0^1 (ax + b)\,\mathrm{d}x = 1$$

and

$$\int_0^1 x(ax + b)\,\mathrm{d}x = 1$$

find the value of $a + b$.

A –1
B 0
C 1
D 2
E 3
F 4
G 5

7.5

TMUA 2016 · Paper 2 · Q1

Find the value of

$$\int_1^2 \left(x^2 - \frac{4}{x^2}\right)^2 \mathrm{d}x$$

A $\frac{43}{15}$

B 3

C $\frac{97}{15}$

D $\frac{103}{15}$

E $\frac{163}{15}$

F 18

7.6

TMUA 2017 · Paper 2 · Q1

Given that $y = \dfrac{(1-3x)^2}{2x^{\frac{3}{2}}}$, which one of the following is a correct expression for $\dfrac{dy}{dx}$?

A $\frac{9}{4}x^{-\frac{1}{2}} + \frac{3}{2}x^{-\frac{3}{2}} - \frac{3}{4}x^{-\frac{5}{2}}$

B $\frac{9}{4}x^{-\frac{1}{2}} - \frac{3}{2}x^{-\frac{3}{2}} + \frac{3}{4}x^{-\frac{5}{2}}$

C $\frac{9}{4}x^{-\frac{1}{2}} - \frac{3}{2}x^{-\frac{3}{2}} - \frac{3}{4}x^{-\frac{5}{2}}$

D $-\frac{9}{4}x^{-\frac{1}{2}} + \frac{3}{2}x^{-\frac{3}{2}} + \frac{3}{4}x^{-\frac{5}{2}}$

E $-\frac{9}{4}x^{-\frac{1}{2}} + \frac{3}{2}x^{-\frac{3}{2}} - \frac{3}{4}x^{-\frac{5}{2}}$

F $-\frac{9}{4}x^{-\frac{1}{2}} - \frac{3}{2}x^{-\frac{3}{2}} - \frac{3}{4}x^{-\frac{5}{2}}$

7.7

TMUA 2018 · Paper 2 · Q1

The function f is given, for $x > 0$, by

$$\mathrm{f}(x) = \frac{x^3 - 4x}{2\sqrt{x}}$$

Find the value of $\mathrm{f}'(4)$.

A 3

B 9

C 9.5

D 12

E 39.5

F 88

7.8

TMUA 2021 · Paper 2 · Q1

Find the value of

$$\int_1^4 \left(3\sqrt{x} + \frac{4}{x^2}\right) \mathrm{d}x$$

A -0.75
B 7.125
C 11
D 17
E 18
F 21.875
G 34.5

7.9

TMUA 2022 · Paper 2 · Q1

Determine the number of stationary points on the curve with equation

$$y = 3x^4 + 4x^3 + 6x^2 - 5$$

A 0
B 1
C 2
D 3
E 4

7.10

TMUA 2017 · Paper 1 · Q2

The function f is given by

$$f(x) = \left(\frac{2}{x} - \frac{1}{2x^2}\right)^2 \quad (x \neq 0)$$

What is the value of $f''(1)$?

A -3
B -1
C 5
D 17
E 29
F 80

7.11

TMUA 2021 · Paper 1 · Q2

The curve $y = x^3 - 6x + 3$ has turning points at $x = \alpha$ and $x = \beta$, where $\beta > \alpha$.

Find

$$\int_{\alpha}^{\beta} x^3 - 6x + 3 \, \mathrm{d}x$$

A $-8\sqrt{2}$

B -10

C $-10 + 6\sqrt{2}$

D 0

E $12 - 8\sqrt{2}$

F $6\sqrt{2}$

G 12

7.12

TMUA Specimen · Paper 2 · Q2

The gradient of the curve $y = \frac{(3x-2)^2}{x\sqrt{x}}$ at the point where $x = 2$ is

A $\frac{3}{2}\sqrt{2}$

B $3\sqrt{2}$

C $4\sqrt{2}$

D $\frac{9}{2}\sqrt{2}$

E $6\sqrt{2}$

7.13

TMUA 2016 · Paper 2 · Q2

$$f(x) = \frac{(x^2 + 5)(2x)}{\sqrt[4]{x^3}}, \qquad x > 0$$

Which one of the following is equal to $f'(x)$?

A $8x^{\frac{9}{4}} + \frac{40}{3}x^{\frac{1}{4}}$

B $\frac{9}{2}x^{\frac{5}{4}} + \frac{5}{2}x^{-\frac{3}{4}}$

C $8x^{\frac{9}{4}} + \frac{40}{3}x^{-\frac{1}{4}}$

D $\frac{8}{13}x^{\frac{13}{4}} + 8x^{\frac{5}{4}}$

7.14

TMUA 2023 · Paper 2 · Q2

Evaluate

$$\int_9^{16}\left(\frac{1}{\sqrt{x}}+\sqrt{x}\right)^2 \mathrm{d}x - \int_9^{16}\left(\frac{1}{\sqrt{x}}-\sqrt{x}\right)^2 \mathrm{d}x$$

A 0
B 2
C 4
D 7
E 14
F 28
G 75
H 175

7.15

TMUA 2016 · Paper 1 · Q3

A line is drawn normal to the curve $y = \frac{2}{x^2}$ at the point on the curve where $x = 1$.

This line cuts the x-axis at P and the y-axis at Q.

The length of PQ is

A $\frac{3\sqrt{5}}{2}$

B $\frac{3\sqrt{17}}{4}$

C $\frac{7\sqrt{17}}{4}$

D $\frac{35}{4}$

E $\frac{35\sqrt{5}}{2}$

F $\frac{3\sqrt{17}}{2}$

7.16

TMUA 2022 · Paper 1 · Q3

Given the following statements about a function f

- $\mathrm{f}''(x) = a$ for all x
- $\mathrm{f}(0) = 1,\ \mathrm{f}(1) = 2$
- $\int_0^1 \mathrm{f}(x)\,\mathrm{d}x = 1$

find the value of a.

A -6
B -3
C -2
D 2
E 3
F 6

7.17

TMUA 2023 · Paper 1 · Q3

For any integer $n \geq 0$,

$$\int_n^{n+1} f(x)\,\mathrm{d}x = n+1$$

Evaluate

$$\int_0^3 f(x)\,\mathrm{d}x + \int_1^3 f(x)\,\mathrm{d}x + \int_2^3 f(x)\,\mathrm{d}x + \int_4^3 f(x)\,\mathrm{d}x + \int_5^3 f(x)\,\mathrm{d}x$$

A -2
B 0
C 1
D 4
E 18
F 27

7.18

TMUA 2016 · Paper 1 · Q5

What is the total area enclosed between the curve $y = x^2 - 1$, the x-axis and the lines $x = -2$ and $x = 2$?

A $\frac{4}{3}$
B $\frac{8}{3}$
C 4
D $\frac{16}{3}$
E 12
F 16

7.19

TMUA 2022 · Paper 1 · Q6

Given that

$$\int_{\log_2 5}^{\log_2 20} x \, \mathrm{d}x = \log_2 M$$

what is the value of M?

A 4
B 15
C 16
D 20
E 25
F 100
G 10 000

7.20

TMUA 2018 · Paper 2 · Q6

Which one of the following functions provides a **counterexample** to the statement:

if $\mathrm{f}'(x) > 0$ for all real x, **then** $\mathrm{f}(x) > 0$ for all real x.

A $\mathrm{f}(x) = x^2 + 1$
B $\mathrm{f}(x) = x^2 - 1$
C $\mathrm{f}(x) = x^3 + x + 1$
D $\mathrm{f}(x) = 1 - x$
E $\mathrm{f}(x) = 2^x$

7.21

TMUA 2020 · Paper 2 · Q6

The function $\mathrm{f}(x)$ is defined for all real values of x.

Which of the following conditions on $\mathrm{f}(x)$ is/are **necessary** to ensure that

$$\int_{-5}^{0} \mathrm{f}(x) \, \mathrm{d}x = \int_{0}^{5} \mathrm{f}(x) \, \mathrm{d}x$$

Condition I: $\mathrm{f}(x) = \mathrm{f}(-x)$ for $-5 \leq x \leq 5$

Condition II: $\mathrm{f}(x) = c$ for $-5 \leq x \leq 5$, where c is a constant

Condition III: $\mathrm{f}(x) = -\mathrm{f}(-x)$ for $-5 \leq x \leq 5$

A none of them
B I only
C II only
D III only
E I and II only
F I and III only
G II and III only
H I, II and III

7.22

TMUA 2021 · Paper 2 · Q6

Consider the following two statements about the polynomial $\mathrm{f}(x)$:

P: $\mathrm{f}(x) = 0$ for exactly three real values of x

Q: $\mathrm{f}'(x) = 0$ for exactly two real values of x

Which one of the following is correct?

A P is **necessary** but **not sufficient** for Q.

B P is **sufficient** but **not necessary** for Q.

C P is **necessary and sufficient** for Q.

D P is **not necessary** and **not sufficient** for Q.

7.23

TMUA 2019 · Paper 1 · Q7

A curve has equation

$$y = (2q - x^2)(2qx + 3)$$

The gradient of the curve at $x = -1$ is a function of q.

Find the value of q which minimises the gradient of the curve at $x = -1$.

A -1

B $-\frac{3}{4}$

C $-\frac{1}{2}$

D 0

E $\frac{1}{2}$

F $\frac{3}{4}$

G 1

7.24

TMUA 2021 · Paper 1 · Q7

The function f is such that $f(0) = 0$, and $xf(x) > 0$ for all non-zero values of x.

It is given that

$$\int_{-2}^{2} f(x)\,dx = 4$$

and

$$\int_{-2}^{2} |f(x)|\,dx = 8$$

Evaluate

$$\int_{-2}^{0} f(|x|)\,dx$$

A -8
B -6
C -4
D -2
E 2
F 4
G 6
H 8

7.25

TMUA 2022 · Paper 1 · Q7

Find the finite area enclosed between the line $y = 0$ and the curve $y = x^2 - 4|x| - 12$

A $\frac{128}{3}$
B $\frac{176}{3}$
C $\frac{256}{3}$
D 108
E 144
F 288

7.26

TMUA 2021 · Paper 2 · Q8

Consider the following statement about the polynomial $\mathrm{p}(x)$, where a and b are real numbers with $a < b$:

$(*)$ There exists a number c with $a < c < b$ such that $\mathrm{p}'(c) = 0$.

Which one of the following is true?

A The condition $\mathrm{p}(a) = \mathrm{p}(b)$ is **necessary and sufficient** for $(*)$

B The condition $\mathrm{p}(a) = \mathrm{p}(b)$ is **necessary** but **not sufficient** for $(*)$

C The condition $\mathrm{p}(a) = \mathrm{p}(b)$ is **sufficient** but **not necessary** for $(*)$

D The condition $\mathrm{p}(a) = \mathrm{p}(b)$ is **not necessary** and **not sufficient** for $(*)$

7.27

TMUA 2018 · Paper 1 · Q9

Find the complete set of values of the constant c for which the cubic equation

$$2x^3 - 3x^2 - 12x + c = 0$$

has three distinct real solutions.

A $-20 < c < 7$

B $-7 < c < 20$

C $c > 7$

D $c > -7$

E $c < 20$

F $c < -20$

7.28

TMUA 2019 · Paper 1 · Q9

p is a positive constant.

Find the area enclosed between the curves $y = p\sqrt{x}$ and $x = p\sqrt{y}$

A $\frac{2}{3}p^{\frac{5}{2}} - \frac{1}{2}p^2$

B $\frac{4}{3}p^{\frac{5}{2}} - p^2$

C $\frac{p^4}{6}$

D $\frac{p^4}{3}$

E $\frac{2}{3}p^3 - \frac{1}{2}p^4$

F $\frac{4}{3}p^3 - p^4$

G $2p^4$

7.29

TMUA 2021 · Paper 2 · Q9

Consider the following statements about a polynomial $\mathrm{f}(x)$:

I $\mathrm{f}(x) = px^3 + qx^2 + rx + s$, where $p \neq 0$.

II There is a real number t for which $\mathrm{f}'(t) = 0$.

III There are real numbers u and v for which $\mathrm{f}(u)\,\mathrm{f}(v) < 0$.

Which of these statements is/are **sufficient** for the equation $\mathrm{f}(x) = 0$ to have a real solution?

	Statement I is sufficient	**Statement II is sufficient**	**Statement III is sufficient**
A	Yes	Yes	Yes
B	Yes	Yes	No
C	Yes	No	Yes
D	Yes	No	No
E	No	Yes	Yes
F	No	Yes	No
G	No	No	Yes
H	No	No	No

7.30

TMUA 2017 · Paper 1 · Q10

A curve C has equation $y = f(x)$ where

$$f(x) = p^3 - 6p^2x + 3px^2 - x^3$$

and p is real.

The gradient of the normal to the curve C at the point where $x = -1$ is M.

What is the greatest possible value of M as p varies?

A $-\frac{3}{2}$

B $-\frac{2}{3}$

C $-\frac{1}{2}$

D $\frac{1}{4}$

E $\frac{2}{3}$

F $\frac{3}{2}$

7.31

TMUA 2019 · Paper 1 · Q10

Evaluate

$$\int_{-1}^{3} |x|(1-x)\,\mathrm{d}x$$

A $\frac{17}{3}$

B $-\frac{17}{3}$

C $\frac{16}{3}$

D $-\frac{16}{3}$

E $\frac{11}{3}$

F $-\frac{11}{3}$

7.32

TMUA 2017 · Paper 2 · Q10

$f(x)$ is a function defined for all real values of x.

Which one of the following is a **sufficient** condition for $\int_1^3 f(x)\,dx = 0$?

A $f(2) = 0$

B $f(1) = f(3) = 0$

C $f(-x) = -f(x)$ for all x

D $f(x+2) = -f(2-x)$ for all x

E $f(x-2) = -f(2-x)$ for all x

7.33

TMUA 2018 · Paper 1 · Q11

The line $y = mx + 5$, where $m > 0$, is normal to the curve $y = 10 - x^2$ at the point (p, q).

What is the value of p?

A $\frac{\sqrt{2}}{6}$

B $-\frac{\sqrt{2}}{6}$

C $\frac{3\sqrt{2}}{2}$

D $-\frac{3\sqrt{2}}{2}$

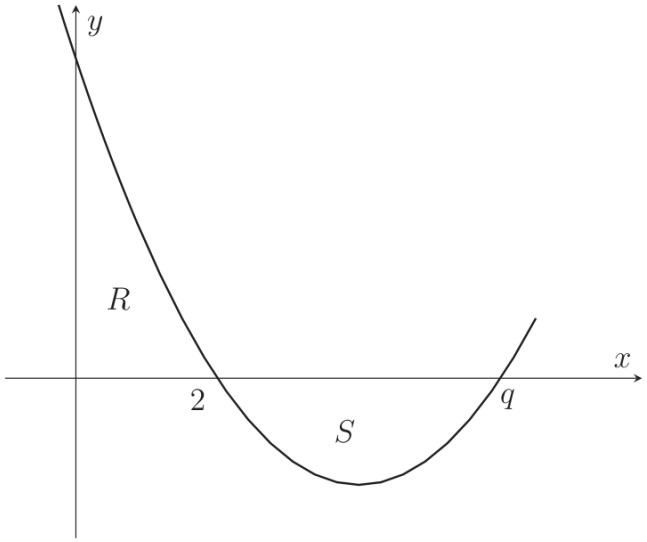
E $\sqrt{5}$

F $-\sqrt{5}$

7.34

TMUA 2020 · Paper 1 · Q11

The quadratic function shown passes through $(2, 0)$ and $(q, 0)$, where $q > 2$.



What is the value of q such that the area of region R equals the area of region S?

A $\sqrt{6}$

B 3

C $\frac{18}{5}$

D 4

E 6

F $\frac{33}{5}$

7.35

TMUA 2021 · Paper 1 · Q11

The function f is given by

$$\mathrm{f}(x) = x^{\frac{1}{7}}(x^2 - x + 1)$$

Find the fraction of the interval $0 < x < 1$ for which $\mathrm{f}(x)$ is decreasing.

A $\frac{2}{15}$

B $\frac{1}{5}$

C $\frac{1}{3}$

D $\frac{1}{2}$

E $\frac{2}{3}$

F $\frac{4}{5}$

G $\frac{13}{15}$

7.36

TMUA 2016 · Paper 2 · Q11

$f(x)$ is a polynomial with real coefficients.

The equation $f(x) = 0$ has exactly two real roots, $x = -p$ and $x = p$, where $p > 0$.

Consider the following three statements:

1 $f'(x) = 0$ for exactly one value of x between $-p$ and p

2 The area between the curve $y = f(x)$, the x-axis and the lines $x = -p$ and $x = p$ is given by $2\int_0^p f(x)\,\mathrm{d}x$

3 The graph of $y = -f(-x)$ intersects the x-axis at the points $x = -p$ and $x = p$ only

Which of the above statements **must** be true?

A none

B 1 only

C 2 only

D 3 only

E 1 and 2 only

F 1 and 3 only

G 2 and 3 only

H 1, 2 and 3

7.37

TMUA 2017 · Paper 2 · Q11

The function $f(x)$ is increasing and $f(0) = 0$.

The positive constants a and b are such that $a < b$.

The area of the region enclosed by the curve $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$ is denoted by R.

The function $g(x)$ is defined by $g(x) = f(x) + 2f(b)$.

Which of the following is an expression for the area enclosed by the curve $y = g(x)$, the x-axis and the lines $x = a$ and $x = b$?

A $R + (b - a)f(b)$

B $R + 2(b - a)f(b)$

C $R + 2f(b) - f(a)$

D $R + 2f(b)$

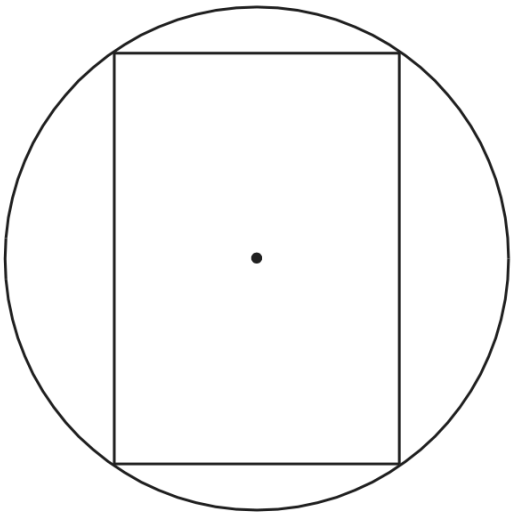
E $R + (f(b))^2$

F $R + (f(b))^2 - (f(a))^2$

G $R + 2(f(b) - f(a))f(b)$

A right circular cylinder is contained within a sphere of radius 5 cm in such a way that the whole of the circumferences of both ends of the cylinder are in contact with the sphere.

The diagram shows a planar cross section through the centre of the sphere and cylinder.



[diagram not to scale]

Find, in cubic centimetres, the maximum possible volume of the cylinder.

A 250π

B 500π

C 1000π

D $\frac{250\sqrt{3}}{3}\pi$

E $\frac{500\sqrt{3}}{9}\pi$

F $\frac{1000\sqrt{3}}{9}\pi$

7.39

TMUA 2017 · Paper 1 · Q12

The polynomial function $f(x)$ is such that $f(x) > 0$ for all values of x.

Given $\int_2^4 f(x)\,dx = A$, which one of the following statements **must** be correct?

A $\int_0^2 [f(x+2)+1]\,dx = A+1$

B $\int_0^2 [f(x+2)+1]\,dx = A+2$

C $\int_2^4 [f(x+2)+1]\,dx = A+1$

D $\int_2^4 [f(x+2)+1]\,dx = A+2$

E $\int_4^6 [f(x+2)+1]\,dx = A+1$

F $\int_4^6 [f(x+2)+1]\,dx = A+2$

7.40

TMUA 2018 · Paper 1 · Q12

A curve has equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = x(x-p)(x-q)(r-x)$$

with $0 < p < q < r$.

You are given that:

$$\int_0^r \mathrm{f}(x)\,\mathrm{d}x = 0$$

$$\int_0^q \mathrm{f}(x)\,\mathrm{d}x = -2$$

$$\int_p^r \mathrm{f}(x)\,\mathrm{d}x = -3$$

What is the total area enclosed by the curve and the x-axis for $0 \le x \le r$?

A 0
B 1
C 4
D 5
E 6
F 10

7.41

TMUA 2019 · Paper 1 · Q12

It is given that

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{24\pi(t-1)}{(1+\sqrt{t}\,)} \quad \text{for } t \geq 1$$

and $V = 7$ when $t = 1$.

Find the value of V when $t = 9$.

A $208\pi + 7$

B $216\pi + 7$

C $224\pi + 7$

D $416\pi + 7$

E $608\pi + 7$

F $744\pi + 7$

7.42

TMUA 2021 · Paper 2 · Q12

Which of the following statements about polynomials f and g is/are true?

I **If** $\mathrm{f}(x) \geq \mathrm{g}(x)$ for all $x \geq 0$, **then** $\int_0^x \mathrm{f}(t)\,\mathrm{d}t \geq \int_0^x \mathrm{g}(t)\,\mathrm{d}t$ for all $x \geq 0$.

II **If** $\mathrm{f}(x) \geq \mathrm{g}(x)$ for all $x \geq 0$, **then** $\mathrm{f}'(x) \geq \mathrm{g}'(x)$ for all $x \geq 0$.

III **If** $\mathrm{f}'(x) \geq \mathrm{g}'(x)$ for all $x \geq 0$, **then** $\mathrm{f}(x) \geq \mathrm{g}(x)$ for all $x \geq 0$.

A none of them

B I only

C II only

D III only

E I and II only

F I and III only

G II and III only

H I, II and III

7.43

TMUA 2022 · Paper 2 · Q12

Place the following integrals in order of size, starting with the smallest.

$$P = \int_0^1 2^{\sqrt{x}}\,\mathrm{d}x$$

$$Q = \int_0^1 2^{x}\,\mathrm{d}x$$

$$R = \int_0^1 \left(\sqrt{2}\right)^{x}\,\mathrm{d}x$$

A $P < Q < R$

B $P < R < Q$

C $Q < P < R$

D $Q < R < P$

E $R < P < Q$

F $R < Q < P$

7.44

TMUA 2016 · Paper 1 · Q13

How many real roots does the equation $3x^5 - 10x^3 - 120x + 30 = 0$ have?

A 1

B 2

C 3

D 4

E 5

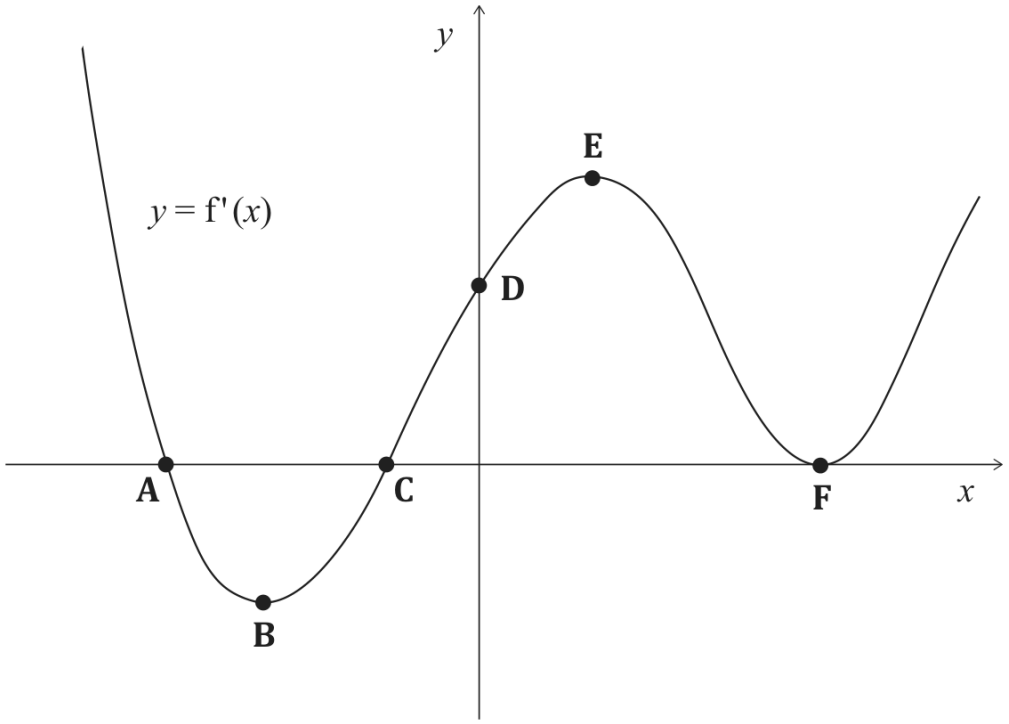
7.45

TMUA 2018 · Paper 1 · Q13

The function $\mathrm{f}(x)$ has derivative $\mathrm{f}'(x)$.

The diagram below shows the graph of $y = \mathrm{f}'(x)$.

Which point corresponds to a local minimum of $\mathrm{f}(x)$?



7.46

TMUA 2020 · Paper 2 · Q13

$\mathrm{f}(x)$ is a function for which

$$\int_0^3 \big(\mathrm{f}(x)\big)^2 \,\mathrm{d}x \;+\; \int_0^3 \mathrm{f}(x)\,\mathrm{d}x \;=\; \int_0^1 \mathrm{f}(x)\,\mathrm{d}x$$

Which of the following claims about $\mathrm{f}(x)$ is/are **necessarily** true?

I $\quad \mathrm{f}(x) \le 0$ for some x with $1 \le x \le 3$

II $\quad \int_0^3 \mathrm{f}(x)\,\mathrm{d}x \le \int_0^1 \mathrm{f}(x)\,\mathrm{d}x$

A neither of them

B I only

C II only

D I and II

7.47

TMUA 2020 · Paper 1 · Q14

The area enclosed between the line $y = mx$ and the curve $y = x^3$ is 6.

What is the value of m ?

A 2

B 4

C $\sqrt{3}$

D $\sqrt{6}$

E $2\sqrt{3}$

F $2\sqrt{6}$

7.48

TMUA 2023 · Paper 1 · Q14

The function

$$f(x) = \frac{2}{3}x^3 + 2mx^2 + n, \quad m > 0$$

has three distinct real roots.

What is the complete range of possible values of n, in terms of m?

A $-\frac{8}{3}m^3 < n < 0$

B $-\frac{4}{3}m^3 < n < 0$

C $0 < n < \frac{3}{2}m^2$

D $0 < n < \frac{40}{3}m^3$

E $n < -\frac{8}{3}m^3$

F $n < \frac{3}{2}m^2$

G $n > -\frac{4}{3}m^3$

H $n > \frac{40}{3}m^3$

7.49

TMUA Specimen · Paper 2 · Q14

The graph of the polynomial function

$$y = ax^5 + bx^4 + cx^3 + dx^2 + ex + f,$$

is sketched, where a, b, c, d, e, and f are real constants with $a \neq 0$.

Which one of the following is **not** possible?

A The graph has two local minima and two local maxima.

B The graph has one local minimum and two local maxima.

C The graph has one local minimum and one local maximum.

D The graph has no local minima or local maxima.

7.50

TMUA 2019 · Paper 2 · Q14

Consider the following statements about the polynomial $\mathrm{p}(x)$, where $a < b$:

I $\mathrm{p}(a) \leq \mathrm{p}(b)$

II $\mathrm{p}'(a) \leq \mathrm{p}'(b)$

III $\mathrm{p}''(a) \leq \mathrm{p}''(b)$

Which of these statements is a **necessary** condition for $\mathrm{p}(x)$ to be increasing for $a \leq x \leq b$?

A none of them

B I only

C II only

D III only

E I and II only

F I and III only

G II and III only

H I, II and III

7.51

TMUA Specimen · Paper 1 · Q15

The smallest possible value of $\int_0^1 (x-a)^2\,dx$ as a varies is

A $\frac{1}{12}$

B $\frac{1}{3}$

C $\frac{1}{2}$

D $\frac{7}{12}$

7.52

TMUA 2016 · Paper 1 · Q15

The least possible value of the gradient of the curve $y = (2x + a)(x - 2a)^2$ at the point where $x = 1$, as a varies, is

A $-\frac{49}{4}$

B -8

C $-\frac{25}{4}$

D $\frac{7}{4}$

E $\frac{47}{16}$

7.53

TMUA 2022 · Paper 1 · Q15

A rectangle is drawn in the region enclosed by the curves p and q, where

$$p(x) = 8 - 2x^2$$

$$q(x) = x^2 - 2$$

such that the sides of the rectangle are parallel to the x- and y-axes.

What is the maximum possible area of the rectangle?

A $\frac{26}{9}$

B $\frac{52}{9}$

C $\frac{4\sqrt{6}}{3}$

D $\frac{8\sqrt{6}}{3}$

E $4\sqrt{2}$

F $8\sqrt{2}$

G $\frac{20\sqrt{10}}{9}$

H $\frac{40\sqrt{10}}{9}$

7.54

TMUA 2018 · Paper 2 · Q15

It is given that $\mathrm{f}(x) = x^3 + 3qx^2 + 2$, where q is a real constant.

The equation $\mathrm{f}(x) = 0$ has 3 distinct real roots.

Which of the following statements is/are **necessarily** true?

I The equation $\mathrm{f}(x) + 1 = 0$ has 3 distinct real roots.

II The equation $\mathrm{f}(x + 1) = 0$ has 3 distinct real roots.

III The equation $\mathrm{f}(-x) - 1 = 0$ has 3 distinct real roots.

A none of them
B I only
C II only
D III only
E I and II only
F I and III only
G II and III only
H I, II and III

7.55

TMUA 2017 · Paper 1 · Q16

The functions f and g are given by $f(x) = 3x^2 + 12x + 4$ and $g(x) = x^3 + 6x^2 + 9x - 8$.

What is the complete set of values of x for which one of the functions is increasing and the other decreasing?

A $x \geq -1$
B $x \leq -1$
C $-3 \leq x \leq -2,\ x \geq -1$
D $x \leq -2,\ x \geq -1$
E $x \leq -3,\ -2 \leq x \leq -1$
F $x \leq -3,\ x \geq -2$
G $-2 \leq x \leq -1$

7.56

TMUA 2018 · Paper 1 · Q16

The curve C has equation $y = x^2 + bx + 2$, where $b \geq 0$.

Find the value of b that minimises the distance between the origin and the stationary point of the curve C.

A $b = 0$
B $b = 1$
C $b = 2$
D $b = \frac{\sqrt{6}}{2}$
E $b = \sqrt{2}$
F $b = \sqrt{6}$

7.57

TMUA 2019 · Paper 1 · Q16

Given that

$$2\int_0^1 \mathrm{f}(x)\,\mathrm{d}x + 5\int_1^2 \mathrm{f}(x)\,\mathrm{d}x = 14$$

and

$$\int_0^1 \mathrm{f}(x+1)\,\mathrm{d}x = 6$$

find the value of

$$\int_0^2 \mathrm{f}(x)\,\mathrm{d}x$$

A -8
B -4
C -2
D 2
E 4
F $\frac{29}{5}$
G $\frac{32}{5}$
H 14

7.58

TMUA 2017 · Paper 2 · Q16

Consider the following statement:

(∗) **If** $f(x)$ is an integer for every integer x, **then** $f'(x)$ is an integer for every integer x.

Which one of the following is a **counterexample** to (∗)?

A $f(x) = \dfrac{x^3 + x + 1}{4}$

B $f(x) = \dfrac{x^4 + x^2 + x}{2}$

C $f(x) = \dfrac{x^4 + x^3 + x^2 + x}{2}$

D $f(x) = \dfrac{x^4 + 2x^3 + x^2}{4}$

7.59

TMUA 2022 · Paper 2 · Q17

A student answered the following question:

a and b are non-zero real numbers.

Prove that the equation $x^3 + ax^2 + b = 0$ has three distinct real roots if $27b\left(b + \frac{4a^3}{27}\right) < 0$

Here is the student's solution:

I We differentiate $y = x^3 + ax^2 + b$ to get $\frac{dy}{dx} = 3x^2 + 2ax = x(3x + 2a)$

Solving $\frac{dy}{dx} = 0$ shows that the stationary points are at $(0, b)$ and $\left(-\frac{2a}{3}, b + \frac{4a^3}{27}\right)$

II If $27b\left(b + \frac{4a^3}{27}\right) < 0$, then b and $b + \frac{4a^3}{27}$ must have opposite signs, and so one of the stationary points is above the x-axis and one is below.

III If the cubic has three distinct real roots, then one of the stationary points is above the x-axis and one is below.

IV Hence if $27b\left(b + \frac{4a^3}{27}\right) < 0$, then the equation has three distinct real roots.

Which one of the following options best describes the student's solution?

A It is a completely correct solution.

B The student has instead proved the converse of the statement in the question.

C The solution is wrong, because the student should have stated step II after step III.

D The solution is wrong, because the student should have shown the converse of the result in step II.

E The solution is wrong, because the student should have shown the converse of the result in step III.

7.60

TMUA 2016 · Paper 1 · Q18

The function $\frac{1-x}{\sqrt[3]{x^2}}$ is defined for all $x \neq 0$.

The complete set of values of x for which the function is decreasing is

A $x \leq -2,\ x > 0$

B $-2 \leq x < 0$

C $x \leq 1,\ x \neq 0$

D $x \geq 1$

E $-2 \leq x \leq 1,\ x \neq 0$

F $x \leq -2,\ x \geq 1$

7.61

TMUA 2019 · Paper 1 · Q18

Find the shortest distance between the curve $y = x^2 + 4$ and the line $y = 2x - 2$.

A 2

B $\sqrt{5}$

C $\frac{6\sqrt{5}}{5}$

D 3

E $\frac{5\sqrt{5}}{3}$

F 5

G 6

7.62

TMUA 2016 · Paper 2 · Q18

Consider this statement about a function $f(x)$:

(*) If $\left(f(x)\right)^2 \leq 1$ for all $-1 \leq x \leq 1$ then $\int_{-1}^{1}\left(f(x)\right)^2 \mathrm{d}x \leq \int_{-1}^{1} f(x)\,\mathrm{d}x$

Which one of the following functions provides a **counterexample** to (*)?

A $f(x) = x + \frac{1}{2}$

B $f(x) = x - \frac{1}{2}$

C $f(x) = x + x^3$

D $f(x) = x - x^3$

E $f(x) = x^2 + x^4$

F $f(x) = x^2 - x^4$

7.63

TMUA 2018 · Paper 2 · Q18

f(x) is a polynomial function defined for all real x.

Which of the following is a **necessary** condition for the inequality

$$\frac{\mathrm{f}(a) + \mathrm{f}(b)}{2} \geq \mathrm{f}\left(\frac{a+b}{2}\right)$$

to be true for all real numbers a and b with $a < b$?

A f$(x) \geq 0$ for all real x

B f$'(x) \geq 0$ for all real x

C f$''(x) \geq 0$ for all real x

D f$(x) \leq 0$ for all real x

E f$'(x) \leq 0$ for all real x

F f$''(x) \leq 0$ for all real x

7.64

TMUA 2023 · Paper 1 · Q19

The solution to the differential equation

$$\frac{dy}{dx} = |-6x| \quad \text{for all } x$$

is $y = f(x) + c$, where c is a constant.

Which one of the following is a correct expression for $f(x)$?

A $-\dfrac{6x}{|x|}$

B $\dfrac{6x}{|x|}$

C $-3x|x|$

D $3x|x|$

E $-3x^2$

F $3x^2$

G $-x^3$

H x^3

7.65

TMUA 2017 · Paper 2 · Q19

Which one of the following is a **sufficient** condition for the equation $x^3 - 3x^2 + a = 0$, where a is a constant, to have exactly one real root?

A $a > 0$

B $a \leqslant 0$

C $a \geqslant 4$

D $a < 4$

E $|a| > 4$

F $|a| \leqslant 4$

G $a = \frac{9}{4}$

H $|a| = \frac{3}{2}$

7.66

TMUA 2023 · Paper 2 · Q19

In this question, $f(x)$ is a non-constant polynomial, and $g(x) = xf'(x)$

$f(x) = 0$ for exactly M real values of x.

$g(x) = 0$ for exactly N real values of x.

Which of the following statements is/are true?

I It is possible that $M < N$

II It is possible that $M = N$

III It is possible that $M > N$

A none of them

B I only

C II only

D III only

E I and II only

F I and III only

G II and III only

H I, II and III

7.67

TMUA 2019 · Paper 1 · Q20

What is the complete range of values of k for which the curves with equations

$$y = x^3 - 12x$$

and

$$y = k - (x - 2)^2$$

intersect at **three** distinct points, of which exactly **two** have positive x-coordinates?

A $-4 < k < 0$

B $-4 < k < 4$

C $-4 < k < 16$

D $-16 < k < 0$

E $-16 < k < 4$

F $-16 < k < 16$

Let f be a polynomial with real coefficients.

The integral $I_{p,q}$ where $p < q$ is defined by

$$I_{p,q} = \int_{p}^{q} (f(x))^2 - (f(|x|))^2 \, \mathrm{d}x$$

Which of the following statements must be true?

1 $I_{p,q} = 0$ **only if** $0 < p$

2 $f'(x) < 0$ for all x **only if** $I_{p,q} < 0$ for all $p < q < 0$

3 $I_{p,q} > 0$ **only if** $p < 0$

A none of them

B 1 only

C 2 only

D 3 only

E 1 and 2 only

F 1 and 3 only

G 2 and 3 only

H 1, 2 and 3

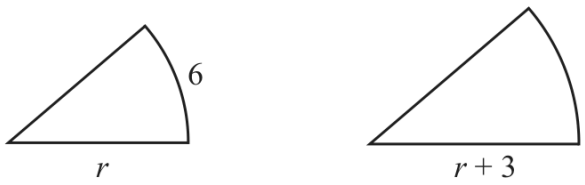
CHAPTER 8 · UNIT P7

Trigonometry 2

9 questions

8.1

TMUA 2022 · Paper 1 · Q4



These sectors of circles are similar.

The arc length of the smaller sector is 6.

The difference between the areas of the sectors is 21.

Find the positive difference between the perimeters of the sectors.

A 4.5
B 7
C 8
D 9
E 10.5
F 14
G 15

8.2

TMUA 2018 · Paper 1 · Q6

Find the number of solutions of the equation

$$x \sin 2x = \cos 2x$$

with $0 \leq x \leq 2\pi$.

A 0
B 1
C 2
D 3
E 4

8.3 TMUA 2020 · Paper 2 · Q9

A student wishes to evaluate the function $f(x) = x \sin x$, where x is in radians, but has a calculator that only works in degrees.

What could the student type into their calculator to correctly evaluate $f(4)$?

A $(\pi \times 4 \div 180) \times \sin(4)$

B $(\pi \times 4 \div 180) \times \sin(\pi \times 4 \div 180)$

C $4 \times \sin(\pi \times 4 \div 180)$

D $(180 \times 4 \div \pi) \times \sin(4)$

E $(180 \times 4 \div \pi) \times \sin(180 \times 4 \div \pi)$

F $4 \times \sin(180 \times 4 \div \pi)$

8.4 TMUA 2016 · Paper 1 · Q10

How many solutions does the equation $x \tan x = 1$ have in the interval $-2\pi \leq x \leq 2\pi$?

A 0
B 1
C 2
D 3
E 4
F 5
G 6

8.5 TMUA 2020 · Paper 1 · Q12

How many real solutions are there to the equation

$$3 \cos x = \sqrt{x}$$

where x is in radians?

A 0
B 1
C 2
D 3
E 4
F 5
G infinitely many

This question uses radians.

Find the number of distinct values of x that satisfy the equation

$$(x+1)(3-x) = 2(1-\cos(\pi x))$$

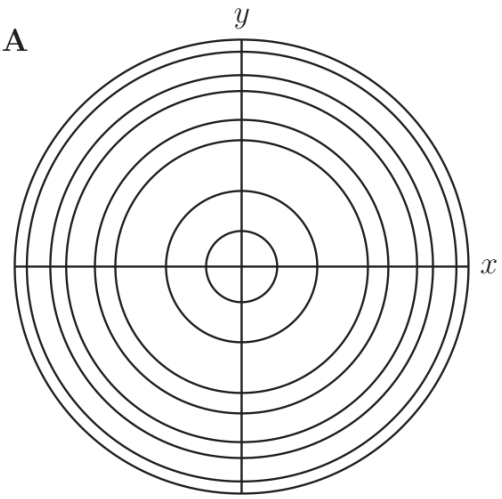
A 2
B 3
C 4
D 5
E 6
F 7

Which of the following sketches shows the graph of

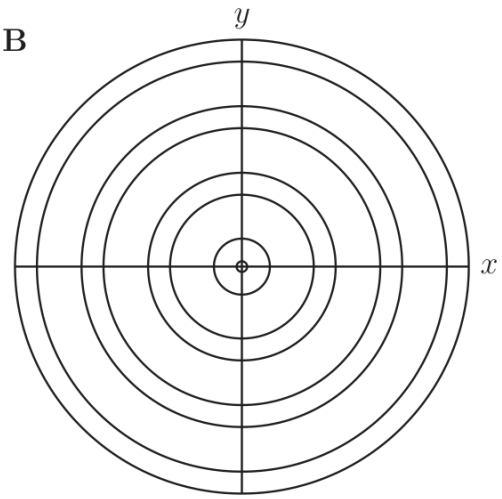
$$\sin(x^2 + y^2) = \frac{1}{2}$$

where $x^2 + y^2 \leq 8\pi$?

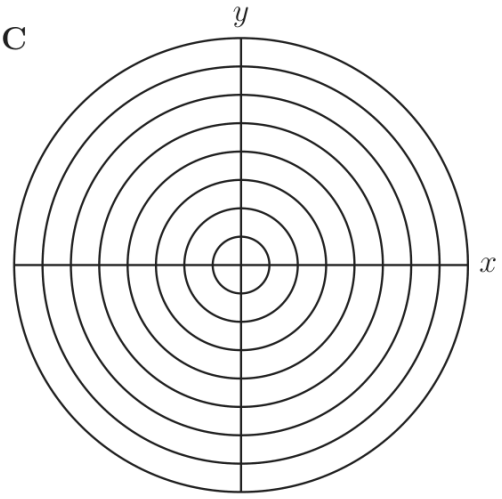
A



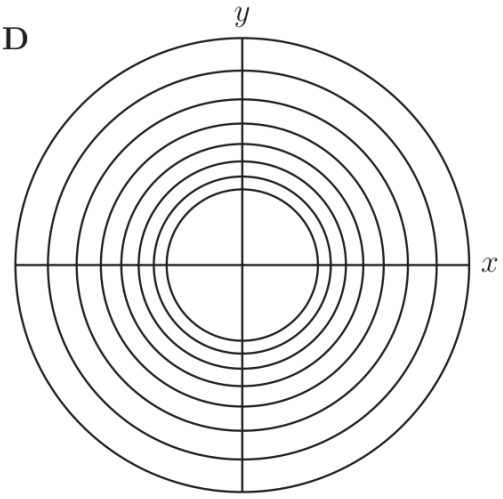
B



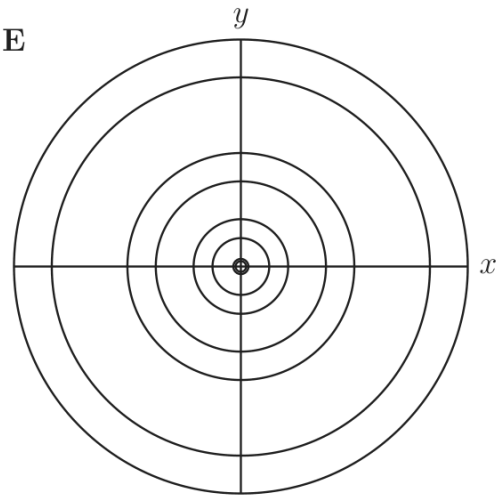
C



D



E



8.8

TMUA 2016 · Paper 2 · Q17

Consider these simultaneous equations, where c is a constant:

$$y = 3\sin x + 2$$
$$y = x + c$$

Which of the following statements is/are true?

1 For some value of c: there is exactly one solution with $0 \leq x \leq \pi$ **and** there is at least one solution with $-\pi < x < 0$.

2 For some value of c: there is exactly one solution with $0 \leq x \leq \pi$ **and** there are no solutions with $-\pi < x < 0$.

3 For some value of c: there is exactly one solution with $0 \leq x \leq \pi$ **and** there are no solutions with $x > \pi$.

A none

B 1 only

C 2 only

D 3 only

E 1 and 2 only

F 1 and 3 only

G 2 and 3 only

H 1, 2 and 3

8.9

TMUA Specimen · Paper 1 · Q18

The angle x is measured in radians and is such that $0 \leq x \leq \pi$.

The total length of any intervals for which $-1 \leq \tan x \leq 1$ and $\sin 2x \geq 0.5$ is

A $\frac{\pi}{12}$

B $\frac{\pi}{6}$

C $\frac{\pi}{4}$

D $\frac{\pi}{3}$

E $\frac{5\pi}{12}$

F $\frac{\pi}{2}$

G $\frac{5\pi}{6}$

CHAPTER 9 · UNIT P8

Sequences and Series

51 questions

9.1

TMUA 2016 · Paper 1 · Q1

It is given that the expansion of $(ax + b)^3$ is $8x^3 - px^2 + 18x - 3\sqrt{3}$, where a, b and p are real constants.

What is the value of p?

A $-12\sqrt{3}$

B $-6\sqrt{3}$

C $-4\sqrt{3}$

D $-\sqrt{3}$

E $\sqrt{3}$

F $4\sqrt{3}$

G $6\sqrt{3}$

H $12\sqrt{3}$

9.2

TMUA 2019 · Paper 2 · Q1

Find the coefficient of the x^4 term in the expansion of

$$x^2\left(2x + \frac{1}{x}\right)^6$$

A 15

B 30

C 60

D 120

E 240

9.3 TMUA 2018 · Paper 1 · Q2

An arithmetic progression has first term a and common difference d.

The sum of the first 5 terms is equal to the sum of the first 8 terms.

Which one of the following expresses the relationship between a and d?

A $a = -\frac{38}{3}d$

B $a = -7d$

C $a = -6d$

D $a = 6d$

E $a = 7d$

F $a = \frac{38}{3}d$

9.4 TMUA 2018 · Paper 2 · Q2

Find the value of the constant term in the expansion of

$$\left(x^6 - \frac{1}{x^2}\right)^{12}$$

A -495

B -220

C -66

D 66

E 220

F 495

9.5 TMUA 2022 · Paper 2 · Q2

Find the coefficient of the x^5 term in the expansion of

$$(1+x)^5 \times \sum_{i=0}^{5} x^i$$

A 1

B 5

C 16

D 25

E 32

9.6

TMUA 2019 · Paper 1 · Q3

Find the coefficient of x in the expression:

$$(1+x)^0 + (1+x)^1 + (1+x)^2 + (1+x)^3 + \cdots + (1+x)^{79} + (1+x)^{80}$$

A 80
B 81
C 324
D 628
E 3240
F 3321
G 6480
H 6642

9.7

TMUA 2021 · Paper 1 · Q3

An arithmetic progression and a convergent geometric progression each have first term $\frac{1}{2}$

The sum of the second terms of the two progressions is $\frac{1}{2}$

The sum of the third terms of the two progressions is $\frac{1}{8}$

What is the sum to infinity of the geometric progression?

A -2
B -1
C $-\frac{1}{2}$
D $-\frac{1}{3}$
E $\frac{1}{3}$
F $\frac{1}{2}$
G 1
H 2

9.8

TMUA 2017 · Paper 2 · Q3

The first term of a geometric progression is $2\sqrt{3}$ and the fourth term is $\frac{9}{4}$

What is the sum to infinity of this geometric progression?

A $-2(2-\sqrt{3})$

B $4(2\sqrt{3}-3)$

C $\dfrac{16(8\sqrt{3}+9)}{37}$

D $\dfrac{4(2\sqrt{3}-3)}{7}$

E $\dfrac{4(2\sqrt{3}+3)}{7}$

F $2(2+\sqrt{3})$

G $4(2\sqrt{3}+3)$

9.9

TMUA 2016 · Paper 1 · Q4

The sequence a_n is defined by the rule:

$$a_n = (-1)^n - (-1)^{n-1} + (-1)^{n+2} \text{ for } n \geq 1.$$

Find the value of

$$\sum_{n=1}^{39} a_n$$

A – 39

B – 3

C – 1

D 0

E 1

F 3

9.10

TMUA 2019 · Paper 1 · Q4

The sequence x_n is given by:

$$x_1 = 10$$
$$x_{n+1} = \sqrt{x_n} \text{ for } n \geq 1$$

What is the value of x_{100}?

[Note that a^{b^c} means $a^{(b^c)}$]

A $10^{2^{99}}$

B $10^{2^{100}}$

C $10^{2^{-99}}$

D $10^{2^{-100}}$

E $10^{-2^{99}}$

F $10^{-2^{100}}$

G $10^{-2^{-99}}$

H $10^{-2^{-100}}$

9.11

TMUA 2020 · Paper 1 · Q4

The 1st, 2nd and 3rd terms of a geometric progression are also the 1st, 4th and 6th terms, respectively, of an arithmetic progression.

The sum to infinity of the geometric progression is 12.

Find the 1st term of the geometric progression.

A 1

B 2

C 3

D 4

E 5

F 6

9.12

TMUA 2023 · Paper 1 · Q4

Evaluate

$$\sum_{n=0}^{\infty} \frac{\sin\left(n\pi + \frac{\pi}{3}\right)}{2^n}$$

A 0

B $\frac{1}{3}$

C $\frac{\sqrt{3}}{3}$

D $\sqrt{3}$

E 3

9.13

TMUA 2019 · Paper 1 · Q5

S is a geometric sequence.

The sum of the first 6 terms of S is equal to 9 times the sum of the first 3 terms of S.

The 7^{th} term of S is 360.

Find the 1^{st} term of S.

A $\frac{40}{27}$

B $\frac{40}{9}$

C $\frac{40}{3}$

D $\frac{45}{16}$

E $\frac{45}{8}$

F $\frac{45}{4}$

9.14

TMUA 2022 · Paper 1 · Q5

The terms x_n of a sequence follow the rule

$$x_{n+1} = \frac{x_n + p}{x_n + q}$$

where p and q are real numbers.

Given that $x_1 = 3,\ x_2 = 5,$ and $x_3 = 7,$ find the value of x_4

A -5

B 5

C $\frac{51}{7}$

D $\frac{15}{2}$

E $\frac{23}{3}$

F 9

G 11

H 13

9.15

TMUA 2023 · Paper 1 · Q6

In the simplified expansion of $(2+3x)^{12}$, how many of the terms have a coefficient that is divisible by 12?

A 0
B 2
C 5
D 10
E 11
F 12
G 13

9.16

TMUA 2016 · Paper 2 · Q6

The sequence of functions $f_1(x), f_2(x), f_3(x), \ldots$ is defined as follows:

$$f_1(x) = x^{10}$$
$$f_{n+1}(x) = xf_n'(x) \quad \text{for } n \geq 1$$

where $f_n'(x) = \frac{df_n(x)}{dx}$

Find the value of

$$\sum_{n=1}^{20} f_n(x)$$

A $\frac{x^{10}(x^{20}-1)}{x-1}$

B $\frac{x^{10}(x^{21}-1)}{x-1}$

C $\left(\frac{10^{20}-1}{9}\right)x^{10}$

D $\left(\frac{10^{21}-1}{9}\right)x^{10}$

E $\left(\frac{(10x)^{20}-1}{10x-1}\right)x^{10}$

F $\left(\frac{(10x)^{21}-1}{10x-1}\right)x^{10}$

G $x^{10}+x^9+x^8+\cdots+x+1$

H $x^{10}+10x^9+(10\times 9)x^8+\cdots+(10\times 9\times \ldots \times 2)x+(10\times 9\times \ldots \times 2\times 1)$

9.17

TMUA 2017 · Paper 2 · Q6

A sequence u_0, u_1, u_2, ... is defined as follows:

$$u_0 = 1$$

$$u_n = \int_0^1 4x u_{n-1}\, dx \quad \text{for } n \geqslant 1$$

What is the value of u_{1000}?

A 2^{1000}

B 4^{1000}

C $\dfrac{4}{1000!}$

D $\dfrac{4}{1001!}$

E $\dfrac{2^{1000}}{1000!}$

F $\dfrac{4^{1000}}{1000!}$

G $\dfrac{2^{1000}}{1001!}$

H $\dfrac{4^{1000}}{1001!}$

9.18

TMUA 2017 · Paper 1 · Q7

The first three terms of an arithmetic progression are p, q and p^2 respectively, where $p < 0$

The first three terms of a geometric progression are p, p^2 and q respectively.

Find the sum of the first 10 terms of the arithmetic progression.

A $\dfrac{23}{8}$

B $\dfrac{95}{8}$

C $\dfrac{115}{8}$

D $\dfrac{185}{8}$

9.19

TMUA 2018 · Paper 1 · Q7

The non-zero constant k is chosen so that the coefficients of x^6 in the expansions of $(1+kx^2)^7$ and $(k+x)^{10}$ are equal.

What is the value of k?

A $\frac{1}{6}$

B 6

C $\frac{\sqrt{6}}{6}$

D $\sqrt{6}$

E $\frac{\sqrt{30}}{30}$

F $\sqrt{30}$

9.20

TMUA 2018 · Paper 2 · Q7

Sequence 1 is an arithmetic progression with first term 11 and common difference 3.

Sequence 2 is an arithmetic progression with first term 2 and common difference 5.

Some numbers that appear in Sequence 1 also appear in Sequence 2. Let N be the 20th such number.

What is the remainder when N is divided by 7?

A 0

B 1

C 2

D 3

E 4

F 5

G 6

9.21

TMUA 2018 · Paper 1 · Q8

The sum to infinity of a geometric progression is 6.

The sum to infinity of the squares of each term in the progression is 12.

Find the sum to infinity of the cubes of each term in the progression.

A 8

B 18

C 24

D $\frac{216}{7}$

E 72

F 216

9.22

TMUA 2022 · Paper 1 · Q8

A geometric sequence has first term a and common ratio r, where a and r are positive integers and r is greater than 1.

The sum of the first n terms of this sequence is denoted by S_n

It is given that the terms of the sequence satisfy

$$S_{30} - S_{20} = kS_{10}$$

for some positive integer k.

What is the smallest possible value of k?

A 2^{10}

B 2^{20}

C 2^{30}

D $\dfrac{2^{10}}{2^{10}-1}$

E $2^{10}(2^{10}-1)$

9.23

TMUA 2022 · Paper 2 · Q8

A selection, S, of n terms is taken from the arithmetic sequence 1, 4, 7, 10, … , 70.

Consider the following statement:

$(*)$ There are two distinct terms in S whose sum is 74.

What is the smallest value of n for which $(*)$ is **necessarily** true?

A 12

B 13

C 14

D 21

E 22

F 23

9.24

TMUA 2023 · Paper 2 · Q9

Consider the following statement about a pentagon P:

(∗) **If** at least one of the interior angles in P is 108°, **then** all the interior angles in P form an arithmetic sequence.

Which of the following is/are true?

I The statement (∗)

II The contrapositive of (∗)

III The converse of (∗)

A none of them

B I only

C II only

D III only

E I and II only

F I and III only

G II and III only

H I, II and III

9.25

TMUA 2017 · Paper 1 · Q11

The sequence x_n is defined by the rules

$$x_1 = 7$$

$$x_{n+1} = \frac{23x_n - 53}{5x_n + 1}$$

The first three terms in the sequence are $7, 3, 1$

What is the value of x_{100}?

A -5

B 0

C 1

D 3

E 7

9.26

TMUA 2022 · Paper 1 · Q11

Evaluate

$$\sum_{n=1}^{100} \log_{10}\left(3^{1-n}\right)$$

A $-4950\log_{10}3$

B $4950\log_{10}3$

C $-5050\log_{10}3$

D $5050\log_{10}3$

E $1-4950\log_{10}3$

F $1+4950\log_{10}3$

G $1-5050\log_{10}3$

H $1+5050\log_{10}3$

9.27

TMUA 2019 · Paper 2 · Q11

An **arithmetic** series has n terms, all of which are **integers**.

The sum of the series is 20.

Which of the following statements **must** be true?

I The first term of the series is even.

II n is even.

III The common difference is even.

A none of them

B I only

C II only

D III only

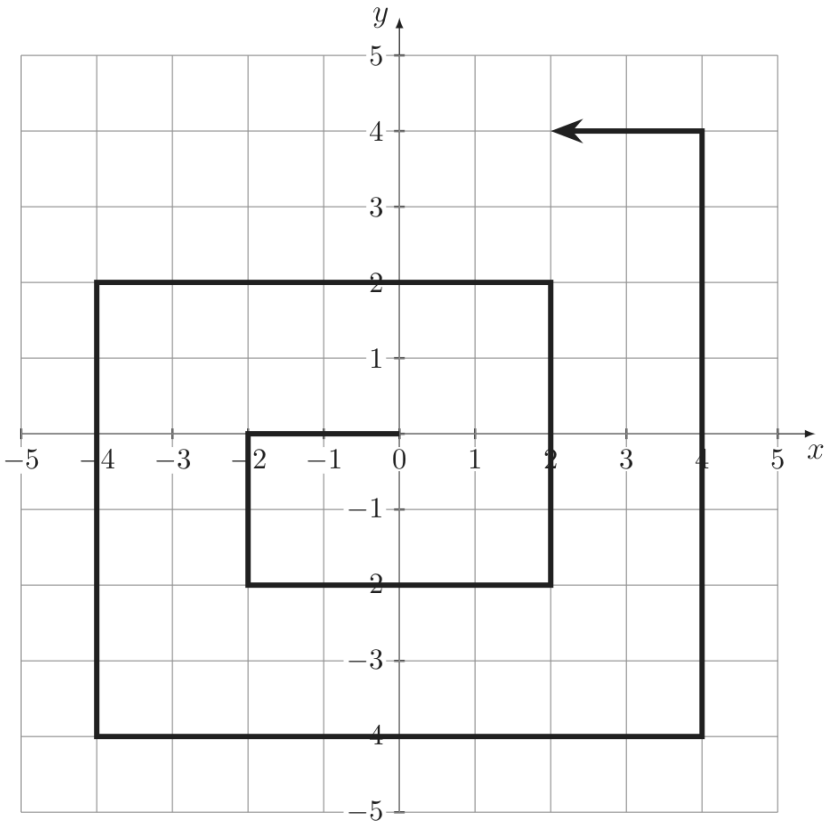
E I and II only

F I and III only

G II and III only

H I, II and III

A spiral line is drawn as shown.



This spiral pattern continues indefinitely.

Which one of the following points is **not** on the spiral line?

A $(99, 100)$

B $(99, -100)$

C $(-99, 100)$

D $(-99, -100)$

E $(100, 99)$

F $(100, -99)$

G $(-100, 99)$

H $(-100, -99)$

9.29

TMUA 2016 · Paper 2 · Q12

The first term of an arithmetic sequence is a and the common difference is d.

The sum of the first n terms is denoted by S_n.

If $S_8 > 3S_6$, what can be deduced about the sign of a and the sign of d?

A both a and d are negative

B a is positive, d is negative

C a is negative, d is positive

D a is negative, but the sign of d cannot be deduced

E d is negative, but the sign of a cannot be deduced

F neither the sign of a nor the sign of d can be deduced

9.30

TMUA 2017 · Paper 1 · Q13

In the expansion of $(a + bx)^5$ the coefficient of x^4 is 8 times the coefficient of x^2.

Given that a and b are non-zero **positive** integers, what is the smallest possible value of $a + b$?

A 3

B 4

C 5

D 9

E 13

F 17

9.31

TMUA 2020 · Paper 1 · Q13

Find the coefficient of x^2y^4 in the expansion of $(1 + x + y^2)^7$

A 6

B 10

C 21

D 35

E 105

F 210

9.32

TMUA 2021 · Paper 1 · Q13

The function f is such that, for every integer n

$$\int_{n}^{n+1} \mathrm{f}(x)\,\mathrm{d}x = n+1$$

Evaluate

$$\sum_{r=1}^{8}\left(\int_{0}^{r} \mathrm{f}(x)\,\mathrm{d}x\right)$$

A 36
B 84
C 120
D 165
E 204
F 288

9.33

TMUA 2016 · Paper 1 · Q14

The terms of an infinite series S are formed by adding together the corresponding terms in two infinite geometric series, T and U.

The first term of T and the first term of U are each 4.

In order, the first three terms of the combined series S are 8, 3, and $\frac{5}{4}$

What is the sum to infinity of S?

A $\frac{32}{5}$
B $\frac{20}{3}$
C $\frac{64}{5}$
D $\frac{40}{3}$
E 16
F 32

An arithmetic sequence T has first term a and common difference d, where a and d are non-zero integers.

Property P is:

> **For some** positive integer m, the sum of the first m terms of the sequence is equal to the sum of the first $2m$ terms of the sequence.

For example, when $a = 11$ and $d = -2$, the sequence T has property P, because

$$11 + 9 + 7 + 5 = 11 + 9 + 7 + 5 + 3 + 1 + (-1) + (-3)$$

i.e. the sum of the first 4 terms equals the sum of the first 8 terms.

Which of the following statements is/are **true**?

I For T to have property P, it is **sufficient** that $ad < 0$.

II For T to have property P, it is **necessary** that d is even.

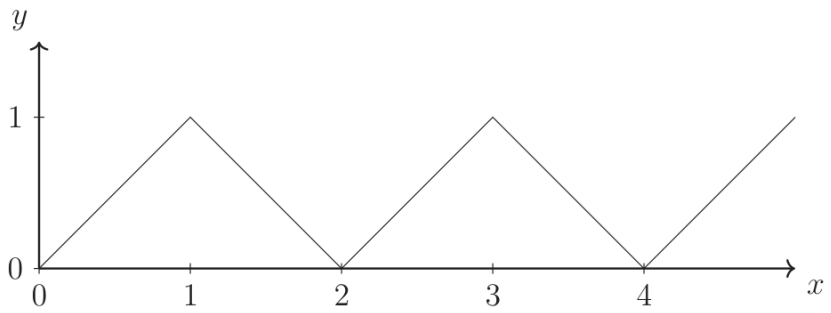
A neither of them

B I only

C II only

D I and II

The diagram shows the graph of $y = \mathrm{f}(x)$.



The graph consists of alternating straight-line segments of gradient 1 and -1 and continues in this way for all values of x.

The function g is defined as

$$\mathrm{g}(x) = \sum_{r=1}^{10} f\left(2^{r-1}x\right)$$

Find the value of

$$\int_0^1 \mathrm{g}(x)\,\mathrm{d}x$$

A $\dfrac{1023}{1024}$

B $\dfrac{1023}{512}$

C 5

D 10

E $\dfrac{55}{2}$

F 55

9.36

TMUA 2017 · Paper 2 · Q15

The function f is defined on the positive integers as follows:

$$f(1) = 5, \text{ and for } n \geqslant 1: \qquad \begin{aligned} f(n+1) &= 3f(n) + 1 \quad \text{if } f(n) \text{ is odd} \\ f(n+1) &= \tfrac{1}{2}f(n) \quad \text{if } f(n) \text{ is even} \end{aligned}$$

The function g is defined on the positive integers as follows:

$$g(1) = 3, \text{ and for } n \geqslant 1: \qquad \begin{aligned} g(n+1) &= g(n) + 5 \quad \text{if } g(n) \text{ is odd} \\ g(n+1) &= \tfrac{1}{2}g(n) \quad \text{if } g(n) \text{ is even} \end{aligned}$$

What is the value of $f(1000) - g(1000)$?

A -6
B -5
C 1
D 2
E 4
F 8

9.37

TMUA 2020 · Paper 2 · Q15

Which one of the following is a **necessary and sufficient** condition for

$$\sum_{k=1}^{n} \sin\left(\frac{k\pi}{3}\right) = \frac{\sqrt{3}}{2}$$

to be true?

A $n = 1$
B n is a multiple of 3
C n is a multiple of 6
D n is 1 more than a multiple of 3
E n is 1 more than a multiple of 6
F n is 1 more than a multiple of 6 or n is 2 more than a multiple of 6

9.38

TMUA 2023 · Paper 2 · Q15

The base 10 number 0.03841 has the value

$$0 \times 10^{-1} + 3 \times 10^{-2} + 8 \times 10^{-3} + 4 \times 10^{-4} + 1 \times 10^{-5} = 0.03841$$

Similarly, the base 2 number 0.01101 has the value

$$0 \times 2^{-1} + 1 \times 2^{-2} + 1 \times 2^{-3} + 0 \times 2^{-4} + 1 \times 2^{-5} = \frac{13}{32}$$

What is the value of the recurring base 2 number $0.\dot{0}0\dot{1}1\dot{} = 0.001100110011\ldots$?

A $\frac{1}{3}$

B $\frac{1}{5}$

C $\frac{1}{15}$

D $\frac{2}{15}$

E $\frac{4}{15}$

F $\frac{3}{16}$

G $\frac{5}{16}$

H $\frac{6}{31}$

9.39

TMUA 2021 · Paper 1 · Q16

Consider the expansion of

$$(a + bx)^n$$

The third term, in **ascending** powers of x, is $105x^2$

The fourth term, in **ascending** powers of x, is $210x^3$

The fourth term, in **descending** powers of x, is $210x^3$

Find the value of $\left(\frac{a}{b}\right)^2$

A $\frac{1}{4}$

B $\frac{4}{9}$

C $\frac{25}{36}$

D $\frac{5}{6}$

E 1

9.40

TMUA Specimen · Paper 2 · Q16

The sequence a_n is given by the rule:

$$a_1 = 2$$
$$a_{n+1} = a_n + (-1)^n \text{ for } n \geq 1$$

What is

$$\sum_{n=1}^{100} a_n$$

A 150

B 250

C -4750

D 5150

E $4\left(1 - \left(\frac{1}{2}\right)^{100}\right)$

F $4\left(\left(\frac{3}{2}\right)^{100} - 1\right)$

9.41

TMUA 2018 · Paper 2 · Q16

In this question, x_1, x_2, x_3, ... is an **arithmetic progression**, all of whose terms are integers.

Let n be a positive integer. If the median of the first n terms of the sequence is an integer, which of the following three statements **must** be true?

I The median of the first $n + 2$ terms is an integer.

II The median of the first $2n$ terms is an integer.

III The median of x_2, x_4, x_6, ..., x_{2n} is an integer.

A none of them

B I only

C II only

D III only

E I and II only

F I and III only

G II and III only

H I, II and III

9.42

TMUA 2023 · Paper 2 · Q16

A sequence is defined by:

$$u_1 = a$$

$$u_2 = b$$

$$u_{n+2} = u_n + u_{n+1} \quad \text{for } n \geq 1$$

where a and b are positive integers. The highest common factor of a and b is 7.

Which of the following statements **must** be true?

I u_{2023} is a multiple of 7

II If u_1 is not a factor of u_2, then u_1 is not a factor of u_n for any $n > 1$

III The highest common factor of u_1 and u_5 is 7

A none of them

B I only

C II only

D III only

E I and II only

F I and III only

G II and III only

H I, II and III

9.43

TMUA 2017 · Paper 1 · Q17

The two functions $F(n)$ and $G(n)$ are defined as follows for positive integers n:

$$F(n) = \frac{1}{n}\int_0^n (n - x)\, dx$$

$$G(n) = \sum_{r=1}^{n} F(r)$$

What is the smallest positive integer n such that $G(n) > 150$?

A 22

B 23

C 24

D 25

E 26

9.44

TMUA 2023 · Paper 1 · Q17

A circle C_n is defined by

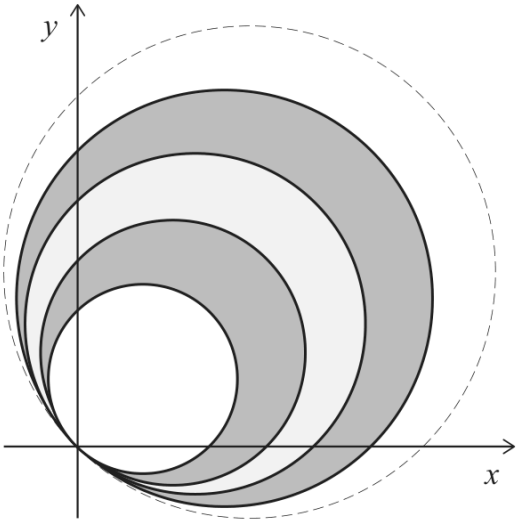
$$x^2 + y^2 = 2n(x + y)$$

where n is a positive integer.

C_1 and C_2 are drawn and the area between them is shaded.

Next, C_3 and C_4 are drawn and the area between them is shaded.

This is shown in the diagram.



[diagram not to scale]

This process continues until 100 circles have been drawn.

What is the total shaded area?

A 100π

B 500π

C 2500π

D 5050π

E 10100π

F 40400π

9.45

TMUA 2023 · Paper 2 · Q17

The ceiling of x, written $\lceil x \rceil$, is defined to be the value of x rounded up to the nearest integer.

For example: $\lceil \pi \rceil = 4$, $\lceil 2.1 \rceil = 3$, $\lceil 8 \rceil = 8$

What is the value of the following integral?

$$\int_0^{99} 2^{\lceil x \rceil}\, dx$$

A 2^{99}

B $2^{99} - 1$

C $2^{99} - 2$

D 2^{100}

E $2^{100} - 1$

F $2^{100} - 2$

9.46

TMUA 2023 · Paper 1 · Q18

You are given that

$$S = 4 + \frac{8k}{7} + \frac{16k^2}{49} + \frac{32k^3}{343} + \cdots + 4\left(\frac{2k}{7}\right)^n + \cdots$$

The value for k is chosen as an integer in the range $-5 \leq k \leq 5$

All possible values for k are equally likely to be chosen.

What is the probability that the value of S is a finite number greater than 3?

A $\frac{1}{11}$

B $\frac{1}{10}$

C $\frac{3}{11}$

D $\frac{3}{10}$

E $\frac{5}{11}$

F $\frac{1}{2}$

G $\frac{7}{11}$

H $\frac{7}{10}$

9.47

TMUA Specimen · Paper 1 · Q19

A geometric series has first term 4 and common ratio r, where $0 < r < 1$.

The first, second, and fourth terms of this geometric series form three successive terms of an arithmetic series.

The sum to infinity of the geometric series is

A $\frac{1}{2}(\sqrt{5} - 1)$

B $2(3 - \sqrt{5})$

C $2(1 + \sqrt{5})$

9.48

TMUA 2016 · Paper 1 · Q19

The coefficient of x^3 in the expansion of $(1 + 2x + 3x^2)^6$ is equal to twice the coefficient of x^4 in the expansion of $(1 - ax^2)^5$.

Find all possible values of the constant a.

A $\pm 2\sqrt{2}$

B $\pm \sqrt{17}$

C $\pm \sqrt{34}$

D $\pm 2\sqrt{17}$

E There are no possible values of a.

9.49

TMUA 2019 · Paper 1 · Q19

Find the value of

$$\sum_{k=0}^{90} \sin(10 + 90k)^\circ$$

A 0

B $\sin 10^\circ$

C $\sin 100^\circ$

D $\sin 190^\circ$

E $\sin 280^\circ$

F 1

9.50

TMUA Specimen · Paper 1 · Q20

The coefficient of x^2 in the expansion of $(4 - x^2)[(1 + 2x + 3x^2)^6 - (1 + 4x^3)^5]$ is

A 28

B 72

C 78

D 192

E 240

F 310

G 312

A sequence of functions f_1, f_2, f_3, ... is defined by

$$f_1(x) = |x|$$
$$f_{n+1}(x) = |f_n(x) + x| \quad \text{for } n \geq 1$$

Find the value of

$$\int_{-1}^{1} f_{99}(x)\,dx$$

A 0
B 0.5
C 1
D 49.5
E 50
F 99
G 99.5
H 100

CHAPTER 10 · UNIT P9

Calculus 2

2 questions

10.1

TMUA 2023 · Paper 2 · Q5

Consider the two statements

R: k is an integer multiple of π

S: $\int_0^k \sin 2x \, dx = 0$

Which of the following statements is true?

A R is **necessary and sufficient** for S.

B R is **necessary** but **not sufficient** for S.

C R is **sufficient** but **not necessary** for S.

D R is **not necessary** and **not sufficient** for S.

10.2

TMUA 2020 · Paper 2 · Q16

The Fundamental Theorem of Calculus (FTC) tells us that for any polynomial f:

$$\frac{d}{dx}\left(\int_0^x f(t)\,dt\right) = f(x)$$

A student calculates $\frac{d}{dx}\left(\int_x^{2x} t^2\,dt\right)$ as follows:

(I) $\int_x^{2x} t^2\,dt = \int_0^{2x} t^2\,dt - \int_0^x t^2\,dt$

(II) By FTC, $\frac{d}{dx}\left(\int_0^x t^2\,dt\right) = x^2$

(III) By FTC, $\frac{d}{dx}\left(\int_0^{2x} t^2\,dt\right) = (2x)^2 = 4x^2$

(IV) So $\frac{d}{dx}\left(\int_x^{2x} t^2\,dt\right) = 4x^2 - x^2$

(V) giving $\frac{d}{dx}\left(\int_x^{2x} t^2\,dt\right) = 3x^2$

Which of the following best describes the student's calculation?

A The calculation is completely correct.

B The calculation is incorrect, and the first error occurs on line (I).

C The calculation is incorrect, and the first error occurs on line (II).

D The calculation is incorrect, and the first error occurs on line (III).

E The calculation is incorrect, and the first error occurs on line (IV).

F The calculation is incorrect, and the first error occurs on line (V).

CHAPTER 11 · UNIT P11

Proof and Logic

61 questions

11.1

TMUA Specimen · Paper 2 · Q3

Consider the following attempt to solve an equation. The steps have been numbered for reference.

$$\sqrt{x+5} = x+3$$

(1)

$$x+5 = x^2+6x+9$$

(2)

$$x^2+5x+4 = 0$$

(3)

$$(x+4)(x+1) = 0$$

$$x = -4 \text{ or } x = -1$$

Which one of the following statements is true?

A Both -4 and -1 are solutions of the equation.

B Neither -4 nor -1 are solutions of the equation.

C One solution is correct and the incorrect solution arises as a result of step (1).

D One solution is correct and the incorrect solution arises as a result of step (2).

E One solution is correct and the incorrect solution arises as a result of step (3).

11.2

TMUA 2018 · Paper 2 · Q3

Consider the following statement:

> A car journey consists of two parts. In the first part, the average speed is u km/h. In the second part, the average speed is v km/h. Hence the average speed for the whole journey is $\frac{1}{2}(u+v)$ km/h.

Which of the following examples of car journeys provide(s) a **counterexample** to the statement?

I In the first part of the journey, the car travels at a constant speed of 50 km/h for 100 km. In the second part of the journey, the car travels at a constant speed of 40 km/h for 100 km.

II In the first part of the journey, the car travels at a constant speed of 50 km/h for one hour. In the second part of the journey, the car travels at a constant speed of 40 km/h for one hour.

III In the first part of the journey, the car travels at a constant speed of 50 km/h for 80 km. In the second part of the journey, the car travels at a constant speed of 40 km/h for 100 km.

A none of them
B I only
C II only
D III only
E I and II only
F I and III only
G II and III only
H I, II and III

11.3

TMUA 2019 · Paper 2 · Q3

a, b and c are real numbers.

Given that $ab = ac$, which of the following statements **must** be true?

I $a = 0$
II $b = 0$ **or** $c = 0$
III $b = c$

A none of them
B I only
C II only
D III only
E I and II only
F I and III only
G II and III only
H I, II and III

11.4

TMUA 2020 · Paper 2 · Q3

A student makes the following claim:

For all integers n, the expression $4\left(\frac{9n+1}{2}-\frac{3n-1}{2}\right)$ is divisible by 3.

Here is the student's argument:

$$4\left(\frac{9n+1}{2}-\frac{3n-1}{2}\right)=2\left(2\left(\frac{9n+1}{2}-\frac{3n-1}{2}\right)\right) \quad \text{(I)}$$

$$=2(9n+1-3n-1) \quad \text{(II)}$$

$$=2(6n) \quad \text{(III)}$$

$$=12n \quad \text{(IV)}$$

$$=3(4n) \quad \text{(V)}$$

which is always a multiple of 3. (VI)

So the expression $4\left(\frac{9n+1}{2}-\frac{3n-1}{2}\right)$ is always divisible by 3.

Which one of the following is true?

A The argument is correct.

B The argument is incorrect, and the first error occurs on line (I).

C The argument is incorrect, and the first error occurs on line (II).

D The argument is incorrect, and the first error occurs on line (III).

E The argument is incorrect, and the first error occurs on line (IV).

F The argument is incorrect, and the first error occurs on line (V).

G The argument is incorrect, and the first error occurs on line (VI).

11.5

TMUA 2022 · Paper 2 · Q3

Consider the following statement about the positive integer n

if n is prime, **then** n^2+2 is **not** prime

Which of the following is a **counterexample** to this statement?

I $n=2$

II $n=3$

III $n=4$

A none of them

B I only

C II only

D III only

E I and II only

F I and III only

G II and III only

H I, II and III

11.6

TMUA 2023 · Paper 2 · Q3

Consider the claim:

For all positive real numbers x and y,

$$\sqrt{x^y} = x^{\sqrt{y}}$$

Which of the following is/are a **counterexample** to the claim?

I $x = 1,\ y = 16$

II $x = 2,\ y = 8$

III $x = 3,\ y = 4$

A none of them

B I only

C II only

D III only

E I and II only

F I and III only

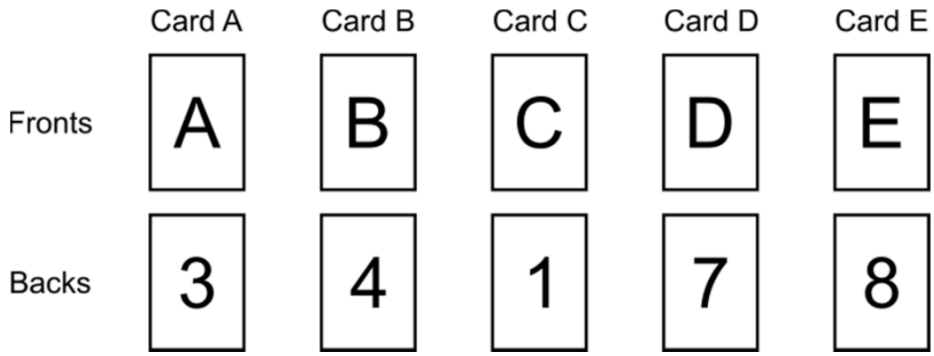
G II and III only

H I, II and III

11.7

TMUA Specimen · Paper 2 · Q4

A set of five cards each have a letter printed on their front and a number printed on their back, as follows:



Which one of the five cards (**A**, **B**, **C**, **D** or **E**) provides a counterexample to the following statement?

Every card that has a vowel on its front has an even number on its back.

11.8

TMUA 2016 · Paper 2 · Q4

Five sealed urns, labelled P, Q, R, S, and T, each contain the **same** (non-zero) number of balls. The following statements are attached to the urns.

Urn P This urn contains one or four balls.
Urn Q This urn contains two or four balls.
Urn R This urn contains more than two balls and fewer than five balls.
Urn S This urn contains one or two balls.
Urn T This urn contains fewer than three balls.

Exactly one of the urns has a true statement attached to it.

Which urn is it?

A Urn P
B Urn Q
C Urn R
D Urn S
E Urn T

11.9

TMUA 2019 · Paper 2 · Q4

Consider the following conjecture:

> **If** N is a positive integer that consists of the digit 1 followed by an odd number of 0 digits and then a final digit 1, **then** N is a prime number.

Here are three numbers:

I $N = 101$ (which is a prime number)

II $N = 1001$ (which equals $7 \times 11 \times 13$)

III $N = 10001$ (which equals 73×137)

Which of these provide(s) a counterexample to the conjecture?

A none of them
B I only
C II only
D III only
E I and II only
F I and III only
G II and III only
H I, II and III

11.10

TMUA 2020 · Paper 2 · Q4

Consider the following statement:

> Every positive integer N that is greater than 6 can be written as the sum of two non-prime integers that are greater than 1.

Which of the following is/are **counterexample(s)** to this statement?

I $N = 5$
II $N = 7$
III $N = 9$

A none of them
B I only
C II only
D III only
E I and II only
F I and III only
G II and III only
H I, II and III

11.11

TMUA 2021 · Paper 2 · Q4

Consider the following claim about positive integers a, b and c:

> **if** a is a factor of bc, **then** a is a factor of b **or** a is a factor of c

Which of the following provide(s) a **counterexample** to this claim?

I $a = 5, b = 10, c = 20$
II $a = 8, b = 4, c = 4$
III $a = 6, b = 7, c = 12$

A none of them
B I only
C II only
D III only
E I and II only
F I and III only
G II and III only
H I, II and III

11.12

TMUA 2023 · Paper 2 · Q4

A student attempts to answer the following question.

What is the largest number of consecutive odd integers that are all prime?

The student's attempt is as follows:

I There are two consecutive odd integers that are prime (for example: 17, 19).

II Any three consecutive odd integers can be written in the form $n-2$, n, $n+2$ for some n.

III If n is one more than a multiple of 3, then $n+2$ is a multiple of 3.

IV If n is two more than a multiple of 3, then $n-2$ is a multiple of 3.

V The only other possibility is that n is a multiple of 3.

VI In each case, one of the integers is a multiple of 3, so not prime.

VII Therefore the largest number of consecutive odd integers that are all prime is two.

Which of the following best describes this attempt?

A It is completely correct.

B It is incorrect, and the first error is on line I.

C It is incorrect, and the first error is on line II.

D It is incorrect, and the first error is on line III.

E It is incorrect, and the first error is on line IV.

F It is incorrect, and the first error is on line V.

G It is incorrect, and the first error is on line VI.

H It is incorrect, and the first error is on line VII.

11.13

TMUA 2016 · Paper 2 · Q5

Consider the statement:

(*) A whole number n is prime if it is 1 less or 5 less than a multiple of 6.

How many counterexamples to (*) are there in the range $0 < n < 50$?

A 2

B 3

C 4

D 5

E 6

11.14

TMUA 2017 · Paper 2 · Q5

Consider the following three statements:

1 $10p^2 + 1$ and $10p^2 - 1$ are both prime when p is an odd prime.

2 Every prime greater than 5 is of the form $6n + 1$ for some integer n.

3 No multiple of 7 greater than 7 is prime.

The result $91 = 7 \times 13$ can be used to provide a counterexample to which of the above statements?

A none of them

B **1** only

C **2** only

D **3** only

E **1** and **2** only

F **1** and **3** only

G **2** and **3** only

H **1**, **2** and **3**

11.15

TMUA 2018 · Paper 2 · Q5

The two diagonals of the quadrilateral Q are perpendicular.

Consider the following statements:

I One of the diagonals of Q is a line of symmetry of Q.

II The midpoints of the sides of Q are the vertices of a square.

Which of these statements is/are **necessarily** true for the quadrilateral Q?

A neither of them

B I only

C II only

D I and II

11.16

TMUA 2019 · Paper 2 · Q5

Consider the following statement about the positive integers a, b and n:

$(*)$: ab is divisible by n

The condition 'either a or b is divisible by n' is:

A **necessary** but **not sufficient** for $(*)$

B **sufficient** but **not necessary** for $(*)$

C **necessary** and **sufficient** for $(*)$

D **not necessary** and **not sufficient** for $(*)$

11.17

TMUA 2022 · Paper 2 · Q5

A straight line L passes through $(1, 2)$.

Let P be the statement

if the y-intercept of L is negative, **then** the x-intercept of L is positive.

Which of the following statements **must** be true?

I P

II the converse of P

III the contrapositive of P

A none of them

B I only

C II only

D III only

E I and II only

F I and III only

G II and III only

H I, II and III

11.18

TMUA 2019 · Paper 2 · Q7

For which one of the following statements can the fact that $12^2 + 16^2 = 20^2$ be used to produce a **counterexample**?

A If a, b and c are positive integers which satisfy the equation $a^2 + b^2 = c^2$, and the three numbers have no common divisor, then two of them are odd and the other is even.

B The equation $a^4 + b^2 = c^2$ has no solutions for which a, b and c are positive integers.

C The equation $a^4 + b^4 = c^4$ has no solutions for which a, b and c are positive integers.

D If a, b and c are positive integers which satisfy the equation $a^2 + b^2 = c^2$, then one is the arithmetic mean of the other two.

Consider the following conditions on a **parallelogram** $PQRS$, labelled anticlockwise:

I length of PQ = length of QR

II The diagonal PR intersects the diagonal QS at right angles

III $\angle PQR = \angle QRS$

Which of these conditions is/are individually **sufficient** for the parallelogram $PQRS$ to be a square?

	Condition I is sufficient	**Condition II is sufficient**	**Condition III is sufficient**
A	yes	yes	yes
B	yes	yes	no
C	yes	no	yes
D	yes	no	no
E	no	yes	yes
F	no	yes	no
G	no	no	yes
H	no	no	no

11.20

TMUA 2022 · Paper 2 · Q7

Consider the following claim:

The difference between two consecutive positive cube numbers is always prime.

Here is an attempted proof of this claim:

I $(x+1)^3 = x^3 + 3x^2 + 3x + 1$

II Taking x to be a positive integer, the difference between two consecutive cube numbers can be expressed as $(x+1)^3 - x^3 = 3x^2 + 3x + 1$

III It is impossible to factorise $3x^2 + 3x + 1$ into two linear factors with integer coefficients because its discriminant is negative.

IV Therefore for every positive integer value of x the integer $3x^2 + 3x + 1$ cannot be factorised.

V Hence, the difference between two consecutive cube numbers will always be prime.

Which of the following best describes this proof?

A The proof is completely correct, and the claim is true.

B The proof is completely correct, but there are counterexamples to the claim.

C The proof is wrong, and the first error occurs on line I.

D The proof is wrong, and the first error occurs on line II.

E The proof is wrong, and the first error occurs on line III.

F The proof is wrong, and the first error occurs on line IV.

G The proof is wrong, and the first error occurs on line V.

11.21

TMUA Specimen · Paper 2 · Q8

Consider the following statement about the positive integer n:

Statement (*): *The sum of the four consecutive integers, the smallest of which is n, is a multiple of 6.*

Which one of the following is true?

A Statement (*) is true for all values of n.

B Statement (*) is true for all values of n which are odd, but not for any other values of n.

C Statement (*) is true for all values of n which are multiples of 3, but not for any other values of n.

D Statement (*) is true for all values of n which are multiples of 6, but not for any other values of n.

E Statement (*) is not true for any value of n.

A region is defined by the inequalities $x + y > 6$ and $x - y > -4$

Consider the three statements:

1 $x > 1$

2 $y > 5$

3 $(x + y)(x - y) > -24$

Which of the above statements is/are true for **every** point in the region?

A none

B 1 only

C 2 only

D 3 only

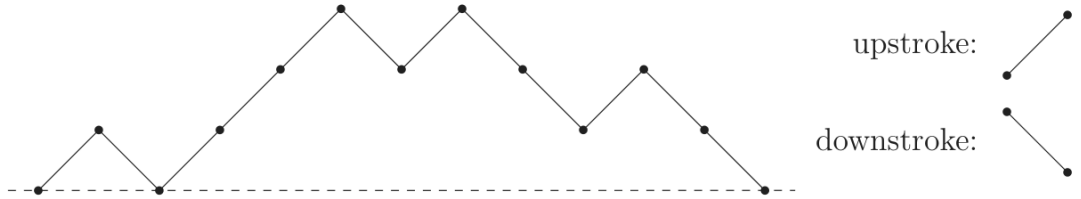
E 1 and 2 only

F 1 and 3 only

G 2 and 3 only

H 1, 2 and 3

The diagram shows an example of a *mountain profile*.



This consists of *upstrokes* which go upwards from left to right, and *downstrokes* which go downwards from left to right. The example shown has six upstrokes and six downstrokes. The horizontal line at the bottom is known as *sea level*.

A *mountain profile of order n* consists of n upstrokes and n downstrokes, with the condition that the profile begins and ends at sea level and **never** goes **below** sea level (although it might reach sea level at any point). So the example shown is a mountain profile of order 6.

Mountain profiles can be coded by using U to indicate an upstroke and D to indicate a downstroke. The example shown has the code UDUUUDUDDUDD. A sequence of U's and D's obtained from a mountain profile in this way is known as a *valid code*.

Which of the following statements is/are true?

I If a valid code is written in reverse order, the result is always a valid code.

II If each U in a valid code is replaced by D and each D by U, the result is always a valid code.

III If U is added at the beginning of a valid code and D is added at the end of the code, the result is always a valid code.

A none of them

B I only

C II only

D III only

E I and II only

F I and III only

G II and III only

11.24

TMUA 2019 · Paper 2 · Q8

a, b and c are real numbers with $a < b < c < 0$

Which of the following statements **must** be true?

I $\quad ac < ab < a^2$

II $\quad b(c + a) > 0$

III $\quad \dfrac{c}{b} > \dfrac{a}{b}$

A none of them

B I only

C II only

D III only

E I and II only

F I and III only

G II and III only

H I, II and III

11.25

TMUA 2023 · Paper 2 · Q8

A student draws a triangle that is acute-angled or obtuse-angled but **not** right-angled.

The student counts the number of straight lines that divide the triangle into two triangles, **at least one** of which is right-angled.

Which of the following statements is/are true?

I The student can draw a triangle for which there is exactly 1 such straight line.

II The student can draw a triangle for which there are exactly 2 such straight lines.

III The student can draw a triangle for which there are exactly 3 such straight lines.

A none of them

B I only

C II only

D III only

E I and II only

F I and III only

G II and III only

H I, II and III

11.26

TMUA Specimen · Paper 2 · Q9

Consider the statement about Fred:

(*) *Every day next week, Fred will do at least one maths problem.*

If statement (*) is **not** true, which of the following is **certainly** true?

A Every day next week, Fred will do more than one maths problem.

B Some day next week, Fred will do more than one maths problem.

C On no day next week will Fred do more than one maths problem.

D Every day next week, Fred will do no maths problems.

E Some day next week, Fred will do no maths problems.

F On no day next week will Fred do no maths problems.

11.27

TMUA 2016 · Paper 2 · Q9

Triangles ABC and XYZ have the **same area**.

Which of these extra conditions, taken independently, would **imply** that they are congruent?

(1) $AB = XY$ **and** $BC = YZ$

(2) $AB = XY$ **and** $\angle ABC = \angle XYZ$

(3) $\angle ABC = \angle XYZ$ **and** $\angle BCA = \angle YZX$

	Condition (1)	Condition (2)	Condition (3)
A	Does not imply congruent	Does not imply congruent	Does not imply congruent
B	Does not imply congruent	Does not imply congruent	Implies congruent
C	Does not imply congruent	Implies congruent	Does not imply congruent
D	Does not imply congruent	Implies congruent	Implies congruent
E	Implies congruent	Does not imply congruent	Does not imply congruent
F	Implies congruent	Does not imply congruent	Implies congruent
G	Implies congruent	Implies congruent	Does not imply congruent
H	Implies congruent	Implies congruent	Implies congruent

Consider the following attempt to prove this true theorem:

Theorem: $a^3 + b^3 = c^3$ has no solutions with a, b and c positive integers.

Attempted proof:

Suppose that there are positive integers a, b and c such that $a^3 + b^3 = c^3$.

I We have $a^3 = c^3 - b^3$.

II Hence $a^3 = (c - b)(c^2 + cb + b^2)$.

III It follows that $a = c - b$ and $a^2 = c^2 + cb + b^2$, since $a \leqslant a^2$ and $c - b \leqslant c^2 + cb + b^2$.

IV Eliminating a, we have $(c - b)^2 = c^2 + cb + b^2$.

V Multiplying out, we have $c^2 - 2cb + b^2 = c^2 + cb + b^2$.

VI Hence $3cb = 0$ so one of b and c is zero.

But this is a contradiction to the original assumption that all of a, b and c are positive. It follows that the equation has no solutions.

Comment on this proof by choosing one of the following options:

A The proof is correct

B The proof is incorrect and the first mistake occurs on line I.

C The proof is incorrect and the first mistake occurs on line II.

D The proof is incorrect and the first mistake occurs on line III.

E The proof is incorrect and the first mistake occurs on line IV.

F The proof is incorrect and the first mistake occurs on line V.

G The proof is incorrect and the first mistake occurs on line VI.

Consider the following attempt to solve the equation $4x\sqrt{2x-1} = 10x - 5$:

$$4x\sqrt{2x-1} = 10x - 5$$

$\Big\downarrow$ (I)

$$4x\sqrt{2x-1} = 5(2x-1)$$

$\Big\downarrow$ (II)

$$16x^2(2x-1) = 25(2x-1)^2$$

$\Big\downarrow$ (III)

$$16x^2 = 25(2x-1)$$

$\Big\downarrow$ (IV)

$$16x^2 - 50x + 25 = 0$$

$\Big\downarrow$ (V)

$$(8x-5)(2x-5) = 0$$

$\Big\downarrow$ (VI)

The solutions of the original equation are $x = \dfrac{5}{8}$ and $x = \dfrac{5}{2}$.

Which one of the following is true?

A The solution is correct.

B Only one of $x = \frac{5}{8}$ and $x = \frac{5}{2}$ is correct and the error arises as a result of step (II).

C Only one of $x = \frac{5}{8}$ and $x = \frac{5}{2}$ is correct and the error arises as a result of step (III).

D Only one of $x = \frac{5}{8}$ and $x = \frac{5}{2}$ is correct and the error arises as a result of step (IV).

E There is another value of x that satisfies the original equation and the error arises as a result of step (II).

F There is another value of x that satisfies the original equation and the error arises as a result of step (III).

G There is another value of x that satisfies the original equation and the error arises as a result of step (IV).

11.30

TMUA 2019 · Paper 2 · Q9

A large circular table has 40 chairs round it.

What is the smallest number of people who can be sitting at the table already such that the next person to sit down **must** sit next to someone?

A 9
B 10
C 13
D 14
E 19
F 20

11.31

TMUA 2022 · Paper 2 · Q9

Consider the following statement:

$$(*)\ \textbf{For all}\ \text{real numbers } x,\ \textbf{if}\ x < k\ \textbf{then}\ x^2 < k$$

What is the complete set of values of k for which $(*)$ is true?

A no real numbers
B $k > 0$
C $k < 1$
D $k \leq 1$
E $0 < k < 1$
F $0 < k \leq 1$
G all real numbers

11.32

TMUA 2016 · Paper 2 · Q10

In this question x and y are non-zero real numbers.

Which one of the following is **sufficient** to conclude that $x < y$?

A $x^4 < y^4$
B $y^4 < x^4$
C $x^{-1} < y^{-1}$
D $y^{-1} < x^{-1}$
E $x^{\frac{3}{5}} < y^{\frac{3}{5}}$
F $y^{\frac{3}{5}} < x^{\frac{3}{5}}$

11.33

TMUA 2019 · Paper 2 · Q10

$PQRS$ is a quadrilateral, labelled anticlockwise.

Which one of the following is a **necessary** but **not sufficient** condition for $PQRS$ to be a parallelogram?

A $PQ = SR$ and PS is parallel to QR

B $PQ = SR$ and PQ is parallel to SR

C $PQ = QR = SR = PS$

D $PR = QS$

11.34

TMUA 2020 · Paper 2 · Q10

The real numbers a, b, c and d satisfy both

$$0 < a + b < c + d$$

and

$$0 < a + c < b + d$$

Which of the following inequalities **must** be true?

I $a < d$

II $b < c$

III $a + b + c + d > 0$

A none of them

B I only

C II only

D III only

E I and II only

F I and III only

G II and III only

H I, II and III

11.35

TMUA 2021 · Paper 2 · Q10

The first seven terms of a sequence of positive integers are:

$$u_1 = 15$$
$$u_2 = 21$$
$$u_3 = 30$$
$$u_4 = 37$$
$$u_5 = 44$$
$$u_6 = 51$$
$$u_7 = 59$$

Consider the following statement about this sequence:

(∗) **If** n is a prime number, **then** u_n is a multiple of 3 **or** u_n is a multiple of 5.

What is the smallest value of n that provides a **counterexample** to (∗)?

A 1
B 2
C 3
D 4
E 5
F 6
G 7

11.36

TMUA 2022 · Paper 2 · Q10

Which of the following statements is/are true?

I **For all** real numbers x and **for all** positive integers n, $x < n$

II **For all** real numbers x, **there exists** a positive integer n such that $x < n$

III **There exists** a real number x such that **for all** positive integers n, $x < n$

A none of them
B I only
C II only
D III only
E I and II only
F I and III only
G II and III only
H I, II and III

11.37

TMUA 2023 · Paper 2 · Q10

Here is an attempt to solve the inequality $x^4 - 2x^2 - 3 < 0$ by completing the square:

$x^4 - 2x^2 - 3 < 0$

I if and only if $x^4 - 2x^2 + 1 < 4$

II if and only if $(x^2 - 1)^2 < 4$

III if and only if $-2 < x^2 - 1 < 2$

IV if and only if $x^2 - 1 < 2$

V if and only if $x^2 < 3$

VI if and only if $-\sqrt{3} < x < \sqrt{3}$

Which of the following statements is true?

A The argument is completely correct.

B The first error occurs in line I.

C The first error occurs in line II.

D The first error occurs in line III.

E The first error occurs in line IV.

F The first error occurs in line V.

G The first error occurs in line VI.

11.38

TMUA 2021 · Paper 2 · Q11

A student attempts to solve the following problem, where a and b are non-zero real numbers:

Show that **if** $a^2 - 4b^3 \geq 0$ **then** there exist real numbers x and y such that $a = xy(x + y)$ and $b = xy$.

Consider the following attempt:

$(x - y)^2 \geq 0$ (I)

so $x^2 + y^2 - 2xy \geq 0$ (II)

so $(x + y)^2 - 4xy \geq 0$ (III)

so $x^2y^2(x + y)^2 - 4x^3y^3 \geq 0$ (IV)

so $a^2 - 4b^3 \geq 0$ (V)

Which of the following best describes this attempt?

A It is completely correct.

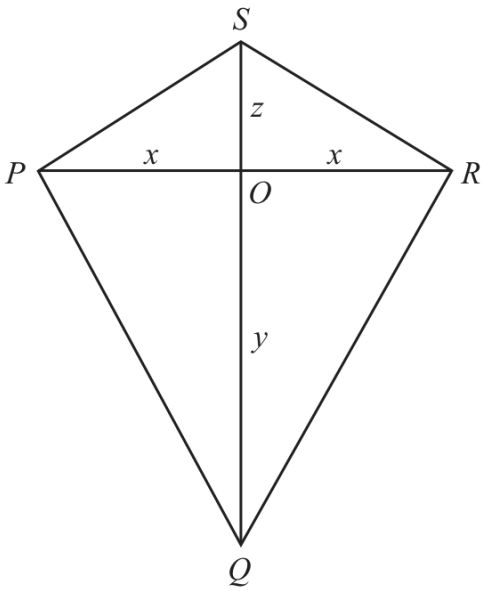
B It is incorrect, but it would be correct if written in the reverse order.

C It is incorrect, but the student has correctly proved the converse.

D It is incorrect because there is an error in line (II).

E It is incorrect because there is an error in line (III).

F It is incorrect because there is an error in line (IV).



The diagram shows a kite $PQRS$ whose diagonals meet at O.

$OP = x$
$OQ = y$
$OR = x$
$OS = z$

Which of the following is **necessary and sufficient** for angle SPQ to be a right angle?

A $x = y = z$

B $2x = y + z$

C $x^2 = yz$

D $y = z$

E $y^2 = x^2 + z^2$

In this question, k is a positive integer.

Consider the following theorem:

If 2^k+1 is a prime, **then** k is a power of 2. $(*)$

Which of the following statements, taken individually, is/are equivalent to $(*)$?

I **If** k is a power of 2, **then** 2^k+1 is prime.

II 2^k+1 is **not** prime **only if** k is **not** a power of 2.

III A **sufficient** condition for k to be a power of 2 is that 2^k+1 is prime.

	Statement I is equivalent to $(*)$	Statement II is equivalent to $(*)$	Statement III is equivalent to $(*)$
A	Yes	Yes	Yes
B	Yes	Yes	No
C	Yes	No	Yes
D	Yes	No	No
E	No	Yes	Yes
F	No	Yes	No
G	No	No	Yes
H	No	No	No

11.41

TMUA 2018 · Paper 2 · Q12

Consider the following statement:

> For any positive integer N there is a positive integer K such that $N(Km+1)-1$ is not prime for any positive integer m.

Which one of the following is the negation of this statement?

A For any positive integer N there is a positive integer K such that there is a positive integer m for which $N(Km+1)-1$ is prime.

B For any positive integer N there is a positive integer K such that there is a positive integer m for which $N(Km+1)-1$ is not prime.

C For any positive integer N there is a positive integer K such that for any positive integer m, $N(Km+1)-1$ is not prime.

D For any positive integer N, any positive integer K and any positive integer m, $N(Km+1)-1$ is not prime.

E There is a positive integer N such that for any positive integer K there is a positive integer m for which $N(Km+1)-1$ is not prime.

F There is a positive integer N such that for any positive integer K there is a positive integer m for which $N(Km+1)-1$ is prime.

G There is a positive integer N such that for any positive integer K and any positive integer m, $N(Km+1)-1$ is prime.

H There is a positive integer N and a positive integer K for which there is no positive integer m for which $N(Km+1)-1$ is prime.

11.42

TMUA Specimen · Paper 2 · Q13

Five runners competed in a race: Fred, George, Hermione, Lavender, and Ron.

> Fred beat George.
>
> Hermione beat Lavender.
>
> Lavender beat George.
>
> Ron beat George.

Assuming there were no ties, how many possible finishing orders could there have been, given only this information?

A 1

B 6

C 12

D 18

E 24

F 120

11.43

TMUA 2016 · Paper 2 · Q13

In this question, a, b, and c are positive integers.

The following is an attempted proof of the false statement:

If a divides bc, then a divides b or a divides c.

['a divides bc' means 'a is a factor of bc']

Which line contains the error in this proof?

1. The statement is equivalent to 'if a does not divide b and a does not divide c then a does not divide bc'.
2. Suppose a does not divide b and a does not divide c. Then the remainder when dividing b by a is r, where $0 < r < a$, and the remainder when dividing c by a is s, where $0 < s < a$.
3. So $b = ax + r$ and $c = ay + s$ for some integers x and y.
4. Thus $bc = a(axy + xs + yr) + rs$.
5. So the remainder when dividing bc by a is rs.
6. Since $r > 0$ and $s > 0$, it follows that $rs > 0$.
7. Hence a does not divide bc.

A Line 1
B Line 2
C Line 3
D Line 4
E Line 5
F Line 6

11.44

TMUA 2017 · Paper 2 · Q13

The positive real numbers $a \times 10^{-3}$, $b \times 10^{-2}$ and $c \times 10^{-1}$ are each in standard form, and

$$(a \times 10^{-3}) + (b \times 10^{-2}) = (c \times 10^{-1}).$$

Which of the following statements (I, II, III, IV) **must** be true?

I $a > 9$
II $b > 9$
III $a < c$
IV $b < c$

A I only
B II only
C I and II only
D I and III only
E I and IV only
F II and III only
G II and IV only
H I, II, III and IV

11.45

TMUA 2021 · Paper 2 · Q13

A region R in the (x, y)-plane is defined by the simultaneous inequalities

$$y - x < 3$$
$$y - x^2 < 1$$

Which of the following statements is/are true for **every** point in R?

I $\quad -1 < x < 2$

II $\quad (y - x)(y - x^2) < 3$

III $\quad y < 5$

A none of them

B I only

C II only

D III only

E I and II only

F I and III only

G II and III only

H I, II and III

11.46

TMUA 2022 · Paper 2 · Q13

Consider the statement (*) about a real number x:

(*) **There exists** a real number y such that $x - xy + y$ is negative.

For how many real values of x is (*) true?

A no values of x

B exactly one value of x

C exactly two values of x

D all except exactly two values of x

E all except exactly one value of x

F all values of x

11.47

TMUA 2023 · Paper 2 · Q13

Let x be a real number.

Which **one** of the following statements is a **sufficient** condition for **exactly** three of the other four statements?

A $x \geq 0$

B $x = 1$

C $x = 0$ **or** $x = 1$

D $x \geq 0$ **or** $x \leq 1$

E $x \geq 0$ **and** $x \leq 1$

11.48

TMUA Specimen · Paper 2 · Q15

For any real numbers a, b, and c where $a \geq b$, consider these three statements:

1 $-b \geq -a$

2 $a^2 + b^2 \geq 2ab$

3 $ac \geq bc$

Which of the statements 1, 2, and 3 **must** be true?

A none

B 1 only

C 2 only

D 3 only

E 1 and 2 only

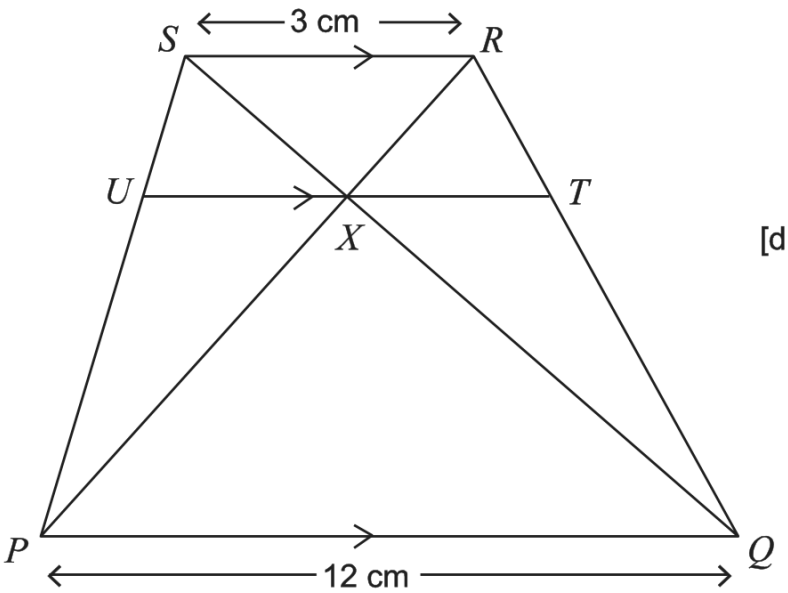
F 1 and 3 only

G 2 and 3 only

H 1,2 and 3

11.49

TMUA 2016 · Paper 2 · Q16



[diagram not to scale]

In the figure, $PQRS$ is a trapezium with PQ parallel to SR.

The diagonals of the trapezium meet at X.

U lies on SP and T lies on RQ such that UT is a line segment through X parallel to PQ.

The length of PQ is 12 cm and the length of SR is 3 cm.

What, in centimetres, is the length of UT?

A 4.2

B 4.5

C 4.8

D 5.25

E 6

11.50

TMUA 2022 · Paper 2 · Q16

In this question, $a_1, \ldots, a_{100}$ and $b_1, \ldots, b_{100}$ and $c_1, \ldots, c_{100}$ are three sequences of integers such that

$$a_n \leq b_n + c_n$$

for each n.

Which of the following statements **must** be true?

I (minimum of $a_1, \ldots, a_{100}$) $\leq$ (minimum of $b_1, \ldots, b_{100}$) + (minimum of $c_1, \ldots, c_{100}$)

II (minimum of $a_1, \ldots, a_{100}$) $\geq$ (minimum of $b_1, \ldots, b_{100}$) + (minimum of $c_1, \ldots, c_{100}$)

III (maximum of $a_1, \ldots, a_{100}$) $\leq$ (maximum of $b_1, \ldots, b_{100}$) + (maximum of $c_1, \ldots, c_{100}$)

A none of them

B I only

C II only

D III only

E I and II only

F I and III only

G II and III only

H I, II and III

11.51

TMUA Specimen · Paper 2 · Q17

Let S be a set of positive integers, for example S could consist of 3, 4, and 8.

A positive integer n is called an S-number **if and only if** for every factor m of n with $m > 1$, the number m is a multiple of some number in S.

So in the above example, 9 is an S-number; this is because the factors of 9 greater than 1 are 3 and 9, and each of these is a multiple of 3.

Positive integer n is therefore **not** an S-number **if and only if**

A for **every** (positive) factor m of n with $m > 1$, there is a number in S which is not a factor of m.

B for **every** (positive) factor m of n with $m > 1$, there is no number in S which is a factor of m.

C for **every** (positive) factor m of n with $m > 1$, every number in S is a factor of m.

D for **some** (positive) factor m of n with $m > 1$, there is a number in S which is not a factor of m.

E for **some** (positive) factor m of n with $m > 1$, there is no number in S which is a factor of m.

F for **some** (positive) factor m of n with $m > 1$, every number in S is a factor of m.

11.52

TMUA 2017 · Paper 2 · Q17

A set S of whole numbers is called *stapled* **if and only if** for every whole number a which is in S there exists a prime factor of a which divides at least one other number in S.

Let T be a set of whole numbers. Which of the following is true **if and only if** T is **not** stapled?

A For every number a which is in T, there is no prime factor of a which divides every other number in T.

B For every number a which is in T, there is no prime factor of a which divides at least one other number in T.

C For every number a which is in T, there is a prime factor of a which does not divide any other number in T.

D For every number a which is in T, there is a prime factor of a which does not divide at least one other number in T.

E There exists a number a which is in T such that there is no prime factor of a which divides every other number in T.

F There exists a number a which is in T such that there is no prime factor of a which divides at least one other number in T.

G There exists a number a which is in T such that there is a prime factor of a which does not divide any other number in T.

H There exists a number a which is in T such that there is a prime factor of a which does not divide at least one other number in T.

11.53

TMUA 2018 · Paper 2 · Q17

A positive integer is called a *squaresum* **if and only if** it can be written as the sum of the squares of two integers. For example, 61 and 9 are both squaresums since $61 = 5^2 + 6^2$ and $9 = 3^2 + 0^2$.

A prime number is called *awkward* **if and only if** it has a remainder of 3 when divided by 4. For example, 23 is awkward since $23 = 5 \times 4 + 3$.

A (true) theorem due to Fermat states that:

> A positive integer is a squaresum **if and only if** each of its awkward prime factors occurs to an even power in its prime factorisation.

It follows that 5×23^2 is a squaresum, since 23 occurs to the power 2, but 5×23^3 is not, since 23 occurs to the power 3.

Which one of the following statements is **not** true?

A Every square number is a squaresum.

B If N and M are squaresums, then so is NM.

C If NM is a squaresum, then N and M are squaresums.

D If N is not a squaresum, then kN is a squaresum for some number k which is a product of awkward primes.

11.54

TMUA 2019 · Paper 2 · Q17

A multiple-choice test question offered the following four options relating to a certain statement:

A The statement is true **if and only if** $x > 1$

B The statement is true **if** $x > 1$

C The statement is true **if and only if** $x > 2$

D The statement is true **if** $x > 2$

Given that **exactly one** of these options was correct, which one was it?

11.55

TMUA 2020 · Paper 2 · Q18

In this question, $\mathrm{f}(x) = ax^3 + bx^2 + cx + d$ and $\mathrm{g}(x) = px^3 + qx^2 + rx + s$ are cubic polynomials.

If $f(x) - g(x) > 0$ for every real x, which of the following is/are **necessarily** true?

I $a > p$

II **if** $b = q$ **then** $c = r$

III $d > s$

A none of them

B I only

C II only

D III only

E I and II only

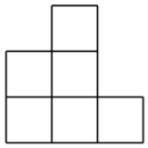
F I and III only

G II and III only

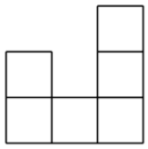
H I, II and III

Some identical unit cubes are used to construct a three-dimensional object by gluing them together face to face.

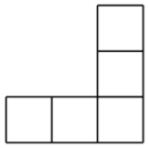
Sketches of this object are made by looking at it from the right-hand side, from the front and from above. These sketches are called the side elevation, the front elevation, and the plan view respectively.



This is the side elevation of the object.



This is the front elevation of the object.



This is the plan view of the object.

How many cubes were used to construct the object?

A exactly 6

B either 6 or 7

C exactly 7

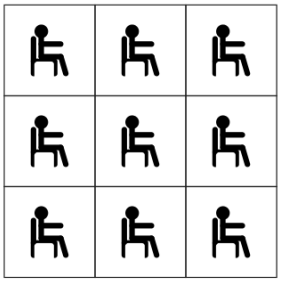
D either 7 or 8

E exactly 8

F either 8 or 9

G exactly 9

Nine people are sitting in the squares of a 3 by 3 grid, one in each square, as shown. Two people are called *neighbours* if they are sitting in squares that share a side. (People in diagonally adjacent squares, which only have a point in common, are not called neighbours.)



Each of the nine people in the grid is either a truth-teller who **always** tells the truth, or a liar who **always** lies.

Every person in the grid says: 'My neighbours are all liars'.

Given only this information, what are the **smallest** number and the **largest** number of people who could be telling the truth?

	smallest	**largest**
A	1	4
B	2	4
C	2	5
D	3	4
E	3	5
F	4	4
G	4	5
H	5	5

11.58

TMUA Specimen · Paper 2 · Q20

Five logicians each make a statement, as follows:

Mr P: Of these five statements, an odd number are true.

Ms Q: Both statements made by women are true.

Mr R: My first name is Robert and Mr P's statement is true.

Ms S: Exactly one statement made by a man is true.

Mr T: Neither statement made by a woman is true.

How many of the five statements can be simultaneously true?

A none

B 1 only

C 2 only

D 3 only

E 4 only

F none or 1 only

G 1 or 2 only

H 2 or 3 only

11.59

TMUA 2016 · Paper 2 · Q20

Each interior angle of a regular polygon with n sides is $\frac{3}{4}$ of each interior angle of a second regular polygon with m sides.

How many pairs of positive integers n and m are there for which this statement is true?

A none

B 1

C 2

D 3

E 4

F 5

G 6

H infinitely many

11.60

TMUA 2017 · Paper 2 · Q20

I have forgotten my 5-character computer password, but I know that it consists of the letters `a`, `b`, `c`, `d`, `e` in some order. When I enter a potential password into the computer, it tells me exactly how many of the letters are in the correct position.

When I enter `abcde`, it tells me that none of the letters are in the correct position. The same happens when I enter `cdbea` and `eadbc`.

Using the best strategy, how many **further** attempts must I make in order to **guarantee** that I can **deduce** the correct password?

A None: I can deduce it immediately

B One

C Two

D Three

E More than three

11.61

TMUA 2020 · Paper 2 · Q20

x is a real number and f is a function.

Given that **exactly one** of the following statements is true, which one is it?

A $x \geq 0$ **only if** $\mathrm{f}(x) < 0$

B $x < 0$ **if** $\mathrm{f}(x) \geq 0$

C $x \geq 0$ **only if** $\mathrm{f}(x) \geq 0$

D $\mathrm{f}(x) < 0$ **if** $x < 0$

E $\mathrm{f}(x) \geq 0$ **only if** $x \geq 0$

F $\mathrm{f}(x) \geq 0$ **if and only if** $x < 0$

CHAPTER 12 · UNIT S3

Statistical Calculations

4 questions

12.1

TMUA 2022 · Paper 2 · Q6

A list consists of n integers.

Consider the following statements:

P: n is odd.

Q: The median of the list is one of the numbers in the list.

Which one of the following is true?

A P is **necessary and sufficient** for Q.

B P is **necessary** but **not sufficient** for Q.

C P is **sufficient** but **not necessary** for Q.

D P is **not necessary** and **not sufficient** for Q.

12.2

TMUA 2018 · Paper 1 · Q17

There are two sets of data: the mean of the first set is 15, and the mean of the second set is 20.

One of the pieces of data from the first set is exchanged with one of the pieces of data from the second set.

As a result, the mean of the first set of data increases from 15 to 16, and the mean of the second set of data decreases from 20 to 17.

What is the mean of the set made by combining all the data?

A $16\frac{1}{4}$

B $16\frac{1}{3}$

C $16\frac{1}{2}$

D $16\frac{2}{3}$

E $16\frac{3}{4}$

12.3

TMUA 2020 · Paper 2 · Q17

A set of six **distinct** integers is split into two sets of three.

The first set of three integers has a mean of 10 and a median of 8.

The second set of three integers has a mean of 12 and a median of 9.

What is the smallest possible range of the set of all six integers?

A 8
B 10
C 11
D 12
E 14
F 15

12.4

TMUA Specimen · Paper 2 · Q18

A group of five numbers are such that:

- their mean is 0
- their range is 20

What is the largest possible median of the five numbers?

A 0
B 4
C $4\frac{1}{2}$
D $6\frac{1}{2}$
E 8
F 20

CHAPTER 13 · UNIT P13

Numerical Methods

6 questions

13.1

TMUA 2019 · Paper 1 · Q8

The function f is such that $0 < \mathrm{f}(x) < 1$ for $0 \leq x \leq 1$.

The trapezium rule with n equal intervals is used to estimate $\int_0^1 \mathrm{f}(x)\,\mathrm{d}x$ and produces an underestimate.

Using the same number of equal intervals, for which one of the following does the trapezium rule produce an overestimate?

A $\int_0^1 (\mathrm{f}(x) + 1)\,\mathrm{d}x$

B $\int_0^1 2\mathrm{f}(x)\,\mathrm{d}x$

C $\int_{-1}^0 \mathrm{f}(x+1)\,\mathrm{d}x$

D $\int_{-1}^0 \mathrm{f}(-x)\,\mathrm{d}x$

E $\int_0^1 (1 - \mathrm{f}(x))\,\mathrm{d}x$

13.2

TMUA 2021 · Paper 1 · Q10

Use the trapezium rule with 3 strips to estimate

$$\int_{\frac{1}{2}}^{2} 2\log_{10} x\,\mathrm{d}x$$

A $\log_{10}\dfrac{\sqrt{6}}{2}$

B $\log_{10}\dfrac{3}{2}$

C $\log_{10}\dfrac{9}{4}$

D $\log_{10} 3$

E $\log_{10}\dfrac{81}{16}$

F $\log_{10}\dfrac{\sqrt{23}}{2}$

13.3

TMUA 2023 · Paper 1 · Q10

The trapezium rule with 4 strips is used to estimate the integral:

$$\int_{-2}^{2} \sqrt{4-x^2}\,\mathrm{d}x$$

What is the positive difference between the estimate and the exact value of the integral?

A $2\left(\pi-2-2\sqrt{3}\right)$

B $2\left(\pi-1-\sqrt{3}\right)$

C $2\left(2\pi-1-\sqrt{3}\right)$

D $4\left(\pi-1-\sqrt{3}\right)$

E $2\pi-3\sqrt{3}$

F $4\pi-6\sqrt{3}$

13.4

TMUA 2020 · Paper 2 · Q12

Which one of **A**–**F** correctly completes the following statement?

Given that $a<b$, and $\mathrm{f}(x)>0$ for all x with $a<x<b$, the trapezium rule produces an overestimate for $\int_a^b \mathrm{f}(x)\,\mathrm{d}x$ ...

A ... **if** $\mathrm{f}'(x)>0$ **and** $\mathrm{f}''(x)<0$ for all x with $a<x<b$

B ... **only if** $\mathrm{f}'(x)>0$ **and** $\mathrm{f}''(x)<0$ for all x with $a<x<b$

C ... **if and only if** $\mathrm{f}'(x)>0$ **and** $\mathrm{f}''(x)<0$ for all x with $a<x<b$

D ... **if** $\mathrm{f}'(x)<0$ **and** $\mathrm{f}''(x)>0$ for all x with $a<x<b$

E ... **only if** $\mathrm{f}'(x)<0$ **and** $\mathrm{f}''(x)>0$ for all x with $a<x<b$

F ... **if and only if** $\mathrm{f}'(x)<0$ **and** $\mathrm{f}''(x)>0$ for all x with $a<x<b$

13.5

TMUA 2019 · Paper 2 · Q13

A student approximates the integral $\int_a^b \sin^2 x\,\mathrm{d}x$ using the trapezium rule with 4 strips. The resulting approximation is an overestimate.

Which of the following is/are **necessarily** true?

I If the student approximates $\int_{-b}^{-a} \sin^2 x\,\mathrm{d}x$ in the same way, the result will be an overestimate.

II If the student approximates $\int_a^b \cos^2 x\,\mathrm{d}x$ in the same way, the result will be an underestimate.

A neither of them

B I only

C II only

D I and II

It is given that $f(x) = -2x^2 + 10$

Consider the following three curves:

$(\mathbf{1})$ $y = f(x)$

$(\mathbf{2})$ $y = f(x + 1)$

$(\mathbf{3})$ the curve $y = f(x + 1)$ reflected in the line $y = 6$

The trapezium rule is used to estimate the area under each of these three curves between $x = 0$ and $x = 1$.

State whether the trapezium rule gives an overestimate or underestimate for each of these areas.

	$(\mathbf{1})$	$(\mathbf{2})$	$(\mathbf{3})$
A	underestimate	underestimate	underestimate
B	underestimate	underestimate	overestimate
C	underestimate	overestimate	underestimate
D	underestimate	overestimate	overestimate
E	overestimate	underestimate	underestimate
F	overestimate	underestimate	overestimate
G	overestimate	overestimate	underestimate
H	overestimate	overestimate	overestimate

ANSWERS

Answer key

Check here first, then read the worked solution

1. Algebra and Functions 1

1.1 D	**1.2** A	**1.3** E	**1.4** H	**1.5** E	**1.6** A	**1.7** A	**1.8** C
1.9 B	**1.10** G	**1.11** C	**1.12** D	**1.13** A	**1.14** F	**1.15** F	**1.16** D
1.17 C	**1.18** F	**1.19** D	**1.20** A	**1.21** A	**1.22** C	**1.23** D	**1.24** E
1.25 D	**1.26** A	**1.27** C	**1.28** A	**1.29** F	**1.30** D	**1.31** E	**1.32** F
1.33 D	**1.34** A	**1.35** D	**1.36** D	**1.37** E	**1.38** E	**1.39** C	**1.40** B

2. Coordinate Geometry 1

2.1 F	**2.2** B	**2.3** D	**2.4** E	**2.5** E	**2.6** B	**2.7** A	**2.8** E
2.9 B	**2.10** B	**2.11** C	**2.12** B	**2.13** E	**2.14** D	**2.15** F	**2.16** F
2.17 F	**2.18** B	**2.19** C	**2.20** E	**2.21** F			

3. Exponentials and Logarithms

3.1 B	**3.2** D	**3.3** A	**3.4** A	**3.5** C	**3.6** A	**3.7** F	**3.8** C
3.9 E	**3.10** F	**3.11** E	**3.12** E	**3.13** E	**3.14** H	**3.15** A	**3.16** D
3.17 F	**3.18** B	**3.19** C	**3.20** E	**3.21** A	**3.22** C	**3.23** B	**3.24** F
3.25 C	**3.26** F	**3.27** B	**3.28** C	**3.29** D			

4. Trigonometry 1

4.1 C	**4.2** D	**4.3** F	**4.4** D	**4.5** B	**4.6** D	**4.7** B	**4.8** D
4.9 D	**4.10** F	**4.11** A	**4.12** B	**4.13** E	**4.14** E	**4.15** E	**4.16** D
4.17 F	**4.18** C	**4.19** C	**4.20** B	**4.21** F	**4.22** B	**4.23** D	**4.24** E
4.25 F	**4.26** B	**4.27** D	**4.28** C	**4.29** D	**4.30** B	**4.31** C	**4.32** E
4.33 D	**4.34** B	**4.35** F	**4.36** B	**4.37** D	**4.38** E	**4.39** E	

5. Algebra and Functions 2

5.1 B	**5.2** E	**5.3** F	**5.4** C	**5.5** E	**5.6** C	**5.7** D	**5.8** E
5.9 A	**5.10** C	**5.11** B	**5.12** B	**5.13** B	**5.14** D	**5.15** E	**5.16** A
5.17 A	**5.18** B	**5.19** B	**5.20** E	**5.21** B	**5.22** B	**5.23** F	**5.24** B
5.25 E							

6. Probability

6.1 C	**6.2** C	**6.3** B	**6.4** F				

7. Calculus 1

7.1 C	**7.2** D	**7.3** C	**7.4** F	**7.5** A	**7.6** A	**7.7** C	**7.8** D
7.9 B	**7.10** C	**7.11** F	**7.12** B	**7.13** B	**7.14** F	**7.15** C	**7.16** F
7.17 C	**7.18** C	**7.19** F	**7.20** C	**7.21** A	**7.22** D	**7.23** F	**7.24** G
7.25 E	**7.26** C	**7.27** B	**7.28** D	**7.29** C	**7.30** E	**7.31** F	**7.32** D
7.33 C	**7.34** E	**7.35** A	**7.36** D	**7.37** B	**7.38** E	**7.39** B	**7.40** F
7.41 C	**7.42** B	**7.43** F	**7.44** C	**7.45** C	**7.46** D	**7.47** E	**7.48** A
7.49 B	**7.50** B	**7.51** A	**7.52** C	**7.53** H	**7.54** G	**7.55** E	**7.56** F
7.57 C	**7.58** C	**7.59** E	**7.60** A	**7.61** B	**7.62** D	**7.63** C	**7.64** D
7.65 E	**7.66** H	**7.67** E	**7.68** D				

8. Trigonometry 2

8.1 C	**8.2** E	**8.3** F	**8.4** E	**8.5** D	**8.6** B	**8.7** A	**8.8** H
8.9 B							

9. Sequences and Series

9.1 H	**9.2** E	**9.3** C	**9.4** B	**9.5** E	**9.6** E	**9.7** G	**9.8** G
9.9 B	**9.10** C	**9.11** D	**9.12** C	**9.13** E	**9.14** H	**9.15** E	**9.16** C
9.17 A	**9.18** B	**9.19** A	**9.20** B	**9.21** D	**9.22** B	**9.23** C	**9.24** D
9.25 A	**9.26** A	**9.27** A	**9.28** G	**9.29** F	**9.30** C	**9.31** F	**9.32** C
9.33 D	**9.34** A	**9.35** C	**9.36** D	**9.37** D	**9.38** B	**9.39** B	**9.40** A
9.41 F	**9.42** B	**9.43** D	**9.44** E	**9.45** F	**9.46** E	**9.47** D	**9.48** B
9.49 C	**9.50** G	**9.51** E					

10. Calculus 2

10.1 A	**10.2** D						

11. Proof and Logic

11.1 C	**11.2** F	**11.3** A	**11.4** C	**11.5** C	**11.6** C	**11.7** A	**11.8** C
11.9 D	**11.10** G	**11.11** C	**11.12** G	**11.13** C	**11.14** B	**11.15** A	**11.16** B
11.17 F	**11.18** B	**11.19** H	**11.20** F	**11.21** C	**11.22** B	**11.23** D	**11.24** E
11.25 D	**11.26** E	**11.27** D	**11.28** D	**11.29** F	**11.30** D	**11.31** A	**11.32** E
11.33 A	**11.34** F	**11.35** E	**11.36** G	**11.37** A	**11.38** C	**11.39** C	**11.40** G
11.41 F	**11.42** C	**11.43** E	**11.44** B	**11.45** A	**11.46** E	**11.47** C	**11.48** E
11.49 C	**11.50** D	**11.51** E	**11.52** F	**11.53** C	**11.54** D	**11.55** G	**11.56** F
11.57 E	**11.58** D	**11.59** E	**11.60** B	**11.61** C			

12. Statistical Calculations

12.1 C	**12.2** A	**12.3** E	**12.4** E				

13. Numerical Methods

13.1 E	**13.2** B	**13.3** B	**13.4** D	**13.5** D	**13.6** B		

Algebra and Functions 1

1.1 Answer D

TMUA Specimen · Paper 1 · Q1

We can rearrange the first equation to

$$x = 3y - 1.$$

Substituting this into the second equation gives

$$3(3y-1)^2 - 7(3y-1)y = 5$$

which expands to

$$27y^2 - 18y + 3 - 21y^2 + 7y = 5$$

so

$$6y^2 - 11y - 2 = 0.$$

We can either factorise this as $(y-2)(6y+1) = 0$, giving $y = 2$ or $y = -\frac{1}{6}$ or use the quadratic formula to obtain

$$y = \frac{11 + \sqrt{11^2 - 4 \times 6 \times (-2)}}{12} = \frac{11 \pm 13}{12}$$

giving $y = 2$ or $y = -\frac{1}{6}$ again.

These give $x = 5$ or $x = -\frac{3}{2}$, so the sum is 3.5, and the answer is D.

Alternatively, we could have rearranged the first equation to get

$$y = \frac{x+1}{3}$$

and then substitute this into the second equation. This has the advantage that we obtain the values of x directly, but the disadvantage that there are fractions involved throughout.

1.2 Answer A

TMUA 2019 · Paper 1 · Q1

We could try substituting in $(1, -1)$ and $(-1, 3)$ into the six given expressions: when we substitute in $x = 1$, only A, C, D and F give -1, so the answer must be one of these. When we substitute in $x = -1$ into these four, only A and C give 3, so the answer must be one of these two. Finally, we need a turning point at $x = -1$, so we differentiate these two: A gives $-2x - 2$, which is zero at $x = -1$, which C gives $2x - 2$, which is -4 at $x = -1$. So the correct answer is A.

Alternatively, we could work out the answer without reference to the given options.

Let $\mathrm{f}(x) = ax^2 + bx + c$. We are given $\mathrm{f}(1) = -1$ and $\mathrm{f}(-1) = 3$, so

$$a + b + c = -1$$
$$a - b + c = 3$$

Adding these gives $2a + 2c = 2$, so $a + c = 1$ and hence $b = -2$.

We then differentiate to get $\mathrm{f}'(x) = 2ax + b$, so as $\mathrm{f}'(-1) = 0$, we require $-2a + b = 0$, so $a = -1$ and $c = 2$. Hence the answer is A.

1.3 Answer E

TMUA 2020 · Paper 2 · Q1

We find the points where the line and curve meet by solving the equations simultaneously; this gives $x^2 + kx + 2 = x - 2$. Rearranging gives

$$x^2 + (k-1)x + 4 = 0.$$

The discriminant of this is $(k-1)^2 - 4 \times 1 \times 4 = (k-1)^2 - 16$. The line and curve cross or touch when the discriminant is ≥ 0, so we must solve

$$(k-1)^2 - 16 \geq 0.$$

Expanding and simplifying this gives

$$k^2 - 2k - 15 \geq 0,$$

which we can factorise to get

$$(k-5)(k+3) \geq 0.$$

The equation $(k-5)(k+3) = 0$ has roots $k = -3$ and $k = 5$, so the quadratic is non-negative when $k \leq -3$ or $k \geq 5$. Thus the correct solution is E.

1.4 Answer H

TMUA 2023 · Paper 2 · Q1

We first multiply both sides of the equation by $11(\sqrt{x} - 6)(\sqrt{x} + 6)$ to eliminate all of the fractions, giving

$$11\big(\sqrt{x} + 6\big) - 11\big(\sqrt{x} - 6\big) = 3\big(\sqrt{x} - 6\big)\big(\sqrt{x} + 6\big).$$

We can now expand all of the brackets to give

$$11\sqrt{x} + 66 - 11\sqrt{x} + 66 = 3x - 108,$$

which simplifies to

$$240 = 3x.$$

Dividing by 3 gives $x = 80$, which is option H.

1.5 Answer E

TMUA 2016 · Paper 1 · Q2

We can use the factor theorem to find the value of a; alternatively we can just divide by $(x+2)$ and deduce a that way, as we will need to divide by $(x+2)$ anyway to complete the factorisation. We show both methods.

The factor theorem states that in this case, $\mathrm{f}(-2) = 0$, where $\mathrm{f}(x)$ is the polynomial. So $3(-2)^3 + 13(-2)^2 + 8(-2) + a = 0$, giving $-24 + 52 - 16 + a = 0$, so $a = -12$. We can now divide the polynomial by $(x + 2)$.

Alternatively, dividing the given polynomial by $(x + 2)$ by considering the x^3, x^2 and x terms, but ignoring the constant term, gives:

$$3x^3 + 13x^2 + 8x + a = (x + 2)(3x^2 + 7x - 6)$$

and so $a = -12$, though that is actually not needed to answer the question.

Finally, we can factorise $3x^2 + 7x - 6$ to determine the answer. Alternatively, we can use the suggested options: only options D and E give a constant term in the quadratic of -6, so it must be one of those. D expands to $3x^2 - 7x - 6$ and E expands to $3x^2 + 7x - 6$, so the answer is E.

1.6 Answer A

TMUA 2019 · Paper 1 · Q2

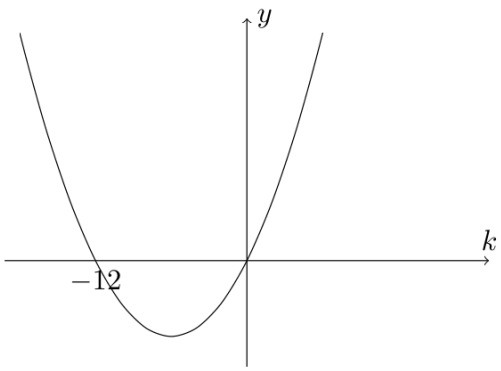
Let us write the expression as $x^2+(k+2)x+(1-2k)$ so that the coefficient of x and the constant are clear. The expression is positive for large values of x, and so is positive for all values of x if the discriminant is negative. Hence we require

$$(k+2)^2 - 4(1-2k) < 0.$$

Expanding and simplifying this gives

$$k^2 + 12k < 0,$$

so $k(k+12) < 0$. If we sketch a graph of the function $k(k+12)$, we obtain



We see that $k(k+12) < 0$ for $-12 < k < 0$, so the correct option is A.

1.7 Answer A

TMUA 2023 · Paper 1 · Q2

As the graphs do not meet, the simultaneous equations $y = x^2 + 5x + 6$ and $y = mx - 3$ have no solutions. Therefore $x^2 + 5x + 6 = mx - 3$ has no solutions, so the quadratic

$$x^2 + (5-m)x + 9 = 0$$

has no solutions and must therefore have a negative discriminant. This gives

$$(5-m)^2 - 4 \times 1 \times 9 = 25 - 10m + m^2 - 36 = m^2 - 10m - 11 < 0.$$

Factorising this quadratic in m gives

$$(m-11)(m+1) < 0.$$

This quadratic has roots -1 and 11, and as the coefficient of m^2 in $m^2 - 10m - 11$ is positive, the quadratic in m is negative between the roots. So the complete range of possible values of m is $-1 < m < 11$, which is option A.

1.8 Answer C

TMUA 2019 · Paper 2 · Q2 (also 2020 · Paper 1 · Q2)

Since $2x+1 = 2(x+\frac{1}{2})$ is a factor of $\mathrm{f}(x) = 2x^3 + px^2 + q$, it follows that $\mathrm{f}(-\frac{1}{2}) = 0$. Similarly, $\mathrm{f}(2) = 0$.

Substituting these values in to $\mathrm{f}(x)$ gives

$$2(-\tfrac{1}{2})^3 + p(-\tfrac{1}{2})^2 + q = 0$$
$$2(2)^3 + p(2^2) + q = 0$$

so

$$-\tfrac{1}{4} + \tfrac{1}{4}p + q = 0$$
$$16 + 4p + q = 0$$

Multiplying the first equation by 16 gives

$$-4 + 4p + 16q = 0.$$

Subtracting $16+4p+q=0$ from this gives $-20+15q=0$, so $q = \frac{4}{3}$, hence $4p = -16 - q = -\frac{52}{3}$. Therefore $2p + q = -\frac{26}{3} + \frac{4}{3} = -\frac{22}{3}$, which is option C.

1.9 Answer B

TMUA 2017 · Paper 1 · Q4

We are dividing a polynomial by a linear expression and asking for the remainder, so it would make sense to use the remainder theorem: if the polynomial $\mathrm{f}(x)$ is divided by $x-a$, then the remainder is $\mathrm{f}(a)$.

In this case, we start with $(3x^2+8x-3)$ and multiply it by $(px-1)$. This gives our polynomial $\mathrm{f}(x) = (3x^2+8x-3)(px-1)$. We then divide $\mathrm{f}(x)$ by $(x+1)$ to get a remainder of $\mathrm{f}(-1) = 24$ (using the remainder theorem).

Substituting $x=-1$ into $\mathrm{f}(x)$ gives

$$\mathrm{f}(-1) = (3-8-3)(-p-1) = 8(p+1)$$

Thus $8(p+1) = 24$, giving $p=2$, and the correct answer is B.

> ***Commentary:*** *We might have been tempted to multiply out the brackets in* $\mathrm{f}(x)$*. But this does not change the value of* $\mathrm{f}(-1)$*, and the remainder theorem does not say anything about the coefficients of* x*,* x^2 *and so on, so there is no benefit to be gained from expanding* $\mathrm{f}(x)$*. It would only take more effort and increase the chances of making a mistake.*

1.10 Answer G

TMUA 2018 · Paper 1 · Q4

We can solve the equations to find x by substituting for y.

The second equation gives $y = a - x$, so the first equation becomes

$$3x^2 + 2x(a-x) = 4$$

This rearranges to

$$x^2 + 2ax - 4 = 0$$

and the discriminant of this equation is

$$(2a)^2 - 4 \times 1 \times (-4) = 4a^2 + 16$$

This is always positive, as $4a^2 \geqslant 0$ for all real values of a, so there are always two distinct real roots, and the correct answer is G.

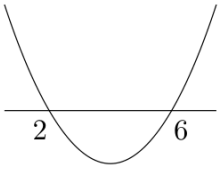
1.11 Answer C

TMUA 2017 · Paper 1 · Q5

Commentary: *The start of this seems like a fairly standard question: find the values of x which satisfy a pair of inequalities. The options offered, though, seem far from standard, so we will have to think carefully once we have found the set S.*

We start by solving the two inequalities separately.

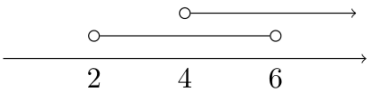
The quadratic factorises as $x^2 - 8x + 12 = (x-2)(x-6)$, so we are solving $(x-2)(x-6) < 0$
We can sketch the graph of $y = (x-2)(x-6)$:



so we see that the first inequality is satisfied when $2 < x < 6$

The second inequality is a linear inequality which immediately simplifies to $2x > 8$ or $x > 4$

We can combine these two inequalities using a number line:



So the set S is given by $4 < x < 6$

Our problem now is that none of the given options says $4 < x < 6$. Most of them are in the form of quadratic inequalities. So we assume that one of the quadratic inequalities probably gives $4 < x < 6$ when we 'solve' it. That, though, seems like a lot of work. We would do better to work in reverse and try to *derive* the quadratic inequality from $4 < x < 6$.

We could think about our earlier sketch, which helped us solve $(x-2)(x-6) < 0$ and gave $2 < x < 6$. If we replace the 2 by a 4, we see that the inequality $(x-4)(x-6) < 0$ will give $4 < x < 6$. We can expand this to get $x^2 - 10x + 24 < 0$, and so the correct answer is C.

1.12 Answer D

TMUA 2018 · Paper 1 · Q5

For this question, we make use of the remainder theorem: the remainder when $\mathrm{f}(x)$ is divided by $(x-a)$ is $\mathrm{f}(a)$.

In this case, we take $a = -2$ and $a = -3$, so

$$\mathrm{f}(-2) = R$$
$$\mathrm{f}(-3) = S$$

Substituting these into the given formula for $\mathrm{f}(x)$ gives

$$\mathrm{f}(-2) = -8 \ \ + 4a - 2b + c = R$$
$$\mathrm{f}(-3) = -27 + 9a - 3b + c = S$$

We want the largest possible value of $R - S$, so we subtract these two equations to get

$$19 - 5a + b = R - S$$

To get the largest possible value, we want a to be as small as possible and b to be as large as possible, so we take $a = 1$, $b = 3$ (and then $c = 2$), giving $R - S = 19 - 5 + 3 = 17$

Thus the correct answer is D.

1.13 Answer A

TMUA 2020 · Paper 1 · Q5

The line of symmetry can be determined by completing the square of the right hand side of the given equation for S:

$$\begin{aligned} px^2 + 6x - q &= p(x^2 + \tfrac{6}{p}x) - q \\ &= p\Big((x + \frac{3}{p})^2 - \frac{9}{p^2}\Big) - q \\ &= p(x + \frac{3}{p})^2 - \frac{9}{p} - q \end{aligned}$$

so the line of symmetry is at $x = -\frac{3}{p}$. Therefore $-\frac{3}{p} = -\frac{1}{4}$, giving $p = 12$.

The curve touches the x-axis at exactly one point, so we must also have $-\frac{9}{p} - q = 0$, giving $q = -\frac{9}{p} = -\frac{9}{12} = -\frac{3}{4}$.

Therefore $p + 8q = 12 - 6 = 6$, which is option A.

1.14 Answer F

TMUA 2021 · Paper 1 · Q5

We can rewrite f(9), f(16) and f(24) using the information given about the function:

$$\begin{aligned} \text{f}(9) &= \text{f}(3 \times 3) \\ &= \text{f}(3)\text{f}(3) && \text{as 9 is a multiple of 3} \\ \text{f}(16) &= \text{f}(16 \times 1) \\ &= 16 && \text{as 16 is not a multiple of 3} \\ \text{f}(24) &= \text{f}(3 \times 8) \\ &= \text{f}(3)\text{f}(8) && \text{as 24 is a multiple of 3} \\ &= \text{f}(3)\text{f}(8 \times 1) \\ &= \text{f}(3) \times 8 && \text{as 8 is not a multiple of 3} \\ &= 8\text{f}(3) \end{aligned}$$

Substituting these into the given equation then gives

$$\big(\text{f}(3)\big)^2 + 16 - 8\text{f}(3) = 0$$

which is a quadratic in f(3); we can factorise it to give

$$\big(\text{f}(3) - 4\big)^2 = 0$$

which has one (repeated) solution: f(3) = 4. Hence the answer is option F.

1.15 Answer F

TMUA 2023 · Paper 1 · Q5

There are (at least) two ways to calculate the area of the rectangle $RSTU$: we could find the lengths of the sides RU and RS and then multiply them, or we could find the area of the square $MNOP$ and subtract the areas of the shaded triangles. We demonstrate both approaches.

We ignore the units, taking lengths to be in cm and areas to be in cm^2.

We first note that the symmetry of the shape means that $MU = MR$, $NR = NS$ and so on.

Also, as the perimeter of $MNOP$ is 40, the side length is 10, so $NR = 10 - x$.

Approach 1: Finding the lengths of the rectangle's sides

The triangle RMU is right-angled, with $MR = MU = x$. By Pythagoras's Theorem, this means that $RU^2 = x^2 + x^2 = 2x^2$, so $RU = x\sqrt{2}$.

Now $NR = 10 - x$, so by a similar argument, $RS = (10 - x)\sqrt{2}$.

Multiplying these gives the area as $A = x\sqrt{2} \times (10 - x)\sqrt{2} = 2x(10 - x)$.

Approach 2: Finding the area by subtraction

The area of $MNOP$ is $10^2 = 100$. The area of MRU and the area of OST are each $\frac{1}{2}x^2$, so the total area of these two triangles is x^2. Likewise, the area of NRS and the area of PTU are each $\frac{1}{2}(10 - x)^2$, so the total area of these two triangles is $(10 - x)^2$. Therefore the area of the rectangle $RSTU$ is

$$100 - x^2 - (10 - x)^2 = 100 - x^2 - (100 - 20x + x^2) = 20x - 2x^2 = 2x(10 - x).$$

We are told that $A = 20$, so we have to solve the equation $2x(10 - x) = 20$. Expanding and rearranging gives $2x^2 - 20x + 20 = 0$; dividing by 2 simplifies this to $x^2 - 10x + 10 = 0$. We can use the quadratic formula to find the possible values of x; they are

$$x = \frac{10 \pm \sqrt{(-10)^2 - 4 \times 1 \times 10}}{2} = \frac{10 \pm \sqrt{60}}{2} = 5 \pm \sqrt{15}.$$

The larger of these is $5 + \sqrt{15}$, and so the answer is option F.

1.16 Answer D

TMUA Specimen · Paper 1 · Q6

Since $x + 2$ is a factor, substituting $x = -2$ into the polynomial must yield zero by the factor theorem:

$$(-2)^3 + 4c(-2)^2 + (-2)(c + 1)^2 - 6 = 0.$$

Simplifying gives

$$-8 + 16c - 2(c^2 + 2c + 1) - 6 = 0$$

so

$$-2c^2 + 12c - 16 = 0.$$

Dividing by -2 now gives

$$c^2 - 6c + 8 = 0$$

so $(c - 2)(c - 4) = 0$ and the roots are $c = 2$ and $c = 4$, with a sum of 6. Hence the answer is D.

We could also find the sum of the roots directly from the quadratic $c^2 - 6c + 8 = 0$ without solving it: if the roots are $c = p$ and $c = q$, then the quadratic can be written as $(c - p)(c - q) = 0$, which expands to $c^2 - (p + q)c + pq = 0$. So the sum of the roots is the negative of the c coefficient, which is 6, and the product of the roots is the constant, which is 7.

1.17 Answer C

TMUA 2016 · Paper 1 · Q6

We can either work in fractions or percentages. This solution works in fractions.

The fraction of red paint in P is $\frac{3}{10}$ and in Q is $\frac{1}{5}$. Let the fraction of red paint in R be x.

The combination is in the ratio $12 : 5 : 3$, with $\frac{1}{4}$ red paint, so the composition is as follows:

Paint	Volume	Red
P	12	$12 \times \frac{3}{10}$
Q	5	$5 \times \frac{1}{5}$
R	3	$3x$
total	20	$20 \times \frac{1}{4}$

We can now sum the individual red amounts, giving:

$$12 \times \tfrac{3}{10} + 5 \times \frac{1}{5} + 3x = 20 \times 14$$

so $\frac{23}{5} + 3x = 5$, giving $3x = \frac{2}{5} = 40\%$, so $x = 13\frac{1}{3}\%$.

The correct answer is C.

1.18 Answer F

TMUA Specimen · Paper 2 · Q6

The area of the rectangle is height times width, so

$$\text{height} = \frac{\text{area}}{\text{width}}.$$

To obtain the greatest possible height, we require the greatest possible area and the least possible width. The upper bound for the area is 5650 cm, and the lower bound for the height is 79.5 cm, so the greatest possible height is $\frac{5650}{79.5}$ cm, which is option F.

1.19 Answer D

TMUA 2020 · Paper 1 · Q8

We can find the maximum value of $f(x)$ by completing the square or by differentiating.

In both methods, we start by expanding the brackets to write $f(x) = -x^2 + (p-2)x + 2p$.

We can then complete the square:

$$\begin{aligned} f(x) &= -(x^2 - (p-2)x) + 2p \\ &= -\left((x - \tfrac{1}{2}(p-2))^2 - \tfrac{1}{4}(p-2)^2\right) + 2p \\ &= -(x - \tfrac{1}{2}(p-2))^2 + \tfrac{1}{4}(p-2)^2 + 2p \end{aligned}$$

so the maximum value occurs when $x = \frac{1}{2}(p-2)$ and the value at this point is $\frac{1}{4}(p-2)^2 + 2p$.

Alternatively, we can differentiate to get

$$f'(x) = -2x + (p-2)$$

so the stationary point is at $x = \frac{1}{2}(p-2)$; substituting this value into the function $f(x)$ gives the same maximum value as before.

A third approach is to note that the stationary point of a quadratic lies midway between its two roots, that is midway between p and -2, so at $x = \frac{1}{2}(p-2)$, and then substitute to find the minimum value.

We now require

$$\tfrac{1}{4}(p-2)^2 + 2p < 4.$$

Multiplying by 4 and rearranging gives $(p-2)^2 + 8p - 16 < 0$; we next expand the brackets and simplify to give $p^2 + 4p - 12 < 0$, so $(p+6)(p-2) < 0$. This quadratic inequality is satisfied exactly when $-6 < p < 2$, which is option D.

1.20 Answer A

TMUA 2021 · Paper 1 · Q8

A straight line meets a quadratic at exactly one point if and only if there is a repeated root to the simultaneous equations. Substituting for y in the first pair gives

$$2x + 3 = x^2 + bx + c.$$

This rearranges to $x^2 + (b-2)x + (c-3) = 0$. As this has a repeated root, its discriminant must be zero, so

$$(b-2)^2 - 4(c-3) = 0.$$

The second line likewise gives $(b-4)^2 - 4(c+2) = 0$. Expanding and simplifying these two equations gives the following simultaneous equations for b and c:

$$b^2 - 4b - 4c = -16$$

$$b^2 - 8b - 4c = -8$$

Subtracting these two equations gives $4b = -8$ so $b = -2$ and hence $c = 7$. Therefore $b - c = -9$, which is option A.

1.21 Answer A

The roots of the equation, using the quadratic formula, are

$$x = \frac{11 \pm \sqrt{11^2 - 8c}}{4}$$

and these differ by

$$x = \frac{2\sqrt{11^2 - 8c}}{4} = 2$$

so we require $\sqrt{11^2 - 8c} = 4$, or $121 - 8c = 16$. Thus $8c = 105$, so $c = \frac{105}{8}$ and the answer is A.

As $x^2 - 14x + 9 = (x - \alpha)(x - \beta)$, the roots of the quadratic $x^2 - 14x + 9 = 0$ are α and β.

Approach 1: calculate α and β

The most obvious approach to answering this question is to calculate α and β directly using the quadratic formula:

$$\alpha, \beta = \frac{14 \pm \sqrt{14^2 - 4 \times 1 \times 9}}{2} = \frac{14 \pm 2\sqrt{7^2 - 9}}{2} = 7 \pm \sqrt{7^2 - 9} = 7 \pm \sqrt{40} = 7 \pm 2\sqrt{10}$$

But now we need to find the square root of these expressions. We can guess that it has the form $p + q\sqrt{10}$ for some integers p and q. Squaring this gives

$$\left(p + q\sqrt{10}\right)^2 = (p^2 + 10q^2) + 2pq\sqrt{10}$$

We cannot make $p^2 + 10q^2 = 7$, but if we take $p = 2$ and $q = \pm 1$, we get $14 \pm 4\sqrt{10}$, which is 2α or 2β. So if we divide by $\sqrt{2}$, taking $p = \sqrt{2}$ and $q = \pm\frac{1}{\sqrt{2}}$, we will obtain $\pm\alpha$ and $\pm\beta$. Therefore

$$\sqrt{\alpha}, \sqrt{\beta} = \pm(\sqrt{2} \pm \sqrt{5})\,.$$

However, we must take the positive values for α and β, so we must have

$$\sqrt{\alpha}, \sqrt{\beta} = \sqrt{5} \pm \sqrt{2})\,.$$

Therefore

$$\begin{aligned}
\left(x - \sqrt{\alpha}\right)\left(x - \sqrt{\beta}\right) &= x^2 - \left(\sqrt{\alpha} + \sqrt{\beta}\right) + \sqrt{\alpha}\sqrt{\beta} \\
&= x^2 - \left((\sqrt{5} + \sqrt{2}) + (\sqrt{5} - \sqrt{2})\right)x + \left(\sqrt{5} + \sqrt{2}\right)\left(\sqrt{5} - \sqrt{2}\right) \\
&= x^2 - 2\sqrt{5}x + 3 \\
&= x^2 - \sqrt{20}x + 3
\end{aligned}$$

which is option C.

Approach 2: use properties of α and β

An alternative way of approaching this question is as follows.

If we expand the two quadratics of interest, we have

$$\begin{aligned}
(x - \alpha)(x - \beta) &= x^2 - (\alpha + \beta)x + \alpha\beta = x^2 - 14x + 9 \\
\left(x - \sqrt{\alpha}\right)\left(x - \sqrt{\beta}\right) &= x^2 - \left(\sqrt{\alpha} + \sqrt{\beta}\right)x + \sqrt{\alpha}\sqrt{\beta}
\end{aligned}$$

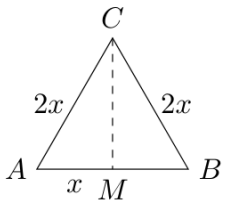
Thus $\alpha + \beta = 14$ and $\alpha\beta = 9$, and we need to determine $\sqrt{\alpha} + \sqrt{\beta}$ and $\sqrt{\alpha}\sqrt{\beta}$. We can take the square root of $\alpha\beta = 9$ to obtain $\sqrt{\alpha}\sqrt{\beta} = 3$.

Next,

$$\left(\sqrt{\alpha} + \sqrt{\beta}\right)^2 = \alpha + \beta + 2\sqrt{\alpha}\sqrt{\beta} = 14 + 2 \times 3 = 20$$

using the values we have determined already, and so $\sqrt{\alpha} + \sqrt{\beta} = \sqrt{20}$. Therefore the correct answer is option C, as before.

The volume of the prism is the length times the cross-sectional area, so we need to work out the area of the equilateral triangle:



We could use trigonometry and the formula for the area of a triangle, area $= \frac{1}{2}ab\sin C$; this gives the area as $\frac{1}{2}(2x)(2x)\sin 60° = x^2\sqrt{3}$.

Alternatively, we could find the length CM using Pythagoras's theorem, giving $CM^2 = (2x)^2 - x^2 = 3x^2$, so $CM = x\sqrt{3}$. Thus the area of the triangle is $\frac{1}{2}AB \times CM = x^2\sqrt{3}$.

Thus the volume of the prism is $T = x^2d\sqrt{3}$.

The total surface area of the prism is twice the area of the triangle, plus the area of the three rectangular faces, so

$$T = 2x^2\sqrt{3} + 3(2xd) = 2x^2\sqrt{3} + 6xd.$$

These expressions for T are equal, so

$$x^2d\sqrt{3} = 2x^2\sqrt{3} + 6xd.$$

Collecting the d terms to the left hand side gives

$$x^2d\sqrt{3} - 6xd = 2x^2\sqrt{3}$$

so

$$d(x^2\sqrt{3} - 6x) = 2x^2\sqrt{3}$$

hence

$$d = \frac{2x^2\sqrt{3}}{x^2\sqrt{3} - 6x} = \frac{2x\sqrt{3}}{x\sqrt{3} - 6}.$$

If we now divide the numerator and denominator by $\sqrt{3}$, we obtain

$$d = \frac{2x}{x - 2\sqrt{3}},$$

which is option D.

1.24 Answer E

TMUA 2021 · Paper 1 · Q12

We can complete the square with the first function: we have

$$x^4 - p^2x^2 = (x^2 - \tfrac{1}{2}p^2)^2 - \tfrac{1}{4}p^4$$

This attains its minimum value when the parenthesised expression is 0, that is when $x = \pm\frac{1}{\sqrt{2}}p$. At that point, $x^4 - p^2x^2 = -\frac{1}{4}p^4 = -9$, so $p^4 = 36$. Since p is a real number, we must have $p^2 = 6$ (rather than -6).

We now consider $x^2 - px + 6$. We can again complete the square to get

$$x^2 - px + 6 = (x - \tfrac{1}{2}p)^2 - \tfrac{1}{4}p^2 + 6.$$

The minimum of this function again occurs when the parenthesised expression is 0, that is when $x = \frac{1}{2}p$. At this point,

$$x^2 - px + 6 = -\tfrac{1}{4}p^2 + 6 = -\tfrac{1}{4} \times 6 + 6 = \tfrac{9}{2}$$

and so the answer is option E.

Alternatively, we could have used calculus instead of completing the square to find the minima; the results would be the same.

1.25 Answer D

TMUA 2022 · Paper 1 · Q12

The formula for y_k is given in completed square form. The minimum of the graph $y = a(x-b)^2+c$, where $a > 0$, is at $x = b$, and the minimum value is c. Therefore the minimum of y_k is at $(\frac{k}{2}, \frac{k^2}{2} + 4k + 3)$.

The minimum of the minimum y-coordinates is therefore the minimum of $\frac{k^2}{2} + 4k + 3$. We can find this minimum either by completing the square or by differentiating this with respect to k. Let us use the latter method here as it seems simpler. We have

$$\frac{\mathrm{d}}{\mathrm{d}k}\left(\frac{k^2}{2} + 4k + 3\right) = k + 4$$

so the minimum occurs at $k = -4$, giving

$$\left(\frac{k}{2}, \frac{k^2}{2} + 4k + 3\right) = (-2, -5)$$

so that $a + b = -7$, which is option D.

1.26 Answer A

TMUA Specimen · Paper 2 · Q12

This is an application of the remainder theorem. If we divide $p(x)$ by $(x - 1)$, we get a quotient $q(x)$ and a remainder r. We can write this as

$$p(x) = (x - 1)q(x) + r.$$

If we now substitute $x = 1$ into this identity, we obtain

$$p(1) = r,$$

so we must have $r = 2$ (since we are told $p(1) = 2$). Therefore

$$p(x) = (x - 1)q(x) + 2 \text{ for some polynomial } q(x),$$

which is option A.

It may be the case that some or all of B, D, E, F, G and H are also true, but we cannot deduce this from the given information. C must be false, as if we substitute $x = 1$, we obtain $2 = -2$.

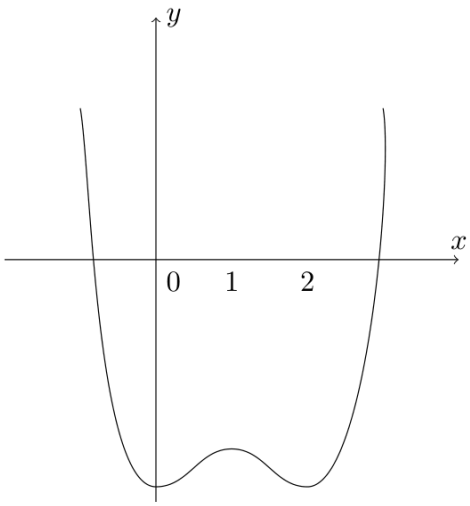
1.27 Answer C

TMUA Specimen · Paper 1 · Q13

We can determine the answer to this question by locating the stationary points of the function $\mathrm{f}(x) = x^4 - 4x^3 + 4x^2 - 10$. We have

$$\mathrm{f}'(x) = 4x^3 - 12x^2 + 8x = 4x(x^2 - 3x + 2) = 4x(x-1)(x-2).$$

There are therefore stationary points at $x = 0$, $x = 1$ and $x = 2$. Substituting these values into $\mathrm{f}(x)$ gives us the y-coordinates of these points on the graph of $y = \mathrm{f}(x)$: they are $(0, -10)$, $(1, -9)$ and $(2, -10)$. Since $\mathrm{f}(x)$ tends to $+\infty$ as x tends to $+\infty$ or $-\infty$, the graph looks roughly like this:



There are thus only two real solutions to $\mathrm{f}(x) = 0$, and the answer is option C.

1.28 Answer A

TMUA 2022 · Paper 1 · Q13

Expanding the brackets gives

$$2 - a^3b^3 + \frac{4}{a^3b^3} - 2 = \sqrt{2}$$

so

$$\frac{4}{a^3b^3} - a^3b^3 = \sqrt{2}.$$

We are interested in the least possible value of ab, so we can find the least possible value of a^3b^3 and take the cube root. Let us simplify the notation by writing $x = a^3b^3$, so $\frac{4}{x} - x = \sqrt{2}$.

Multiplying by x gives a quadratic: $4 - x^2 = \sqrt{2}\,x$, which we can rearrange to $x^2 + \sqrt{2}\,x - 4 = 0$ and then solve using the quadratic formula:

$$x = \frac{-\sqrt{2} \pm \sqrt{2 + 16}}{2} = \frac{-\sqrt{2} \pm 3\sqrt{2}}{2} = -2\sqrt{2} \text{ or } \sqrt{2}.$$

Since we want the least possible value of x, we take $x = a^3b^3 = -2\sqrt{2} = -2^{\frac{3}{2}}$, and so the minimum possible value of ab is $(-2^{\frac{3}{2}})^{\frac{1}{3}} = -2^{\frac{1}{2}} = -\sqrt{2}$, which is option A.

1.29 Answer F

TMUA 2017 · Paper 2 · Q14

We can complete the square for the original quadratic:

$$x^2 - 2bx + c = (x - b)^2 - b^2 + c$$

so the vertex P is at $(b, c - b^2)$.

The graph given shows that $b > 0$ and $c - b^2 > 0$.

The new quadratic is

$$x^2 - 2Bx + c = (x - B)^2 - B^2 + c$$

so the new vertex is at $(B, c - B^2)$.

Now $B > b$, so the new vertex is to the right of P. As $B > b > 0$, $B^2 > b^2$ so $c - B^2 < c - b^2$, and the new vertex is below P.
Therefore the correct answer is F.

1.30 Answer D

TMUA 2016 · Paper 2 · Q15

We start by finding the roots of $f(x) = 0$. They are given by the quadratic formula, and are

$$r_1, r_2 = \frac{2p \pm \sqrt{(-2p)^2 - 4q}}{2} = p \pm \sqrt{p^2 - q}.$$

The roots are real if and only if the discriminant is non-negative, so if and only if $p^2 - q \geqslant 0$.

The difference between the roots is $r_2 - r_1 = 2\sqrt{p^2 - q}$, so the condition $(*)$ is true if and only if

$$p^2 - q \geqslant 0 \text{ and } 2 < 2\sqrt{p^2 - q} < 4$$

so it is true if and only if

$$p^2 - q \geqslant 0 \text{ and } 1 < \sqrt{p^2 - q} < 2$$

which is true if and only if

$$p^2 - q \geqslant 0 \text{ and } 1 < p^2 - q < 4$$

The first condition is implied by the second condition, so both conditions are true if and only if $1 < p^2 - q < 4$, which we can rearrange to $q + 1 < p^2 < q + 4$. Subtracting 1 gives $q < p^2 - 1 < q + 3$, which is condition D.

We can also show that the other four statements are not true if and only if $(*)$ is true, as follows.

Statement A fails, because the left inequality $q < p^2$ is not the same as $q < p^2 - 1$. Statement A is true if $(*)$ is true, but $(*)$ could be true without statement A being true; in other words "statement A is true only if $(*)$ is true" is false.

Statement B fails, because this requires $p > 0$, whereas we could have $p < 0$; thus, statement B is true only if $(*)$ is true, but "statement B is true if $(*)$ is true" is false.

Statement C fails, because this allows for the difference between the roots to be exactly 2 or 4. So statement C is true if $(*)$ is true, but "statement C is true only if $(*)$ is true" is false.

Finally, statement E fails, because the second inequality is equivalent to $p^2 - 1 < q + 4$, so similarly to statement A, statement E is true if $(*)$ is true, but "statement E is true only if $(*)$ is true" is false.
So the correct option is D.

1.31 Answer E

TMUA Specimen · Paper 1 · Q16

We begin by simplifying this expression; we write each term as a product of prime powers to obtain:

$$\begin{aligned}\frac{10^{c-2d}\times 20^{2c+d}}{8^c\times 125^{c+d}} &= \frac{(2\times 5)^{c-2d}\times(2^2\times 5)^{2c+d}}{(2^3)^c\times(5^3)^{c+d}}\\ &= \frac{2^{c-2d+2(2c+d)}\times 5^{c-2d+(2c+d)}}{2^{3c}\times 5^{3(c+d)}}\\ &= \frac{2^{5c)}\times 5^{3c-d}}{2^{3c}\times 5^{3c+3d}}\\ &= 2^{2c}\times 5^{-4d}\end{aligned}$$

For this to be an integer, we require $2c$ and $-4d$ to be non-negative integers. Since c and d are non-zero integers, we need $c>0$ and $d<0$, which is option E.

(In fact, this is an "if and only if" condition; options C, D and F would make the expression non-integer, as would A and G; while conditions B and H are necessary, they are not sufficient: if $d<0$, it is still possible that $c<0$, so it is not true that the given expression is (necessarily) an integer if $d<0$.)

1.32 Answer F

TMUA 2019 · Paper 2 · Q16

For the quadratic to intersect the x-axis at two distinct points, we require the discriminant to be positive. In this case, this gives

$$q^2-4p^2>0.$$

We can either deduce the correct graph by solving this inequality, or decide between the given options by checking various points. We show both approaches here.

First, we can write the inequality as $(q-2p)(q+2p)>0$. Therefore, this inequality holds when either $q-2p>0$ and $q+2p>0$, or when $q-2p<0$ and $q+2p<0$. The first case gives $q>2p$ and $q>-2p$. As $p>0$, this simplifies to $q>2p$. The second case gives $q<2p$ and $q<-2p$, which simplifies to $q<-2p$. So the solution is $q>2p$ or $q<-2p$, and this is shown in graph F.

Alternatively, we can check values of p and q. When $p=1$, $q=0$, $q^2-4p^2=-4<0$, so this point cannot lie in the region. This eliminates options A, C, E and G. Next, we test a point with $q<0$, say $p=1$, $q=-4$. This gives $q^2-4p^2=12>0$, so this point is in the region. This eliminates options B and D, so we are left with F or H. Finally, we can check a point which distinguishes the two, say $p=2$, $q=3$, which gives $q^2-4p^2=-7<0$, so this point is not in the region. Therefore the correct answer is graph F.

1.33 Answer D

TMUA Specimen · Paper 1 · Q17

The quadratic equation can be written as $ax^2+(a-2)x-2=0$, which has discriminant

$$(a-2)^2-4a(-2)=a^2-4a+4+8a=a^2+4a+4=(a+2)^2.$$

The quadratic has real distinct roots when the disciminant is positive, so when $(a+2)^2>0$.

Now $(a+2)^2=0$ when $a=-2$, and $(a+2)^2\geqslant 0$ for all values of a, so $(a+2)^2>0$ when $a\neq -2$, which is option D.

1.34 Answer A

TMUA 2020 · Paper 1 · Q17

A graph-based (exam-suitable) approach is as follows: Thinking about a sketch graph of $y = \sqrt{x}$, any negative value of m will lead to a single point of intersection, so we must be restricted to $m > 0$. This excludes options D and F. When $c \leq 0$, there can be at most one point of intersection. The only option which rejects all possible values of m when $c < 0$ is A, so that must be the correct option. (This option does not make sense when $c = 0$, but we ignore that.)

An algebraic solution is as follows.

The line and the graph intersect when $mx + c = \sqrt{x}$, where $x \geq 0$. If $m = 0$, there is at most one solution, so we cannot have $m = 0$. The equation rearranges to give a quadratic in $\sqrt{x}$, namely $mx - \sqrt{x} + c = 0$. For there to be two distinct non-negative solutions, we first require the discriminant to be positive, so $1 - 4mc > 0$, or $4mc < 1$. We can write the roots explicitly as

$$\sqrt{x} = \frac{1 \pm \sqrt{1 - 4mc}}{2m}$$

and we require both of these to be non-negative. The midpoint of the two roots is $\frac{1}{2m}$, so if $m < 0$, one or both of the roots is negative; we must therefore exclude this case. So we must have $m > 0$ and $4mc < 1$. If $c < 0$, then $1 - 4mc > 1$, so $\frac{1-\sqrt{1-4mc}}{2m} < 0$; this therefore gives only one non-negative root.

We are thus left with two possibilities: $m > 0$ and $c = 0$, or $m > 0$ and $4mc < 1$, which rearranges as $0 < m < \frac{1}{4c}$. The closest offered option is A.

1.35 Answer D

TMUA 2023 · Paper 2 · Q18

Writing $u = x^2$, the equation becomes $u^2 + bu + c = 0$. This has two distinct real roots u_1 and u_2 if and only if $b^2 - 4c > 0$, or $b^2 > 4c$, so this is a necessary condition.

The roots of the original equation are then given by $x = \pm\sqrt{u_1}$ and $x = \pm\sqrt{u_2}$. So for the original equation to have four distinct real roots, we require u_1 and u_2 to be distinct *positive* real numbers. We have

$$u_1, u_2 = \frac{-b \pm \sqrt{b^2 - 4c}}{2}$$

so the smaller of the two is $\frac{1}{2}\left(-b - \sqrt{b^2 - 4c}\right)$. If $b > 0$, then this will be negative, so we require $b < 0$. Supposing $b < 0$, we also require $-b > \sqrt{b^2 - 4c}$, which becomes on squaring (as both sides are positive) $b^2 > b^2 - 4c$, so $c > 0$.

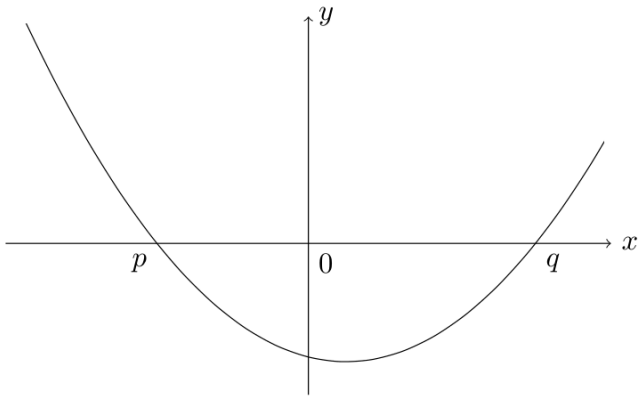
Putting these all together, the original equation will have four distinct real roots if and only if all of the following hold:

- $b^2 > 4c$
- $b < 0$
- $c > 0$

As $c > 0$, we can deduce that $b^2 > 4c$ if and only if $b > 2\sqrt{c}$ or $b < -2\sqrt{c}$. Since we require $b < 0$, the first is not possible, so the pair of conditions $b < 0$ and $b^2 > 4c$ is equivalent to the single condition $b < -2\sqrt{c}$.

Therefore a necessary and sufficient condition for the original equation to have four distinct real roots is $c > 0$ and $b < -2\sqrt{c}$, which is option D.

We can sketch the graph of $y = x^2 + bx + c$. It intersects the x-axis at p and q, and intersects the y-axis at $c < 0$. So the graph must look like this:



However, the vertex might be either side of the y-axis.

To solve the inequality $x^2 + bcx + c^3 < 0$, we need to find where this equals zero. We can use the quadratic formula for this: $x^2 + bcx + c^3 = 0$ when

$$\begin{aligned} x &= \frac{-bc \pm \sqrt{(bc)^2 - 4c^3}}{2} \\ &= \tfrac{1}{2}(-bc \pm \sqrt{b^2c^2 - 4c^3}) \\ &= \tfrac{1}{2}c(-b \pm \sqrt{b^2 - 4c}) \end{aligned}$$

Going back to the original quadratic, we can solve it to find p and q in terms of b and c:

$$p, q = \tfrac{1}{2}(-b \pm \sqrt{b^2 - 4c})$$

Comparing these, we see that the roots of $x^2 + bcx + c^3$ are pc and qc in some order. So the solution is C or D.

To distinguish between these possibilities, we need to know which is the larger of pc and qc. Note that we were told that $c < 0$. Since $p < q$, it follows that $pc > qc$, and so the answer is D.

It turns out that we didn't actually need our graph sketch.

A somewhat less obvious approach to this question is as follows.

We can divide the second inequality by $c^2 > 0$, so $x^2 + bcx + c^3 < 0$ is equivalent to

$$\frac{x^2}{c^2} + \frac{bx}{c} + c < 0$$

or

$$\left(\frac{x}{c}\right)^2 + b\left(\frac{x}{c}\right) + c < 0$$

This is the same as our original inequality, just with x replaced by $\frac{x}{c}$, so its solution is $p < \frac{x}{c} < q$. Multiplying this by $c < 0$ then gives the solution $pc > x > qc$.

Let us write $f(x) = x^2 - 52x - 52$. We first note that $f(x)$ is a quadratic, and that $f(0) = -52$. We might notice that $f(52) = 52^2 - 52^2 - 52 = -52$, or we could complete the square to find that $f(x) = (x - 26)^2 - 26^2 - 52$, so the line of symmetry is at $x = 26$, again giving $f(52) = -52$.

It follows that $f(x) < 0$ when $0 \leq x \leq 52$ (and other values of x just outside this range).

We next check $f(53)$. We have $f(53) = 53^2 - 52 \times 53 - 52 = 53^2 - 52 \times 54$. Now $52 \times 54 = (53 - 1)(53 + 1) = 53^2 - 1$, so $f(53) = 53^2 - (53^2 - 1) = 1$.

Therefore 53 is the smallest integer for which $f(x) > 0$, and the correct answer is option E.

This is a rather ugly looking expression! Let us consider the first square root to begin with:

$$\sqrt{8-4\sqrt{2}+1}$$

We could simplify this to $\sqrt{9-4\sqrt{2}}$, but it seems likely that the examiners have written it in this way to help us.

One way to approach this question is to square the given expression, see what comes out, and then compare the answer with the squares of the given options. We will try this approach below.

Another approach is to see if we can write the inside of the square root as the square of something. We have $8=(2\sqrt{2})^2$ and $1=1^2$, which suggests that we try $2\sqrt{2}+1$. But as the middle term has a minus sign, we should actually try $2\sqrt{2}-1$. Squaring this gives

$$\left(2\sqrt{2}-1\right)^2=8-4\sqrt{2}+1$$

and so

$$\sqrt{8-4\sqrt{2}+1}=2\sqrt{2}-1.$$

It is important that we take the positive square root; since $2\sqrt{2}>1$, this is the positive square root.

Similarly,

$$\sqrt{9-12\sqrt{2}+8}=3-2\sqrt{2},$$

where again we note that $3>2\sqrt{2}$, so this is the positive square root.

These sum to 2, and hence the answer is E.

The alternative approach was to square the given expression. Writing it as

$$\sqrt{9-4\sqrt{2}}+\sqrt{17-12\sqrt{2}}$$

we can square it to get

$$\begin{aligned}9-4\sqrt{2}+17-12\sqrt{2}+2\sqrt{\left(9-4\sqrt{2}\right)\left(17-12\sqrt{2}\right)}&\\ =26-16\sqrt{2}+2\sqrt{153-68\sqrt{2}-108\sqrt{2}+96}&\\ =26-16\sqrt{2}+2\sqrt{249-176\sqrt{2}}&\end{aligned}$$

Unfortunately, we are now left in an even worse situation than we started with, as we have to find some way of simplifying this square root; can we find a and b with $(a-b\sqrt{2})^2=249-176\sqrt{2}$? It turns out that $11-8\sqrt{2}$ does the job; this, however, is negative (as $11^2<(8\sqrt{2})^2$), so the positive square root is $8\sqrt{2}-11$. Thus the whole expression simplifies to $26-16\sqrt{2}+2(8\sqrt{2}-11)=4$, and so the correct solution is $\sqrt{4}=2$, which is option E.

The first approach was certainly simpler.

The equation has solutions $x = a$ and $x = \frac{1 \pm \sqrt{1-4a}}{2}$.

One case in which there are two distinct values is when $1 - 4a = 0$ and $a \neq \frac{1}{2}$, which occurs when $a = \frac{1}{4}$.

The other case in which there are two distinct values is when a is one of the solutions of the quadratic $x^2 - x + a = 0$. We could find this by equating $a = \frac{1 \pm \sqrt{1-4a}}{2}$, but a much simpler approach is to substitute $x = a$ into the quadratic, giving $a^2 - a + a = 0$, hence $a^2 = 0$ giving $a = 0$.

Therefore there are two such values, $a = 0$ and $a = \frac{1}{4}$, which is option C.

An algebraic approach

We can manipulate the equation algebraically. We could square it immediately, but it is probably simpler to first move one of the square roots to the other side so that there is only one square root on each side, that is $\sqrt{x+p} = p - \sqrt{x}$

We can now square this to obtain

$$x + p = p^2 - 2p\sqrt{x} + x$$

so $2p\sqrt{x} = p^2 - p$

If $p \neq 0$, we can divide by p to obtain $2\sqrt{x} = p - 1$. Since the left side of this must be positive, there will be a solution as long as $p \geqslant 1$.

We also need to check that when $p \geqslant 1$, the solution obtained does satisfy the original equation. We have $\sqrt{x} = \frac{1}{2}(p-1)$ so $x = \frac{1}{4}(p-1)^2$. Then the left side of the original equation becomes

$$\sqrt{\tfrac{1}{4}(p-1)^2 + p} + \sqrt{\tfrac{1}{4}(p-1)^2} = \sqrt{\tfrac{1}{4}(p+1)^2} + \sqrt{\tfrac{1}{4}(p-1)^2} = \tfrac{1}{2}(p+1) + \tfrac{1}{2}(p-1) = p$$

and so this is a real solution.

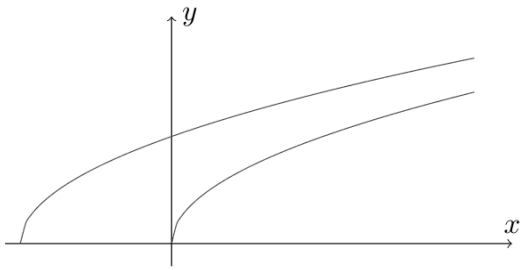
Finally, when $p = 0$ (which we excluded earlier), the equation becomes $\sqrt{x} + \sqrt{x} = 0$, which has the solution $x = 0$.

Thus the correct option is B.

A graphical approach

We can sketch the graph $y = \sqrt{x+p} + \sqrt{x}$. We know that $\sqrt{x}$ is an increasing function, defined for $x \geqslant 0$, taking only non-negative values. It follows that $y = \sqrt{x+p}$ is also an increasing function, defined for $x \geqslant -p$, being a translation of $y = \sqrt{x}$ by p in the negative x-direction.

Here is a sketch of the two graphs for some $p > 0$:



Clearly, there cannot be a solution for $p < 0$, as the left hand side is never negative. For $p \geqslant 0$, the function on the left hand side is defined for all $x \geqslant 0$, and is increasing towards infinity. So the smallest value the function takes is when $x = 0$, which is $\sqrt{p}$, and it takes on every value larger than this. So there will be a solution to the original equation whenever $\sqrt{p} \leqslant p$, which is whenever $p \leqslant p^2$, that is $p^2 - p \geqslant 0$ or $p(p-1) \geqslant 0$. This holds for $p \leqslant 0$ and $p \geqslant 1$. Since we also require $p \geqslant 0$, there is a solution for $p = 0$ and for all $p \geqslant 1$.

Coordinate Geometry 1

2.1 Answer F

TMUA 2021 · Paper 1 · Q1

Approach 1: using the equations of the circles

We can find the equation by solving the equations of the circles simultaneously.

The equations are

$$(x+2)^2+(y-1)^2=r^2$$
$$(x-3)^2+(y+2)^2=r^2$$

(We do not know the radius, so we have called it r; we are given that both circles have the same radius.)

We can expand the brackets to give

$$x^2+4x+4+y^2-2y+1=r^2$$
$$x^2-6x+9+y^2+4y+4=r^2$$

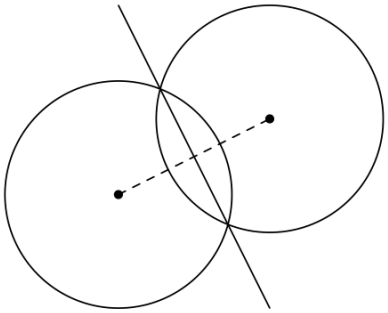
Subtracting these equations then gives an equation which must be satisfied by any points lying on both circles:

$$10x-6y-8=0$$

which simplifies to $5x-3y=4$, and so the answer is F.

Approach 2: using geometry

The circles have the same radius, so by symmetry, the straight line through the two points of intersection must be the perpendicular bisector of the line segment joining the two circle centres, as shown in this diagram:



In our case, the two centres are at $(-2,1)$ and $(3,-2)$, so the midpoint is as $(\frac{1}{2},-\frac{1}{2})$. The gradient of the line joining the centres is $\frac{-2-1}{3-(-2)}=-\frac{3}{5}$. Therefore the gradient of the perpendicular bisector is $\frac{5}{3}$, and it has equation

$$y-(-\tfrac{1}{2})=\tfrac{5}{3}(x-\tfrac{1}{2})$$

Multiplying by 3 gives

$$3y+\tfrac{3}{2}=5(x-\tfrac{1}{2})$$

which rearranges to give

$$5x-3y=4$$

2.2 Answer B

TMUA Specimen · Paper 2 · Q1

We first divide by 2 to get $x^2 + y^2 + \ldots$, then complete the square for x and y:

$$\begin{aligned} & 2x^2 + 2y^2 - 8x + 12y + 15 = 0 \\ \iff \quad & x^2 + y^2 - 4x + 6y + \tfrac{15}{2} = 0 \\ \iff \quad & (x-2)^2 - 2^2 + (y+3)^2 - 3^2 + \tfrac{15}{2} = 0 \\ \iff \quad & (x-2)^2 + (y+3)^2 = \tfrac{11}{2} \end{aligned}$$

so the radius is $\sqrt{\frac{11}{2}}$, which is option B.

2.3 Answer D

TMUA 2022 · Paper 1 · Q2

We rewrite the equation in completed square form: starting from

$$x^2 - 2px + y^2 - 6y - p^2 + 8p + 9 = 0,$$

completing the square for x and y gives

$$(x-p)^2 - p^2 + (y-3)^2 - 9 - p^2 + 8p + 9 = 0,$$

which simplifies to

$$(x-p)^2 + (y-3)^2 - 2p^2 + 8p = 0$$

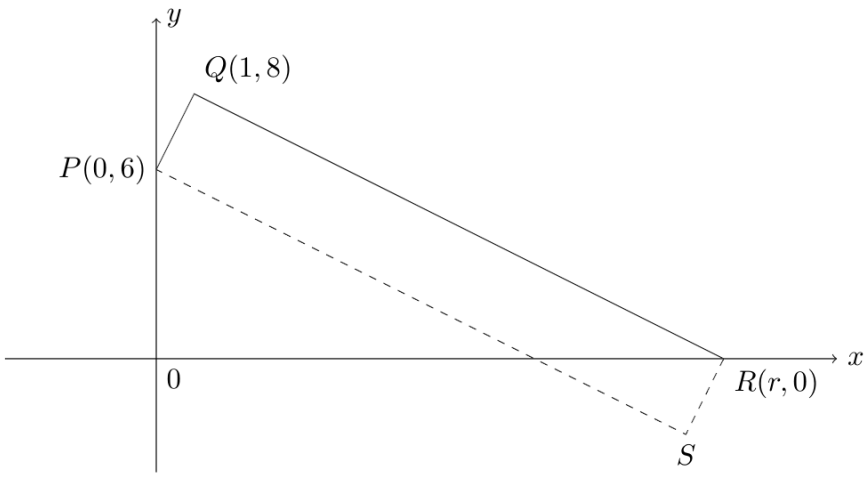
or

$$(x-p)^2 + (y-3)^2 = 2p^2 - 8p.$$

This is a circle with squared radius $2p^2 - 8p$, as long as this is positive. So the complete set of values of p for which the original equation describes a circle in the xy-plane are those for which $2p^2 - 8p > 0$, or $p^2 - 4p > 0$.

This factorises as $p(p-4) > 0$, which has roots $p = 0$, $p = 4$, so the complete set of values in which the inequality holds is $p < 0$ or $p > 4$, which is option D.

A quick sketch (not drawn to scale) may help us here.



PQ has length $\sqrt{1^2+2^2}=\sqrt{5}$ by Pythagoras's theorem.

The gradient of PQ is $\dfrac{8-6}{1-0}=2$, so QR has gradient $-\frac{1}{2}$

We could now find the coordinates of R by finding the equation of the line QR and substituting in $y=0$ to find the x-coordinate of R.

Alternatively, if we set the coordinates of R to be $(r,0)$, we can just use the known gradient of QR to find r:

$$\frac{0-8}{r-1}=-\frac{1}{2}$$

so $\frac{8}{r-1}=\frac{1}{2}$ giving $r-1=16$, hence $r=17$

Therefore the length QR is

$$\sqrt{(r-1)^2+(-8)^2}=\sqrt{16^2+8^2}=8\sqrt{2^2+1^2}=8\sqrt{5}$$

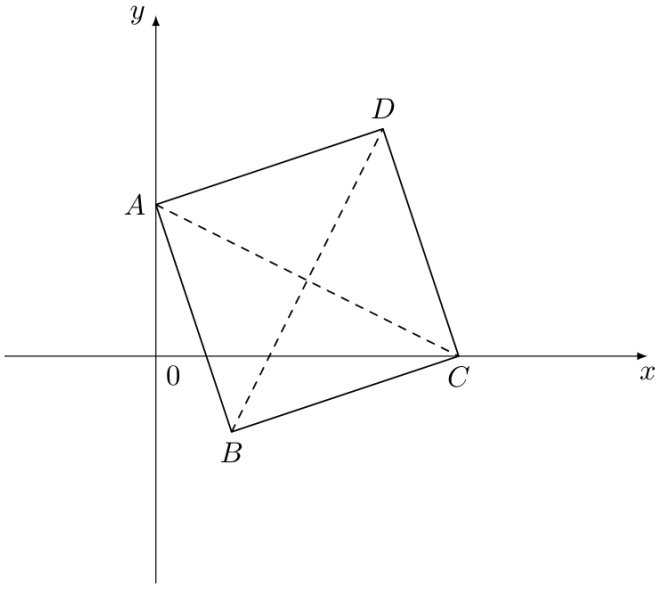
which gives the area of $PQRS$ as $\sqrt{5}\times 8\sqrt{5}=40$, and the answer is E.

Interestingly, it turns out that we only actually needed $r-1$; we did not need to know the actual coordinates of R.

2.5 Answer E

TMUA 2021 · Paper 2 · Q2

We can sketch the situation:



We know that the two diagonals of the square bisect each other at right angles, so this question requires us to find the perpendicular bisector of AC.

The midpoint of AC is $\left(\frac{0+4}{2}, \frac{2+0}{2}\right) = (2, 1)$, which is the centre of the square.

The gradient of AC is $\frac{0-2}{4-0} = -\frac{1}{2}$, so the gradient of BD is 2.

The equation of BD is therefore $y - 1 = 2(x - 2)$, which rearranges to give $y = 2x - 3$, which is option E.

2.6 Answer B

TMUA Specimen · Paper 1 · Q3

We find the equation of the perpendicular bisector first, and then find the x-coordinate of this line when $y = 0$.

The midpoint of the line segment joining $(2, -6)$ and $(5, 4)$ is $(\frac{7}{2}, -1)$, and the gradient of this line segment is $\frac{4-(-6)}{5-2} = \frac{10}{3}$.

Therefore the perpendicular bisector passes through $(\frac{7}{2}, -1)$ and has gradient $-\frac{3}{10}$. Its equation is therefore

$$y - (-1) = -\tfrac{3}{10}\left(x - \tfrac{7}{2}\right).$$

We could expand and simplify this equation, but that is not necessary for our purposes. Instead, we substitute $y = 0$ to obtain

$$1 = -\tfrac{3}{10}\left(x - \tfrac{7}{2}\right)$$

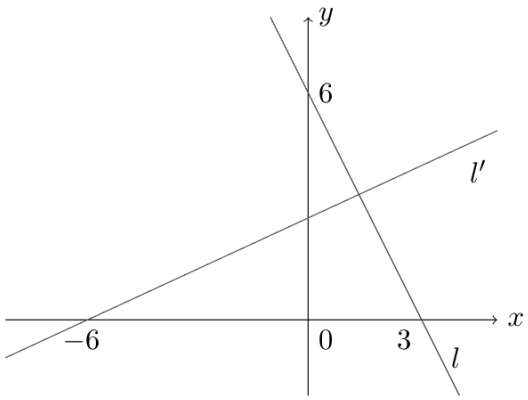
giving

$$x - \tfrac{7}{2} = -\tfrac{10}{3}$$

so $x = \frac{7}{2} - \frac{10}{3} = \frac{21}{6} - \frac{20}{6} = \frac{1}{6}$ showing that the correct answer is option B.

It is always worth drawing a sketch for questions like this, to ensure that we know what area is being referred to.

The line l has a negative gradient, and passes through the point $(0, 6)$; it also intersects the x-axis at $(3, 0)$. So the sketch looks something like this:



We see therefore that the required area is a triangle with base length along the x-axis of $3-(-6) = 9$. Thus to find the area we only need to find the y-coordinate of the point of intersection of the two lines.

The equation of the second line, l', can be given by $y - y_1 = m(x - x_1)$: its gradient is the gradient perpendicular to -2, which is $\frac{1}{2}$, and it passes through $(-6, 0)$, so it has equation

$$l' : y - 0 = \tfrac{1}{2}(x - (-6))$$

which simplifies to $y = \frac{1}{2}x + 3$. We can find the coordinates of intersection by solving the equations of l and l' simultaneously. Substituting for y, we get

$$6 - 2x = \tfrac{1}{2}x + 3$$

so $\frac{5}{2}x = 3$, or $x = \frac{6}{5}$. This looks plausible from the sketch. We then get $y = \frac{1}{2}x + 3 = \frac{18}{5}$ (and we can check that the equation of l gives the same y-value). Thus the area of the triangle is

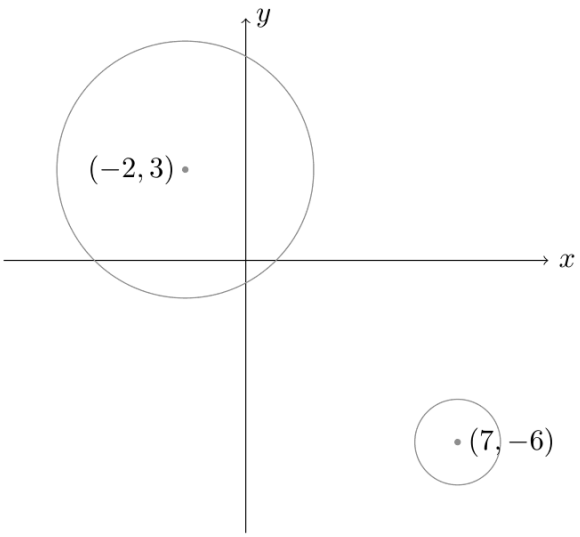
$$\tfrac{1}{2} \times 9 \times \tfrac{18}{5} = \tfrac{81}{5} = 16\tfrac{1}{5}$$

and the correct answer is A.

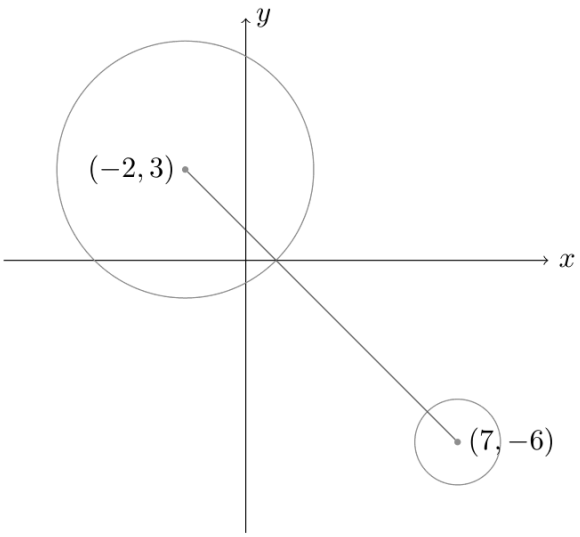
The first circle has centre $(-2, 3)$ and radius $\sqrt{18} = 3\sqrt{2}$

The second circle has centre $(7, -6)$ and radius $\sqrt{2}$

It would help to draw a quick sketch of the situation.



It seems clear from the sketch that to find the shortest distance between the circles, we should draw a line between their centres and find the length of the segment which lies outside both of them:



The distance between the centres is

$$\sqrt{((-2)-7)^2+(3-(-6))^2} = \sqrt{81+81} = 9\sqrt{2}$$

and if we subtract the two radii ($3\sqrt{2}$ and $\sqrt{2}$) for the parts of the line which lie within the circles, we are left with $9\sqrt{2} - 3\sqrt{2} - \sqrt{2} = 5\sqrt{2}$. The correct answer is E.

2.9 Answer B

TMUA 2022 · Paper 2 · Q4

We can rewrite the equation of circle by completing the square to find its centre. We get:

$$(x+f)^2 - f^2 + (y+g)^2 - g^2 + h = 0$$

which we can rearrange as

$$(x+f)^2 + (y+g)^2 = f^2 + g^2 - h.$$

Therefore the centre is at $(-f, -g)$ (and the radius is $\sqrt{f^2+g^2-h}$, but that is not relevant to us here).

Then the distance between the centre of the circle and the point P is given by

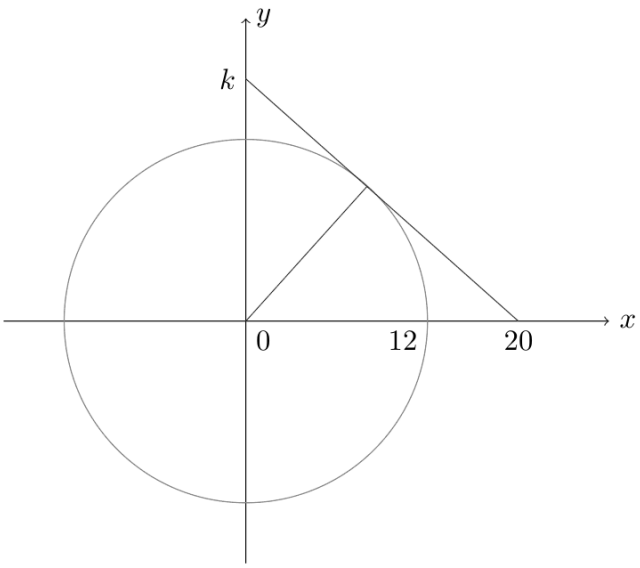
$$L^2 = (p+f)^2 + (q+g)^2$$

so we need the values of f, g, p and q. Hence the correct answer is option B.

2.10 Answer B

TMUA 2017 · Paper 1 · Q6

This calls for a sketch. The circle $x^2 + y^2 = 144$ is centred on the origin and has radius 12:



It's not exact, and it's not drawn to scale, but it's good enough to work with. The blue radius is perpendicular to the tangent line. We are trying to find the value of k.

Approach 1

We could work out the equation of the tangent line. It has gradient m (which we don't know) and passes through $(20, 0)$, so it has equation $y = m(x-20)$. Since it is tangent to the circle, solving the two equations simultaneously must give one repeated root.

We have $x^2 + y^2 = 144$ into which we can substitute $y = m(x-20)$ giving

$$\begin{aligned} & x^2 + m^2(x-20)^2 = 144 \\ \iff \quad & x^2 + m^2(x^2 - 40x + 400) = 144 \\ \iff \quad & (m^2+1)x^2 - 40m^2x + 400m^2 - 144 = 0 \end{aligned}$$

We want this to have exactly one root for x, so we need the discriminant to equal zero, giving

$$(-40m^2)^2 - 4(m^2+1)(400m^2 - 144) = 0.$$

We can take out a factor of 4^2:

$$(10m^2)^2 - (m^2+1)(100m^2 - 36) = 0$$

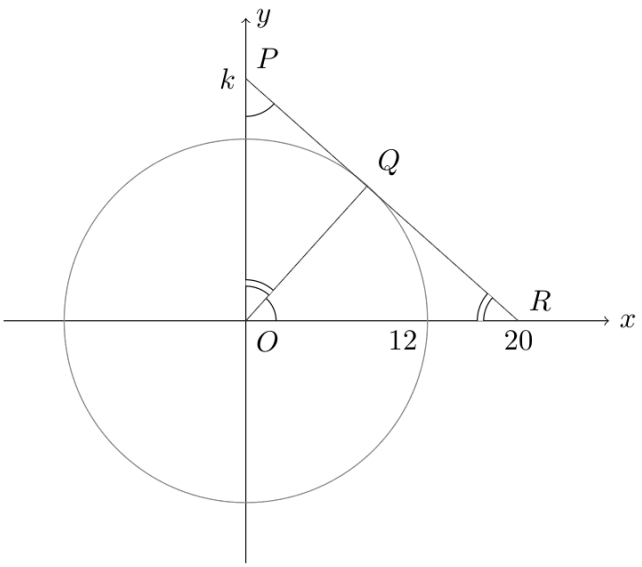
and then expand the brackets to give

$$100m^4 - 100m^4 + 36m^2 - 100m^2 + 36 = 0$$

so $64m^2 = 36$, or $8m = \pm 6$. Since $m < 0$ (as the line intersects the positive y-axis), we have $m = -\frac{6}{8} = -\frac{3}{4}$. Thus the tangent line has equation

Approach 2

We might notice from the sketch that we have some similar triangles, since the radius meets the tangent at right angles. If we label all of the vertices and equal angles, we will be able to describe things more precisely:



So the triangles $\triangle OQR$, $\triangle PQO$ and $\triangle POR$ are all similar. We also know the lengths $OR = 20$ and $OQ = 12$ (as it is a radius). By Pythagoras's Theorem, applied to $\triangle OQR$, we can calculate $QR = 16$ (and we might note that $\triangle OQR$ is a 3-4-5 triangle). We can then use the similarity of triangles $\triangle PQO$ and $\triangle OQR$ to find k:

$$\frac{PO}{QO} = \frac{OR}{QR} \implies \frac{k}{12} = \frac{20}{16}$$

giving $k = 15$, as before.

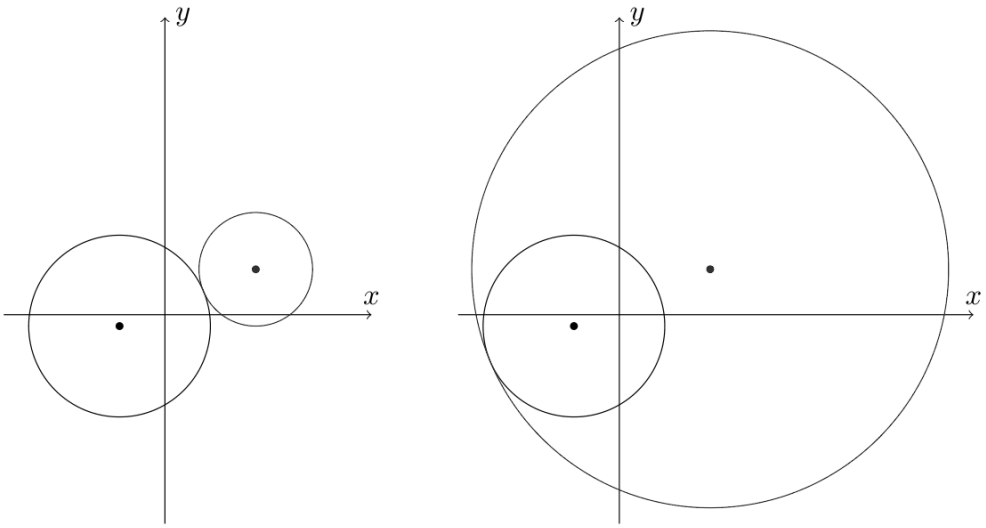
> ***Commentary:*** *You might prefer one or other of these approaches. The second is shorter, but requires some confidence with geometry and recognising similar triangles.*

2.11 Answer C

TMUA 2019 · Paper 1 · Q6

We could try to solve this algebraically or geometrically. An algebraic approach looks difficult, as we have two quadratic equations, and one of them has an unknown coefficient (that is, r^2). So we use a geometric approach instead.

The first circle has centre $(-4, -1)$ and radius 8, while the second circle has centre $(8, 4)$ and radius r. The point $(8, 4)$ does not lie inside the first circle (it is more than $8 - (-4) = 12$ units away), so the two circles will have exactly one point in common when the first circle is tangent to the second circle externally or when it lies inside the second circle and touches it internally. Here is a sketch of these two situations:



In the first case, where $r = r_1$, we see that $r_1 + 8$ is the distance between the centres of the two circles. In the second case, where $r = r_2$, we see that $r_2 - 8$ is the distance between the centres.

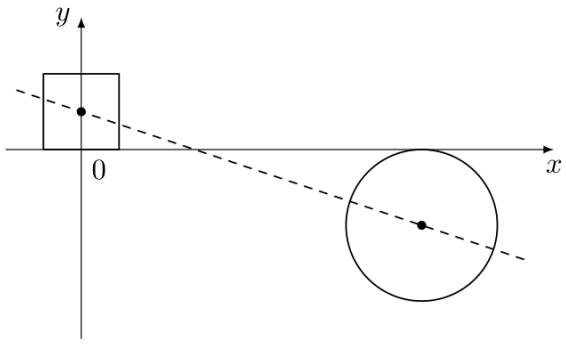
Therefore $r_1 + 8 = r_2 - 8$, so $r_2 - r_1 = 16$, and the answer is C.

Note that we did not even need to work out the actual values of r_1 and r_2. If we had done so, the distance between the centres is $\sqrt{12^2 + 5^2} = 13$, so $r_1 = 13 - 8 = 5$ and $r_2 = 13 + 8 = 21$.

2.12 Answer B

TMUA 2021 · Paper 2 · Q7

We start by drawing the square and the circle. Since the straight line bisects the area of each shape, it must pass through their centres, as shown.



This straight line therefore passes through the centre of the square at $(0, 1)$ and the centre of the circle at $(9, -2)$. The gradient of this line is then $\frac{-2-1}{9-0} = -\frac{1}{3}$, so the line has equation $y = -\frac{1}{3}x + 1$.

The line meets the x-axis where $y = 0$, so $0 = -\frac{1}{3}x + 1$, giving $x = 3$.

Hence the correct answer is B.

2.13 Answer E

TMUA 2023 · Paper 2 · Q7

We can rearrange the equation to give $by = -ax + c$, so

$$y = -\frac{a}{b}x + \frac{c}{b}.$$

Therefore the gradient is $-\frac{a}{b}$ and the y-intercept is $\frac{c}{b}$.

We go through the conditions to determine whether they are necessary and/or sufficient for a positive gradient and positive y-intercept.

A $\frac{c}{b} > 0$ is equivalent to the y-intercept being positive. Next, $\frac{a}{b} < 0$ is equivalent to $-\frac{a}{b} > 0$, which is equivalent to the gradient being positive. So this is a necessary and sufficient condition.

B $\frac{c}{b} < 0$ is equivalent to the y-intercept being negative, so this is neither necessary nor sufficient.

C If we take $a > 0 > c > b$, then $-\frac{a}{b} > 0$ and $\frac{c}{b} > 0$, so the gradient and y-intercept are both positive. Therefore $a > b > c$ is not necessary.

D If we take $a < 0 < c < b$, then $-\frac{a}{b} > 0$ and $\frac{c}{b} > 0$, so the gradient and y-intercept are both positive. Therefore $a < b < c$ is not necessary.

E If $-\frac{a}{b} > 0$ and $\frac{c}{b} > 0$, then $\frac{a}{b} < 0 < \frac{c}{b}$, so a and c must have opposite signs. Therefore this condition is necessary. But if a and c have opposite signs, we know nothing about the sign of b, so we might or might not have $-\frac{a}{b} > 0$. So this condition is not sufficient.

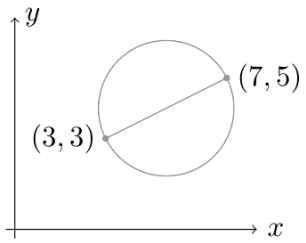
F As in E, if the gradient and y-intercept are both positive, a and c must have opposite signs, so this condition is not necessary.

The correct option is therefore E.

2.14 Answer D

TMUA 2016 · Paper 1 · Q9

The initial circle looks like this:



The centre of the circle is at the midpoint of this diameter, which is $\left(\frac{3+7}{2}, \frac{3+5}{2}\right) = (5, 4)$, and the radius is therefore

$$\sqrt{(5-3)^2 + (4-3)^2} = \sqrt{5}.$$

The transformations do the following to the centre and radius of the circle:

	Centre	Radius
original	$(5, 4)$	$\sqrt{5}$
after translation	$(2, 4)$	$\sqrt{5}$
after reflection	$(2, -4)$	$\sqrt{5}$
after enlargement	$(2, -4)$	$4\sqrt{5}$

Therefore the final circle has equation

$$(x-2)^2 + (y+4)^2 = (4\sqrt{5})^2 = 80$$

and the answer is D.

We clearly need to work out the radius of the circle to get anywhere on this problem. We can do this by completing the square. The equation of the circle, $x^2 + y^2 - 18x - 22y + 178 = 0$, becomes

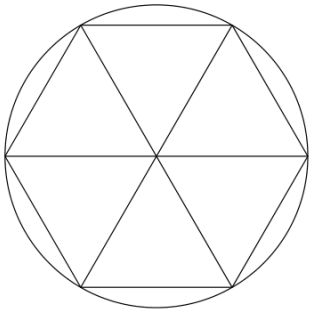
$$(x-9)^2 - 9^2 + (y-11)^2 - 11^2 + 178 = 0$$

With a little bit of arithmetic, we can rearrange this to give

$$(x-9)^2 + (y-11)^2 = 24$$

so the circle has a radius of $r = \sqrt{24}$

We now need to work out the area of the inscribed regular hexagon. We sketch the hexagon and its diameters:

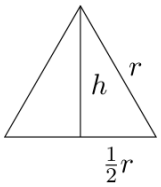


It is clear from the sketch (and even from a not-very-precise sketch) that the hexagon is composed of six triangles, each of which is equilateral and has side length equal to the circle's radius r.

We can work out the area of one of the triangles either using Pythagoras's Theorem or using the formula $\frac{1}{2}ab\sin C$ for the area of a triangle.

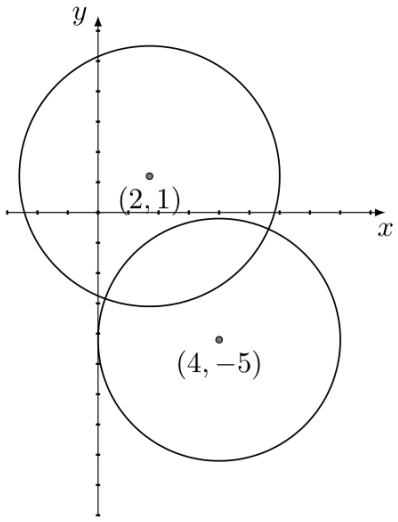
The trigonometric formula $\frac{1}{2}ab\sin C$ gives $\frac{1}{2}r^2\sin 60° = \frac{\sqrt{3}}{4}r^2$. Substituting $r^2 = 24$ then gives each triangle an area of $6\sqrt{3}$, and so the whole hexagon has an area of $36\sqrt{3}$, which is option F.

Alternatively we can proceed using Pythagoras's Theorem. If the height of the triangle is h, then we have:

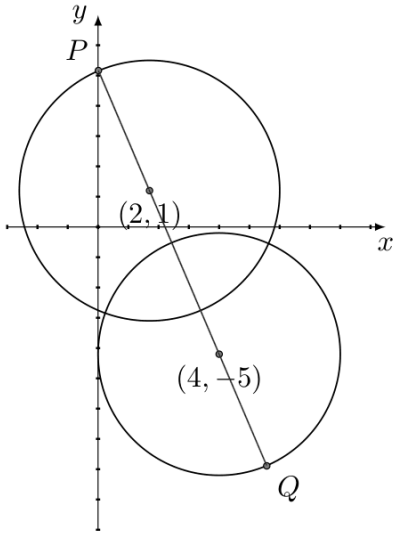


Therefore $h^2 = r^2 - (\frac{1}{2}r)^2$ giving $h = \frac{\sqrt{3}}{2}r$ and the triangle has area $\frac{1}{2}rh = \frac{\sqrt{3}}{4}r^2$ as before.

It would help to sketch the circles. The first circle has centre $(2, 1)$ and radius 4; the second has centre $(4, -5)$ and radius 4. (This is not an accurate sketch, though we have noted that the second circle touches the y-axis.)



It is clear that the furthest apart that P and Q could be is if they are on the line that passes through the two circle centres as shown below:



The distance between P and Q is then the sum of the radius of the first circle, the distance between the centres and the radius of the second circle.

The distance between the centres is $\sqrt{(4-2)^2 + (-5-1)^2} = \sqrt{40} = 2\sqrt{10}$ and each radius is 4, so the maximum distance is $8 + 2\sqrt{10}$, which is option F.

We rewrite these equations in the form $y = mx + c$:

$$y = -\frac{a}{b}x - \frac{c}{b}$$
$$y = -\frac{b}{c}x - \frac{a}{c}$$
$$y = -\frac{c}{a}x - \frac{b}{a}$$

Since these equations are cyclically symmetrical (replacing $a \to b \to c \to a$ leaves the set of equations unchanged), we may assume the two parallel or perpendicular lines are the first two.

A If the first two lines are parallel, then $\frac{a}{b} = \frac{b}{c}$, but this does not imply that $\frac{c}{a}$ is equal to these. For example we could have $a = 1$, $b = 2$, $c = 4$ giving $\frac{a}{b} = \frac{b}{c} = \frac{1}{2}$ but $\frac{c}{a} = 4$.

B The same example shows that the third line need not be perpendicular to the other two. In fact, in this case, since $\frac{a}{c} = \left(-\frac{a}{b}\right)\left(-\frac{b}{c}\right) = \left(-\frac{a}{b}\right)^2 > 0$, the third line will always have a negative gradient (namely $-\frac{c}{a}$), even if the first two also have negative gradients.

C The same example shows that this is not necessarily the case; in fact, as the third line always has a negative gradient, it cannot be parallel to $y = x$.

D Again, the same example shows that the third line might not have gradient -1.

E If the first two lines are perpendicular, we have $\left(-\frac{a}{b}\right)\left(-\frac{b}{c}\right) = \frac{a}{c} = -1$, so $-\frac{c}{a} = 1$. It is not obvious why the lines should meet at a point, so let us return to this after we have looked at the remaining options.

F This is true: the calculation in E shows that the third line has gradient 1.

G This is false: the third line has gradient 1, not -1.

So the correct option is F.

We still need to eliminate option E. Let us consider an example with $a = 1$, $b = 2$; we therefore need $c = -1$ as $-\frac{c}{a} = 1$. Thus the three lines are

$$y = -\frac{1}{2}x - \frac{-1}{2}$$
$$y = -\frac{2}{-1}x - \frac{1}{-1}$$
$$y = -\frac{-1}{1}x - \frac{2}{1}$$

which simplifies to

$$y = -\tfrac{1}{2}x + \tfrac{1}{2}$$
$$y = 2x + 1$$
$$y = x - 2$$

Approach 1

We start by completing the square to find the centre and radius of the circle:

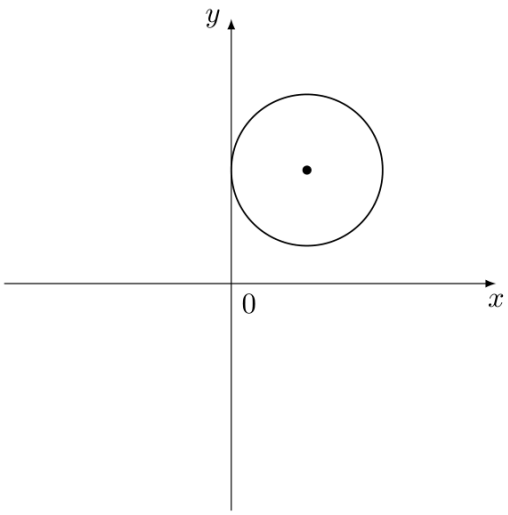
$$(x+\tfrac{1}{2}a)^2-(\tfrac{1}{2}a)^2+(y+\tfrac{1}{2}b)^2-(\tfrac{1}{2}b)^2+c=0$$

which gives

$$(x+\tfrac{1}{2}a)^2+(y+\tfrac{1}{2}b)^2=(\tfrac{1}{2}a)^2+(\tfrac{1}{2}b)^2-c$$

so the circle has centre $(-\frac{1}{2}a,-\frac{1}{2}b)$ and radius $\sqrt{(\frac{1}{2}a)^2+(\frac{1}{2}b)^2-c}$.

Now let us sketch a circle tangent to the y-axis:



We can see that the circle is tangent to the y-axis if and only if its radius is equal to the distance of its centre from the y-axis. This is *not* the same as saying the radius is equal to the x-coordinate of the centre, as the centre may have a negative x-coordinate. As an equation, this condition is then

$$\sqrt{(\tfrac{1}{2}a)^2+(\tfrac{1}{2}b)^2-c}=\left|-\tfrac{1}{2}a\right|$$

which, on squaring, is equivalent to

$$(\tfrac{1}{2}a)^2+(\tfrac{1}{2}b)^2-c=(\tfrac{1}{2}a)^2$$

(This is equivalent to the previous equation because both sides of the first equation are non-negative.)

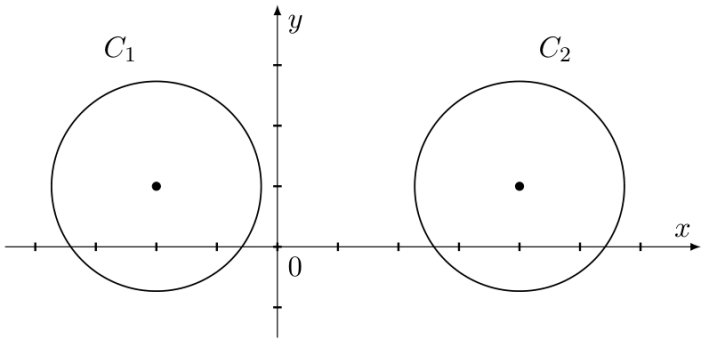
Subtracting $(\frac{1}{2}a)^2$ from both sides gives $\frac{1}{4}b^2-c=0$ or $b^2=4c$, so this condition is necessary and sufficient for the circle to be tangent to the y-axis, which is option B.

Approach 2

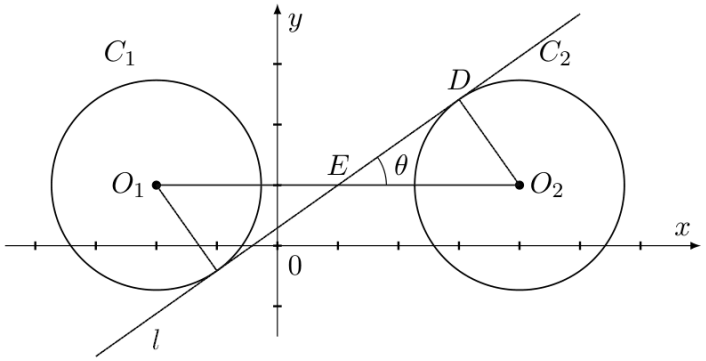
The circle is tangent to the y-axis, $x=0$, if and only if the pair of simultaneous equations $x^2+ax+y^2+by+c=0$ and $x=0$ has a repeated root.

Substituting $x=0$ into the circle equation gives $y^2+by+c=0$, which has discriminant b^2-4c. So there is a repeated root if and only if $b^2-4c=0$, which is equivalent to option B.

It is helpful to sketch the situation before going any further. The circle C_1 has centre $(-2, 1)$ and radius $\sqrt{3}$, while circle C_2 has centre $(4, 1)$ and radius $\sqrt{3}$. A quick sketch shows the configuration:



There are four straight lines which are tangent to both of these circles: two of these are horizontal, and the other two are not. We want the one with a positive gradient, so we sketch it on the graph along with the radii which meet it.



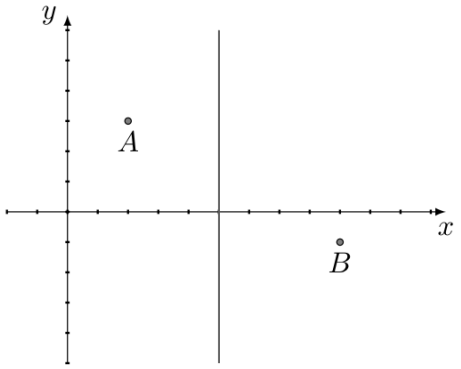
By joining the two circle centres, we see that the angle the line l makes with the x-axis, which is θ, is the same as the angle marked on this diagram. We have also labelled some of the points so that we can refer to them later.

Since the radius at D meets the tangent l at right angles, we can work out the side lengths of the triangle O_2DE and hence $\tan\theta$. The distance between the circle centres is $4 - (-2) = 6$, so the hypotenuse EO_2 has length 3. The circle C_2 has radius $O_2D = \sqrt{3}$, so by Pythagoras's Theorem, $DE^2 = (EO_2)^2 - (DO_2)^2 = 9 - 3 = 6$. Thus $DE = \sqrt{6}$ and hence

$$\tan\theta = \frac{\sqrt{3}}{\sqrt{6}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

and the correct answer is option C.

A sketch is useful here; we also label the points as $A(2,3)$, $B(9,-1)$ and $C(5,k)$. The point C lies on the blue line in this sketch.



We could have C at about $(5,5)$ and have a right angle at A. We could have C a little lower and have a right angle at C. We could have C at about $(5,-5)$ and have a right angle at C, or we could have C a little lower and have a right angle at B. We work through these four possibilities.

- Right angle at A. The gradient of the line AB is $\frac{-1-3}{9-2} = -\frac{4}{7}$, so the gradient of the line AC is $\frac{7}{4}$.

  The equation of the line AC is thus $y-3=\frac{7}{4}(x-2)$. This intersects the line $x=5$ when $y-3=\frac{7}{4}(5-2)$, giving $y=3+\frac{21}{4}=\frac{33}{4}$. So $k=\frac{33}{4}$ gives a right-angled triangle.

- Right angle at B. As in the first case, the gradient of the line BC is $\frac{7}{4}$.

  The equation of the line BC is thus $y+1=\frac{7}{4}(x-9)$. This intersects the line $x=5$ when $y+1=\frac{7}{4}(5-9)$, giving $y=-1-7=-8$. So $k=-8$ gives a right-angled triangle.

- Right angle at C. We could determine the possible values of k either by observing that the lines AC and BC must be perpendicular, or by using Pythagoras's Theorem to solve $AC^2+BC^2=AC^2$. Let us use the perpendicular approach here. We have

$$\text{gradient of } AC = \frac{k-3}{5-2} = \frac{k-3}{3}$$
$$\text{gradient of } BC = \frac{k+1}{5-9} = -\frac{k+1}{4}$$

  These must multiply to -1, so

$$\frac{k-3}{3}\cdot\frac{k+1}{4}=1$$

  which simplifies to $(k-3)(k+1)=12$. We can solve this as a quadratic as usual: it rearranges to $k^2-2k-15=0$ giving $k=5$ and $k=-3$.

Adding these possible values of k gives a total of $\frac{33}{4}-8+5-3=\frac{9}{4}$, which is option E.

Note first that the region where the centre of C_2 may lie is entirely within the circle C_1, as $2^2 + 3^2 = 13 < 25$.

As the radius of C_1 is 5, the circle C_2 will intersect C_1 if and only if the centre of C_2 is at least a distance of 1 from the origin (the centre of C_1). This is because in this case, a line from the centre of C_1 passing through the centre of C_2 will reach the circle C_1 before (or as) it reaches the circle C_2, but a line from the centre of C_1 in the opposite direction will reach C_2 first. If, though, the centre of C_2 is at a distance of less than 1 from the origin, the furthest point on C_2 from the origin will be at a distance of less than $1 + 4 = 5$ from the origin, so C_2 lies entirely inside C_1.

The area of the allowed (a, b) region is $4 \times 6 = 24$. The (a, b) region where the circles do not intersect is a circle of radius 1, with area π, lying entirely within the allowed region, so the region where the circles do intersect has area $24 - \pi$.

Thus the probability of intersection is $\dfrac{24 - \pi}{24}$, which is option F.

Exponentials and Logarithms

3.1 Answer B

TMUA 2021 · Paper 1 · Q4

This function is a quadratic in $u = 2^x$; we can rewrite it as

$$u^2 - 2^3 u + 4$$

or

$$u^2 - 8u + 4.$$

We can complete the square to express this as

$$(u-4)^2 - 4^2 + 4 = (u-4)^2 - 12.$$

The minimum therefore occurs when $u = 4$, at which point the function value is 12.

We must also check that this value can actually be attained: $u = 4$ if and only if $2^x = 4$, which happens when $x = 2$, so this minimum is attained, and the correct answer is B.

The minimum can also be found using calculus: differentiating $u^2 - 8u + 4$ gives $2u - 8$; this is zero when $u = 4$, at which point the function value is $4^2 - 8 \times 4 + 4 = -12$; this is a minimum as the function is a quadratic with positive u^2 coefficient.

3.2 Answer D

TMUA Specimen · Paper 1 · Q5

$$\begin{aligned}
& y = -\log_{10}(1-x) && \\
\iff \quad & -y = \log_{10}(1-x) && \text{negate} \\
\iff \quad & 10^{-y} = 1 - x && \text{exponentiate to base 10} \\
\iff \quad & x + 10^{-y} = 1 && \text{add } x \\
\iff \quad & x = 1 - 10^{-y} && \text{subtract } 10^{-y}
\end{aligned}$$

so the answer is D.

3.3 Answer A

TMUA Specimen · Paper 2 · Q5

The target $\log_3 2$ suggests that we want to take logarithms of both sides of the given approximation to base 3, giving

$$\log_3 2^5 \approx \log_3 3^3$$

so that

$$5 \log_3 2 \approx 3$$

and hence $\log_3 2 \approx \frac{3}{5}$, which is option A.

3.4 Answer A

TMUA 2020 · Paper 2 · Q5

Note that none of these graphs have scales on.

A key question is what happens to y when x is large and positive and when x is large and negative.

If $x = 100$, say, then $y = \dfrac{2^{100}}{1 + 2^{100}}$. Since 2^{100} is a very large positive number, this fraction is almost exactly 1. This excludes graphs B and C, where the value of y increases towards infinity. It also excludes graphs D and E, where the value of y tends towards 0. We are therefore left with graphs A and F.

One way to distinguish between A and F is to compare the value at $x = 0$ with the value when x is large. When $x = 0$, we have $y = \frac{1}{2}$. Since $\frac{1}{2} < 1$ and y is close to 1 when x is large, graph F is not possible, so the answer is graph A.

Alternatively, we can consider what happens when $x = -100$, say. Then $y = \dfrac{2^{-100}}{1 + 2^{-100}}$. Since 2^{-100} is a tiny positive number, very close to 0, this fraction is almost exactly 0. This excludes all of the graphs except for A and B; combining this with our earlier observations shows that the correct graph is A.

3.5 Answer C

TMUA 2020 · Paper 1 · Q6

The denominator is a quadratic in $u = 5^x$, so we can write it as

$$\frac{1}{u^2 - 4u + 7} = \frac{1}{(u-2)^2 + 3}$$

The maximum value of this fraction is obtained when the denominator takes its minimum value, which is when $u = 2$, giving a value of $\frac{1}{3}$. We must also check that this value can be obtained; this is the case as $5^x = 2$ has a real solution.

Therefore the correct answer is option C.

3.6 Answer A

TMUA 2020 · Paper 1 · Q7

We write these two equations in powers of 2, giving:

$$2^{3x} = 2^{3y+9}$$

$$2^{2x+2} = \frac{2^{4y+4}}{2^{3y+9}} = 2^{(4y+4)-(3y+9)} = 2^{y-5}$$

Therefore we have

$$3x = 3y + 9$$

$$2x + 2 = y - 5$$

We can divide the first of these equations by 3 for simplicity, giving

$$x = y + 3$$

$$2x + 2 = y - 5$$

We can now solve these simultaneous equations; subtracting the first from the seconds gives $x + 2 = -8$, so $x = -10$ and $y = -13$, so $x + y = -23$, which is option A.

3.7 Answer F

TMUA 2023 · Paper 1 · Q7

Note that $\mathrm{P}(x)$ and $\mathrm{Q}(x)$ being in the ratio $4:1$ means that $\mathrm{P}(x) = 4\mathrm{Q}(x)$, so

$$2^x + 4 = 4\big(2^{(2x-2)} - 2^{(x+2)} + 16\big).$$

Simplifying the powers, noting that $4 = 2^2$, gives

$$2^x + 4 = 2^{2x} - 2^{x+4} + 64.$$

This is a quadratic in 2^x, so let us write $u = 2^x$. The equation can then be written as

$$u + 4 = u^2 - 16u + 64.$$

This rearranges to $u^2 - 17u + 60 = 0$, which factorises as $(u-5)(u-12) = 0$, so $u = 5$ or $u = 12$. Therefore, as $u = 2^x$, we have $x = \log_2 5$ or $x = \log_2 12$, and the largest possible value of x is $\log_2 12$, which is option F.

3.8 Answer C

TMUA 2016 · Paper 2 · Q7

The proof certainly begins with (1):

> Let $x = \log_a b$ and $y = \log_a c$.

We are then given a choice of writing d in various ways (line (2), (3) or (4)). We are given

$$\log_c d = (\log_a b)^2$$

so we can rewrite this using x as $\log_c d = x^2$. Therefore $d = c^{(x^2)}$, which is line (3).

Our next choice is between lines (5) and (7) (these are the only options, as the correct answer must be C or D); these are equivalent, but expressed differently. How can we get from $d = c^{(x^2)}$ to d being a power of a? We must rewrite c in terms of y: since $y = \log_a c$, it follows that $c = a^y$, so $d = (a^y)^{(x^2)}$, which is line (5).

Without doing any more work, the answer must therefore be option C, and the whole proof reads as follows:

> Suppose that $\log_c d = (\log_a b)^2$.
>
> (1) Let $x = \log_a b$ and $y = \log_a c$. Then (3) $d = c^{(x^2)}$, so (5) $d = (a^y)^{(x^2)}$, so (9) $d = a^{(x^2 y)}$, so (7) $d = (a^x)^{xy}$, and therefore (4) $d = b^{xy}$, hence (*) as required.

3.9 Answer E

TMUA 2017 · Paper 2 · Q7

The graph of $y = \mathrm{f}(x)$ looks like an exponential graph, so all of the four given statements certainly seem plausible.

One approach to this question is to think about the shapes of different exponential graphs. Another is to consider a specific value of x, and see what happens.

If we take $x = 1$, $a^x = a$, and $\mathrm{f}(1) > a$. We can now substitute $x = 1$ into the two expressions for $\mathrm{f}(x)$ offered: $\mathrm{f}(x) = b^x$ gives $\mathrm{f}(1) = b$, so we would need $b > a$. Similarly, $\mathrm{f}(x) = a^{kx}$ gives $\mathrm{f}(1) = a^k$. But it is clear from the solid line that $a > 1$ (as $a^0 = 1$ and $a^1 = a > 1$), so $a^k > a$ requires $k > 1$.

So both statements **1** and **3** could be true, but not **2** or **4**.

We can be even more careful and show that we cannot have just one of **1** and **3** being true. If statement **3** is true, we have $\mathrm{f}(x) = a^{kx} = (a^k)^x$, so writing $b = a^k > a$ shows that statement **1** is true; likewise, if statement **1** is true, we have $\mathrm{f}(x) = b^x = (a^{\log_a b})^x = a^{(\log_a b)x}$ and $\log_a b > 1$, so statement **3** is also true.

Hence the correct answer is E.

3.10 Answer F

TMUA Specimen · Paper 1 · Q8

The options all involve logarithm to base 10, so we start by taking logarithms of the given equation. This gives

$$\log_{10}(a^x b^{2x} c^{3x} = \log_{10} 2$$

so

$$\log_{10}(a^x) + \log_{10}(b^{2x}) + \log_{10}(c^{3x}) = \log_{10} 2.$$

We now use the index rule for logarithms to give

$$x \log_{10} a + 2x \log_{10} b + 3x \log_{10} c = \log_{10} 2$$

which factorises to

$$x(\log_{10} a + 2 \log_{10} b + 3 \log_{10} c) = \log_{10} 2$$

and hence

$$x = \frac{\log_{10} 2}{\log_{10} a + 2 \log_{10} b + 3 \log_{10} c}.$$

This, though, is not one of the offered options; we now have to apply the logarithm rules in reverse to give

$$x = \frac{\log_{10} 2}{\log_{10} a + \log_{10}(b^2) + \log_{10}(c^3)}$$

which simplifies further to

$$x = \frac{\log_{10} 2}{\log_{10}(ab^2c^3)}$$

and so the answer is F.

3.11 Answer E

TMUA Specimen · Paper 2 · Q10

We can start by finding the value of y when x is close to 1. When $x = 1.1$, say, $y = \log_{1.1} 2$, which is equivalent to $1.1^y = 2$. Now $1.1^2 = 1.21$, so $y > 2$ (it looks as though $y \approx 10$). The only graph with this value of y near $x = 1$ is E, so this must be the answer.

Alternatively, we could rearrange the equation to get $x^y = 2$. y must always be positive, otherwise x^y would be less than 1. As x gets large, y must get closer to 0. Putting these together again shows that only graph E works.

3.12 Answer E

TMUA Specimen · Paper 1 · Q11

Writing $y = 2^x$, the equation becomes $y^2 - 8y + 15 = 0$, which factorises to give $(y-3)(y-5) = 0$. Thus $y = 3$ or $y = 5$, so $2^x = 3$ or $2^x = 5$.

The given options all use logarithms to base 10, so we will take logarithms of these two equations to base 10. The first gives $x \log_{10} 2 = \log_{10} 3$, so $x = \dfrac{\log_{10} 3}{\log_{10} 2}$. Likewise, the second equation gives $x = \dfrac{\log_{10} 5}{\log_{10} 2}$. Finally, we can add these to obtain

$$\frac{\log_{10} 3}{\log_{10} 2} + \frac{\log_{10} 15}{\log_{10} 2} = \frac{\log_{10}(3 \times 5}{\log_{10} 2} = \frac{\log_{10} 15}{\log_{10} 2}$$

and the answer is option E.

An alternative is to take logarithms to base 2, giving the roots as $\log_2 3$ and $\log_2 5$. These sum to $\log_2 15$. Unfortunately, this is not one of the options offered, as they are all given using logarithms to base 10. If you know the change of base rule for logarithms, though, which states that $\log_a b = \dfrac{\log_c b}{\log_c a}$, you can put $c = 10$ to obtain the correct option.

3.13 Answer E

TMUA 2016 · Paper 1 · Q11

This is a quadratic in 2^{2x}, so we write $y = 2^{2x}$ and solve for y. The quadratic becomes

$$y^2 + 12 = 8y$$

as $4^{2x} = (2^2)^{2x} = 2^{4x} = (2^{2x})^2$ and $2^{2x+3} = 2^{2x} \times 2^3$.

This rearranges to $y^2 - 8y + 12 = 0$, which factorises as $(y-6)(y-2) = 0$, so $y = 6$ or $y = 2$. Thus $2^{2x} = 6$ or $2^{2x} = 2$, with the value of x being larger in the first than in the second, so $2^{2p} = 6$ and $2^{2q} = 2$. The options are given in terms of log to base 10, so we take $\log_{10}$ of the first equation. Thus $2p \log_{10} 2 = \log_{10} 6$ and $2q = 1$, and hence

$$p = \frac{\log_{10} 6}{2 \log_{10} 2} \quad \text{and} \quad q = \frac{1}{2}.$$

Therefore

$$\begin{aligned} p - q &= \frac{\log_{10} 6}{2 \log_{10} 2} - \frac{1}{2} \\ &= \frac{\log_{10} 6 - \log_{10} 2}{2 \log_{10} 2} \\ &= \frac{\log_{10} 3}{\log_{10} 4} \end{aligned}$$

so the answer is option E.

3.14 Answer H

TMUA 2019 · Paper 1 · Q11

We can write the first equation as $\log_3 x + 2\log_3 y = 1$. Then writing $X = \log_3 x$ and $Y = \log_3 y$, the two equations become

$$X + 2Y = 1$$
$$XY = -3$$

We are interested in the values of X, so we rearrange the second equation to get $Y = -3/X$, and substitute this into the first equation to give

$$X - \frac{6}{X} = 1.$$

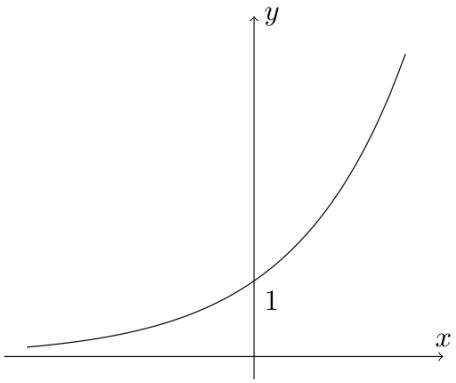
Multiplying this by X gives the quadratic $X^2 - 6 = X$, or $X^2 - X - 6 = 0$. This factorises to $(X-3)(X+2) = 0$, so $X = 3$ or $X = -2$, so $x = 3^X$ is either 27 or $\frac{1}{9}$, and their sum is $27\frac{1}{9}$, which is option H.

We could also have solved this problem by writing Y in terms of X using the first equation, and then substituting this into the second; it would have given the same result.

3.15 Answer A

TMUA 2018 · Paper 2 · Q11

We sketch the graph of $y = 2^x$. We know that $y = mx + c$ is a straight line with gradient m and a y-intercept of c, and we will imagine this line for different values of m and c. Solutions to the equation $2^x = mx + c$ correspond to the x-coordinates where these two graphs meet.



I Recall that 'P **only if** Q' is the same as saying '**if** P **then** Q', so in this case, the statement becomes '**if** the equation $2^x = mx + c$ has a negative real solution, **then** $c > 1$'. But we see that the line $y = -x$, with $c = 0$, has a solution with negative real x, so this statement is false.

II We can rewrite this statement as '**if** $c > 1$ **then** the equation has two distinct real solutions'. So we imagine straight lines passing through some point above $(0, 1)$, and we see that if this straight line has a negative gradient, then it can only intercept the curve once. So this statement is false.

III If $c \leqslant 1$, then a line with a negative gradient will still only intercept the curve once, so this is also false.

Therefore none of the statements is true, and the correct option is A.

3.16 Answer D

TMUA Specimen · Paper 1 · Q14

If the graph is a straight line, then we must have

$$(\log y) = m(\log x) + c$$

for some m and c; this is just the usual straight-line equation with x and y replaced by $\log x$ and $\log y$.

This can be written as

$$\log y = \log(x^m) + c.$$

If we now write $c = \log C$ for some C, then this becomes

$$\log y = \log x^m + \log C = \log(Cx^m).$$

Exponentiating both sides gives

$$y = Cx^m$$

which is in the form of option D.

An alternative to writing $c = \log C$ is just to exponentiate the equation $\log y = \log(x^m) + c$. If we suppose that the base of the logarithms is k, then this gives

$$y = x^m k^c$$

(using $k^{u+v} = k^u k^v$ and $k^{\log u} = u$ for any u and v). Since k^c is a constant, this equation can be rewritten as $y = ax^m$ where $a = k^c$, and we again obtain option D.

3.17 Answer F

TMUA 2017 · Paper 1 · Q14

The equations involve 2^x and $2^{2x} = (2^x)^2$, so we write $a = 2^x$; likewise we write $b = 2^y$, and so our equations become:

$$a + 3b = 3$$
$$a^2 - 9b^2 = 6$$

The first equation gives $a = 3 - 3b$, which we can substitute into the second equation to obtain

$$(3 - 3b)^2 - 9b^2 = 6$$

which expands to $9 - 18b = 6$, and hence $b = \frac{1}{6}$ and $a = \frac{5}{2}$

Another way to solve these simultaneous equations is to note that the second one can be written as $(a + 3b)(a - 3b) = 6$, using the identity for the difference of two squares. Since $a + 3b = 3$, this gives $a - 3b = 2$, from which we can again deduce a and b.

We can now find x and y:

$$x = \log_2 a = \log_2 \tfrac{5}{2} = p$$
$$y = \log_2 b = \log_2 \tfrac{1}{6} = q$$

and hence

$$p - q = \log_2 \frac{5}{2} - \log_2 \frac{1}{6} = \log_2 \left(\frac{5}{2} \div \frac{1}{6} \right) = \log_2 15$$

Thus the correct answer is F.

3.18 Answer B

TMUA 2018 · Paper 1 · Q14

There are a variety of different ways to approach this question. The following is probably the simplest.

We can substitute the coordinates of the points into the equation of the line to obtain

$$\log_2 p = 3m + 4$$
$$4 = m \log_2 p + 4$$

The second equation gives $m \log_2 p = 0$, so either $m = 0$ or $\log_2 p = 0$, so either $m = 0$ or $p = 1$.

If $m = 0$, then the first equation gives $\log_2 p = 4$, so $p = 16$

If $\log_2 p = 0$, so $p = 1$, then the first equation gives $3m + 4 = 0$, which has a solution.

So there are two possible values of p, namely 1 and 16. Hence the answer is B.

> ***Commentary:*** *You might find it helpful to write q (for example) in place of $\log_2 p$, so that the equations become*
>
> $$q = 3m + 4$$
> $$4 = mq + 4$$
>
> *This makes their structure much more obvious.*

3.19 Answer C

TMUA 2021 · Paper 2 · Q14

Writing $X = 2^x$ and $Y = \log_2 y$ gives the simultaneous equations

$$pX + Y = 2$$
$$X + Y = 1$$

Subtracting gives $(p-1)X = 1$, so as long as $p \neq 1$, we have $X = \frac{1}{p-1}$ and so $Y = \frac{p-2}{p-1}$.

Now the equation $\log_2 y = \frac{p-2}{p-1}$ gives a real value for y for any value of $p \neq 1$ (specifically $y = 2^{(p-2)/(p-1)}$), but the equation $2^x = \frac{1}{p-1}$ has a solution if and only if $\frac{1}{p-1} > 0$, which is if and only if $p > 1$.

Therefore the original simultaneous equations have a real solution (x, y) if and only if $p > 1$, which is option C.

3.20 Answer E

TMUA 2018 · Paper 1 · Q15

We note that $3 = \left(\sqrt{3}\right)^2$.

So if we write $y = \left(\sqrt{3}\right)^x$, the equation becomes

$$y^2 - \left(\sqrt{3}\right)^4 y + 20 = 0$$

or

$$y^2 - 9y + 20 = 0$$

This factorises to give $(y-4)(y-5) = 0$ so either $y = 4$ or $y = 5$

We can write $y = \left(3^{\frac{1}{2}}\right)^x = 3^{\frac{1}{2}x}$ so either $3^{\frac{1}{2}x} = 4$ or $3^{\frac{1}{2}x} = 5$

We can now take logs to base 3 to obtain

$$\tfrac{1}{2}x = \log_3 4 \quad \text{or} \quad \tfrac{1}{2}x = \log_3 5$$

so

$$x = 2\log_3 4 \quad \text{or} \quad x = 2\log_3 5$$

The question has asked for the sum of the real solutions of the equation. Both of these are real, so we can add them to get $2\log_3 4 + 2\log_3 5 = 2\log_3 20$ using the log rule $\log ab = \log a + \log b$

Hence the answer is E.

3.21 Answer A

TMUA 2019 · Paper 1 · Q15

We look to see how we can simplify this problem. First, the powers are 9^x and 3^x. We can write $9^x = (3^2)^x = 3^{2x} = (3^x)^2$, but it is not clear how this will help. But without a better idea, let us go with it, and write $u = 3^x$, so the question becomes

$$\frac{2^{(u^2)}}{8^u} = \frac{1}{4}.$$

Next, let us write everything as powers of 2, giving

$$\frac{2^{(u^2)}}{2^{3u}} = \frac{1}{2^2}.$$

Therefore

$$2^{u^2-3u} = 2^{-2}$$

so we require

$$u^2 - 3u = -2.$$

We can rearrange this quadratic to give $u^2 - 3u + 2 = 0$, so $(u-1)(u-2) = 0$ giving $u = 1$ or $u = 2$. Thus $3^x = 1$ or $3^x = 2$, giving two possibilities: $x = 0$ or $x = \log_3 2$.

The question explicitly says "find the real **non-zero** solution", so the correct answer is option A.

3.22 Answer C

TMUA 2020 · Paper 1 · Q15

We can start by rewriting $\log_2\left(\frac{1}{x}\right)$ as $-\log_2 x$, so the equation simplifies to

$$(\log_2 x)^4 + 12(-\log_2 x)^2 - 2^6 = 0,$$

which further simplifies to

$$(\log_2 x)^4 + 12(\log_2 x)^2 - 2^6 = 0.$$

Writing $y = (\log_2 x)^2$ gives the quadratic

$$y^2 + 12y - 64 = 0.$$

This factorises as

$$(y + 16)(y - 4) = 0,$$

so $y = (\log_2 x)^2 = 4$ (as this cannot be negative.

Thus $\log_2 x = 2$ or $\log_2 x = -2$, giving $x = 2^2 = 4$ or $x = 2^{-2} = \frac{1}{4}$. The positive difference between these is $4 - \frac{1}{4} = \frac{15}{4}$, which is option C.

3.23 Answer B

TMUA 2019 · Paper 2 · Q15

We expand the expressions using the logarithm rules, namely $\log(ab) = \log a + \log b$ and $\log a^n = n \log a$ when $a > 1$. This gives

$$\begin{aligned} \log a + 2\log b + \log c &= 7 \\ 2\log a + \log b + 2\log c &= 11 \\ 2\log a + 2\log b + 3\log c &= 15 \end{aligned}$$

This is a set of simultaneous equations! If we write $A = \log a$, $B = \log b$ and $C = \log c$, they become

$$A + 2B + C = 7 \quad (1)$$
$$2A + B + 2C = 11 \quad (2)$$
$$2A + 2B + 3C = 15 \quad (3)$$

Subtracting (3) − (2) and (3) − 2 × (1) to eliminate A gives

$$B + C = 4 \quad (4)$$
$$-2B + C = 1 \quad (5)$$

Subtracting these then gives

$$3B = 3$$

so $B = 1$ (and thus $C = 3$, $A = 2$).

Therefore $\log b = 1$, so the base of logarithms is b (and also $a = b^2$, $c = b^3$). The correct answer is option B.

3.24 Answer F

TMUA 2022 · Paper 2 · Q15

Since $\log_x y = z$, we have $y = x^z$. Likewise $z = y^x$. Combining these gives

$$z = y^x = (x^z)^x = x^{xz}.$$

Therefore raising the equation to the power of $\dfrac{1}{xz}$ gives $z^{1/xz} = x$, hence $\log_z x = \dfrac{1}{xz}$, which is option F.

The first equation involves the sum of two logarithms, so we can rewrite it as:

$$\log_{10}(2(y-1)) = \log_{10} x^2$$

and so $2(y-1) = x^2$.

The second equation becomes $y + 3 - 3x = 1$, as $\log_a 1 = 0$ for any base a.

We therefore have two simultaneous equations:

$$2y - 2 = x^2$$
$$y - 3x + 2 = 0.$$

We want to find the values of y, so we rewrite the second equation to eliminate x:

$$x = \frac{y+2}{3}$$

and so the first equation becomes

$$2y - 2 = \frac{(y+2)^2}{9}$$

or

$$18y - 18 = y^2 + 4y + 4$$

which rearranges to give the quadratic $y^2 - 14y + 22 = 0$. The solutions to this quadratic are

$$\frac{14 \pm \sqrt{14^2 - 4 \times 22}}{2} = 7 \pm \sqrt{7^2 - 22} = 7 \pm \sqrt{27}$$

and so the correct answer is C.

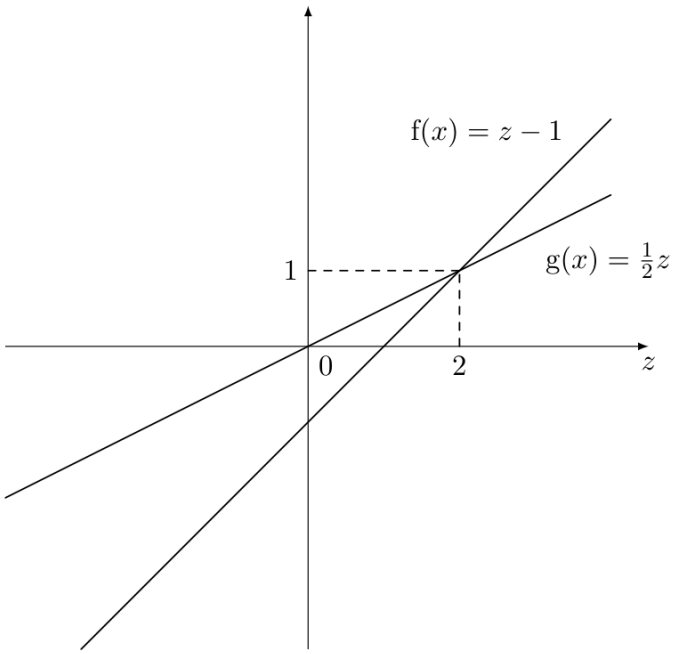
It would be good to check whether both of these solutions are valid (though the question does not require us to do so). To be valid, we require both $x > 0$ and $y > 1$. Since $\sqrt{27} < 6$, we see that $y > 1$. Also, as $x = \frac{1}{3}(y+2)$ and $y > 0$, it follows that $x > 0$. So there are, indeed, two valid solutions.

The expressions for $\mathrm{f}(x)$ and $\mathrm{g}(x)$ look quite complicated, so we start by simplifying them using the laws of logarithms:

$$\begin{aligned}\mathrm{f}(x) &= \log_2\left(\log_2 \sqrt{x}\,\right)\\ &= \log_2(\tfrac{1}{2}\log_2 x)\\ &= -1 + \log_2(\log_2 x)\\ \mathrm{g}(x) &= \log_2\left(\sqrt{\log_2 x}\,\right)\\ &= \tfrac{1}{2}\log_2(\log_2 x)\end{aligned}$$

We now see that both $\mathrm{f}(x)$ and $\mathrm{g}(x)$ can be written as a function of $\log_2(\log_2 x)$. As $x > 1$, $\log_2 x > 0$ and $\log_2(\log_2 x)$ can take any real value.

Let us write $z = \log_2(\log_2 x)$, so that we have $\mathrm{f}(x) = z - 1$ and $\mathrm{g}(x) = \frac{1}{2}z$. We can sketch a graph showing these two functions of z:



We can calculate the intersection of $y = z - 1$ and $y = \frac{1}{2}z$, obtaining $z = 2$, $y = 1$.

We can then see from the graph that $\mathrm{f}(x) \geq \mathrm{g}(x)$ when both are at least 1, and $\mathrm{f}(x) \leq \mathrm{g}(x)$ when both are at most 1, so either $\mathrm{f}(x) \geq \mathrm{g}(x) \geq 1$ or $\mathrm{f}(x) \leq \mathrm{g}(x) \leq 1$ (or both), which is option F.

(Note that the '**or**' in the options is inclusive 'or', so we are allowed to have both of the inequalities being true.)

We could first check to see whether the answer makes sense. If we substitute $n = 1$ into the original equation, we get $\frac{1}{4} < \left(\frac{1}{32}\right)^{10}$, which is clearly nonsense, so there has to be a mistake somewhere in the argument.

There are two straightforward approaches to finding the mistake: one is to check that the logic of each step works, the other is to try a value of n which either does or does not solve the original inequality, and check that each step works for this value of n. Since we know that the solution is wrong, it must fail somewhere, at least when $n = 1$.

Approach 1: checking the logic

Clearly step (V) is correct, so the mistake must be earlier.

Step (I) applies the same function to both sides of an inequality, so it looks fine.

Step (II) is just a simplification of the logarithm expressions using the power rule. The powers are positive integers, and both $\frac{1}{4}$ and $\frac{1}{32}$ are positive, so this step is fine.

Starting from the other end, step (IV) is just a calculation, and we can check quite easily that it is correct.

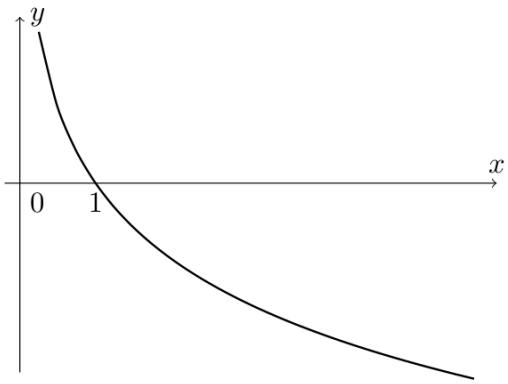
Step (III) looks like it is probably wrong, as we are performing a division with an inequality.

However, with more care, we can calculate $\log_{\frac{1}{2}}\left(\frac{1}{4}\right) = 2$ (as we need to do to check step (IV) anyway), so step (III) is just dividing by 2, and is actually fine.

Unfortunately, though, every step looks fine.

Working through them once more, steps (IV) and (V) are certainly fine, as is step (III). We have argued that step (II) is correct, so by elimination it must be step (I) which is wrong.

It is helpful to sketch a graph of $y = \log_{\frac{1}{2}} x$ to understand why step (I) is wrong. We have $\log_{\frac{1}{2}}(\frac{1}{2}) = 1$, $\log_{\frac{1}{2}} 1 = 0$, $\log_{\frac{1}{2}} 2 = -1$, so the graph looks like this:



So $\log_{\frac{1}{2}} x$ is a *decreasing* function: if $x < y$, then $\log_{\frac{1}{2}} x > \log_{\frac{1}{2}} y$; in other words, $\log_{\frac{1}{2}}$ reverses the direction of an inequality. Therefore step (I) is incorrect: the inequality should have been reversed in this step.

The correct answer is therefore B.

Approach 2: substituting a value of n

We have already shown that $n = 1$ does not satisfy the original inequality, yet it is given as a solution at the end of step (V). So we use this value of n to find at least one error in the argument.

Note that this approach is not guaranteed to work: there may be other errors in the argument that will escape detection by this method, but given that the options tell us that only one step is invalid, it will be sufficient for our purposes.

It will help us with our calculations if we write everything in terms of powers of $\frac{1}{2}$.

The original inequality can be written as

$$\left(\frac{1}{2}\right)^{2n} < \left(\frac{1}{2}\right)^{50}$$

Substituting $n = 1$ gives

$$\left(\frac{1}{2}\right)^{2} < \left(\frac{1}{2}\right)^{50}$$

which is *not* true.

After step (I), the inequality reads

$$\log_{\frac{1}{2}}\left(\frac{1}{2}\right)^{2n} < \log_{\frac{1}{2}}\left(\frac{1}{2}\right)^{50}$$

Substituting $n = 1$ gives

$$\log_{\frac{1}{2}}\left(\frac{1}{2}\right)^{2} < \log_{\frac{1}{2}}\left(\frac{1}{2}\right)^{50}$$

which we can simplify to

$$2 < 50$$

which *is* true.

So something has gone wrong in step (I), and this is where the error lies.

> ***Commentary:*** *There is a slightly subtle question about whether the student intended each of the steps in this argument to be reversible. For example, if a student started with the statement $x + 5 = 8$ and deduced $x > 0$, they have made a correct deduction (though not 'solved the equation'), but the argument is not reversible: $x > 0$ does not imply that $x + 5 = 8$. So we will assume that in this case they intend to say that $1 \leqslant n \leqslant 24$ are* all *solutions to the original equation.*
>
> *It follows that step (I) is where the mistake lies, as $n = 1$ satisfies the inequality after step (I) but not the inequality before it.*
>
> *If we did not make this assumption about the student's intentions, we could repeat the argument above with a value of n which* does *satisfy the original inequality, such as $n = 100$; then the first inequality is true but the second inequality, obtained after step (I), is false, so the proposed implication is false.*

There are different ways of approaching this question. Given that the question asks which one of the statements must be true, it is probably simplest to try to find values which "break" each of the inequalities. For each one, we can either multiply through by the denominators first, or leave the fractions as fractions.

A We can write this as

$$2^{z-x+1} > 2^{x-y} + 2^{z-y}$$

To make this false, we want the left side small and right side large. But $x > y > z$, so the left hand side is less than 2. If we then make $x - y > 1$, the right hand side will be at least 2, and so this statement is not necessarily true. For example, $x = 10$, $y = 5$, $z = 3$ will make the inequality false.

B This becomes $2 > 3^{x-y} + 3^{z-y}$. If we take $x = 10$ and $y = 2$, then the first term on the right side is 3^8 so this inequality is not always true.

C Multiplying by $5^z 5^y$ gives

$$2 \times 5^{x+y} > 5^{x+z} + 5^{2z}$$

Now $5^{x+y} > 5^{x+z}$ and $5^{x+y} > 5^{2z}$ so this must be true.

D This is

$$2 < 7^{x-y} + 7^{z-y}$$

If x and y are very close, then 7^{x-y} is just greater than 1, and then if z is much smaller than y, 7^{z-y} is close to 0, so the sum will be just greater than 1. So for example, $x = 1000$, $y = 999.99$, $z = 1.1$ will do to show that this statement is not always true.

Therefore the correct option is C.

We can combine the logarithms on the right and rearrange the left hand side to give

$$\log_{10}(x-y)^2 = \log_{10}\big((2-2x)(y+5)\big)$$

and so

$$(x-y)^2 = (2-2x)(y+5).$$

Expanding the brackets gives

$$x^2 - 2xy + y^2 = 2y + 10 - 2xy - 10x$$

which rearranges to

$$x^2 + y^2 + 10x - 2y = 10.$$

Completing the squares gives

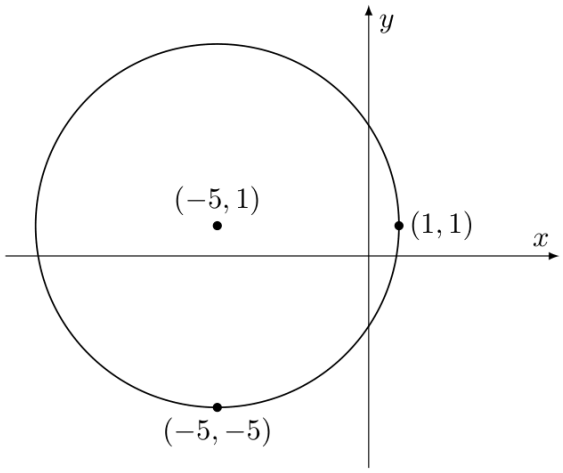
$$(x+5)^2 + (y-1)^2 = 36$$

which is a circle with centre $(-5, 1)$ and radius 6.

The obvious answer is therefore 12π, but we must take care: does every point on the circle satisfy the original equation? The manipulations we have performed make assumptions; in particular, for the logarithms to be defined, we require $x - y > 0$, $2 - 2x > 0$ and $y + 5 > 0$, which we can rearrange to

$$x > y, \qquad x < 1 \qquad \text{and} \qquad y > -5$$

Let us sketch the circle and these three inequalities. Here is the circle with two points on the circumference marked:



The restriction $x < 1$ holds for all of the points on the circle except for the single point $(1, 1)$, and the restriction $y > -5$ holds everywhere except for $(-5, -5)$. The restriction $x > y$ is more significant for us. We can draw the line $x = y$ on our sketch:

Trigonometry 1

4.1 Answer C

TMUA 2022 · Paper 1 · Q1

This equation is a quadratic in $\cos^2\theta$, and it factorises as

$$(2\cos^2\theta - 3)(\cos^2\theta - 1) = 0.$$

Therefore $2\cos^4\theta - 5\cos^2\theta + 3 = 0$ if and only if $\cos^2\theta = \frac{3}{2}$ or $\cos^2\theta = 1$. The first equation has no real solutions, so the original equation holds if and only if $\cos^2\theta = 1$, or $\cos\theta = \pm 1$.

$\cos\theta = 1$ has the solutions $\theta = 0$ and $\theta = 2\pi$ in the interval $0 \leq \theta \leq 2\pi$, and $\cos\theta = -1$ has the solution $\theta = \pi$ in this interval.

Therefore the original equation has three real solutions in the interval, and the correct option is C.

4.2 Answer D

TMUA Specimen · Paper 1 · Q2

We rewrite $\sin^2\theta$ as $1 - \cos^2\theta$ to give

$$1 - \cos^2\theta + 3\cos\theta = 3.$$

This is now a quadratic in $u = \cos\theta$, giving

$$1 - u^2 + 3u = 3$$

or $u^2 - 3u + 2 = 0$. This factorises as $(u-1)(u-2) = 0$, so $u = 1$ or $u = 2$, that is, $\cos\theta = 1$ or $\cos\theta = 2$.

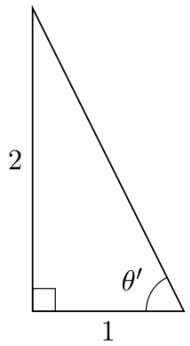
$\cos\theta = 2$ has no real solutions. The solutions of $\cos\theta = 1$ in the given range are $\theta = 0$, $\theta = 2\pi$, $\theta = 4\pi$. So there are three solutions, and the answer is option D.

4.3 Answer F

TMUA 2020 · Paper 2 · Q2

In the range $180° < \theta < 360°$, we have that $\tan\theta > 0$ when $180° < \theta < 270°$. In this smaller range, $\cos\theta < 0$.

Next, in the range $0° < \theta' < 90°$, if $\tan\theta' = 2$, we can sketch a right-angled triangle with this angle:



The hypotenuse has length $\sqrt{2^2 + 1^2} = \sqrt{5}$, so $\cos\theta' = \dfrac{1}{\sqrt{5}} = \dfrac{\sqrt{5}}{5}$.

Therefore for our original angle θ, we have $\cos\theta = -\dfrac{\sqrt{5}}{5}$, which is option F.

4.4 Answer D

TMUA 2016 · Paper 2 · Q3

We use the identity $\sin^2 x + \cos^2 x = 1$ to write the equation in terms of $\sin x$:

$$8\sin^2 x + 4\cos^2 x = 4\sin^2 x + 4 = 7$$

so $4\sin^2 x = 3$, giving $\sin^2 x = \frac{3}{4}$ so $\sin x = \pm\frac{\sqrt{3}}{2}$.

The solutions to $\sin x = \frac{\sqrt{3}}{2}$ lie in the interval $0 < x < \pi$, while the solutions to $\sin x = -\frac{\sqrt{3}}{2}$ lie in the interval $\pi < x < 2\pi$, and thus the largest angle in the range $0 < x < 2\pi$ which satisfies the original equation is the largest solution to $\sin x = -\frac{\sqrt{3}}{2}$. This is $2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$, which is option D.

4.5 Answer B

TMUA 2017 · Paper 2 · Q4

Let's run through every step of the argument with care.

The first step is '$\tan x = \sqrt{3}$ so $x = 60°$'. We should first check that $x = 60°$ is a solution: using our knowledge of trigonometry and special angles, we know that $\tan 60° = \sqrt{3}$, so it is.

But we also know that tan is periodic with period 180°: there are multiple values of x which have $\tan x = \sqrt{3}$, and all solutions to $\tan x = \sqrt{3}$ are therefore of the form $x = 60° + 180n°$ for $n \in \mathbb{Z}$.

So there is unquestionably an error in the student's answer, but we do not yet know whether there is more than one possible value of $\sin 2x$.

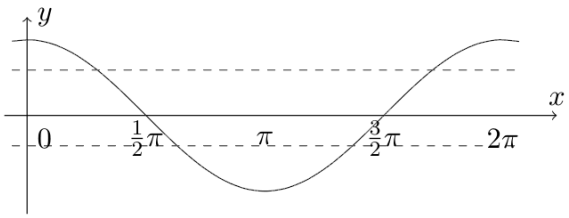
It follows from $x = 60° + 180n°$ that $2x = 120° + 360n°$

It is certainly the case that $\sin 120° = \frac{\sqrt{3}}{2}$. But since sin is periodic with period 360°, we see that $\sin(120° + 360n°) = \frac{\sqrt{3}}{2}$ for every $n \in \mathbb{Z}$.

So there is only one possible value of $\sin 2x$, but the student should have considered other possible values of x for which $\tan x = \sqrt{3}$; this is option B.

4.6 Answer D

It is worth sketching a graph of $y = \cos x$ to understand the meaning of the stated condition. Two lines have been drawn corresponding to different values of c.

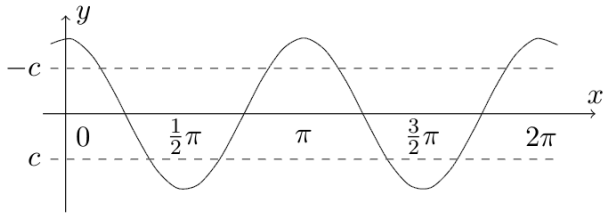


For there to be two solutions for $0 < x < \frac{3}{2}\pi$, we see from the sketch that we need $-1 < c < 0$

We now consider the equation $\cos^2 2x = c^2$, so $\cos 2x = \pm c$

There are two straightforward approaches. The first is to sketch the graph of $y = \cos 2x$ and look at $0 < x < \frac{3}{2}\pi$; the second is to note that the range can be written as $0 < 2x < 3\pi$, so we could write $u = 2x$ and look for solutions to $\cos u = \pm c$ in the range $0 < u < 3\pi$. We will use the first approach here; the second is very similar.

Here is a sketch of $y = \cos 2x$:



We immediately see that, for $0 < x < \frac{3}{2}\pi$, there are six solutions to our equation, so the correct answer is D.

4.7 Answer B

Line A is a known identity, so this is fine.

Line B rearranges the identity to get $\cos^2 x = 1 - \sin^2 x$ and then takes the square root of both sides. However, when we take the square root, we always get the positive (or zero) root, but $\cos x$ is negative for some values of x. So this is where the error is introduced.

Another way of seeing this is to start with the nonsensical $0 = 4$ of line E, and substitute $x = \pi$ in each line to find the first one which fails; it is, indeed, line B, as this gives $-1 = 1$.

4.8 Answer D

TMUA 2021 · Paper 1 · Q6

This is almost a function in $\cos x$ except for the $\sin^2 x$ term, so we rewrite this in terms of $\cos x$:

$$\mathrm{f}(x) = \frac{\cos x + 3}{7 + 5\cos x - (1 - \cos^2 x)} = \frac{\cos x + 3}{6 + 5\cos x + \cos^2 x}$$

We can then factorise the denominator to give

$$\mathrm{f}(x) = \frac{\cos x + 3}{(\cos x + 2)(\cos x + 3)}$$

and the $\cos x + 3$ factor cancels. (We should also check that this can never be zero; if it were, the function would be undefined at some points.) Therefore we can write

$$\mathrm{f}(x) = \frac{1}{\cos x + 2}$$

The maximum value of this function occurs where $\cos x = -1$, giving $\mathrm{f}(x) = 1$, and the minimum value occurs where $\cos x = +1$, giving $\mathrm{f}(x) = \frac{1}{3}$. The difference between the minimum and maximum values is therefore $1 - \frac{1}{3} = \frac{2}{3}$, which is option D.

4.9 Answer D

TMUA 2019 · Paper 2 · Q6

The following is just one approach to checking an argument.

We start by checking the final solutions on line (VI). $x = 0$ makes the LHS (left hand side) equal 1 but the RHS (right hand side) equal -1, so the attempt cannot be correct. Furthermore, this is not a value of x for which $\tan x$ is undefined, so G is not the correct answer. We must then identify where the error arose.

We can now test the incorrect solution $x = 0$ at various stages of the argument to see where things first go wrong. Rather than working line-by-line, we start about half way through to see whether the mistake happens in the first half of the argument or the second half; this is a little more efficient.

Let us try the incorrect solution $x = 0$ in line (III) (which is midway between (I) and (V)): this gives $-1 = -1$ or $-1 = -1$, so $x = 0$ is valid at this point. This tells us that the error must occur on line (I), (II) or (III).

On line (I), $x = 0$ gives $1 = -1$, while on line (II), $x = 0$ gives $(-1) \times (-1) = -1$. So it is most likely that the first error occurs on line (III).

We check the algebra on line (I): all that has happened is that $2\sin x$ has been subtracted from both sides, so this is correct.

We likewise check the algebra on line (II): by multiplying out the brackets, we get the equation on line (I), so this is correct. (We might be concerned about the case $\cos x = 0$ here, but this is not possible on either line (I) or line (II) because of the $\tan x$ terms on each line.)

Finally, we are left with line (III). The implication here is that because two numbers multiply to give -1, one of them must be -1. This would be true if we were dealing with integers, but we are not, so this is an invalid implication.

The correct solution is therefore D: the first error occurs on line (III).

4.10 Answer F

TMUA 2016 · Paper 1 · Q8

We note that the only angle involved is $2x$, and $0° \leqslant x \leqslant 360°$ gives $0° \leqslant 2x \leqslant 720°$.

We start by writing everything in terms of $\sin 2x$, giving:

$$(1 - \sin^2 2x) + \sqrt{3}\sin 2x - \tfrac{7}{4} = 0$$

which rearranges to give

$$\sin^2 2x - \sqrt{3}\sin 2x + \tfrac{3}{4} = 0.$$

We can apply the quadratic formula to this to obtain

$$\sin 2x = \frac{\sqrt{3} \pm \sqrt{3 - 3}}{2} = \frac{\sqrt{3}}{2}$$

and hence the possible values of $2x$ in the range are $2x = 60°$, $120°$, $420°$ and $480°$. Thus the largest possible value of x in the range $0° \leqslant x \leqslant 360°$ is $240°$, and the answer is F.

4.11 Answer A

TMUA 2017 · Paper 1 · Q8

Multiplying out the brackets does not look like a particularly useful approach (what would we do with the terms $\cos x$ and $\sin x \cos x$ in the same inequality?), so we will leave the expression in its factored form.

If we temporarily write $a = 1 - 2\sin x$ and $b = \cos x$ we will be able to see the structure of the problem a little more clearly. We want $ab \geqslant 0$, and this will happen when either $a \geqslant 0$ and $b \geqslant 0$, or when $a \leqslant 0$ and $b \leqslant 0$. So we need to solve each of these simpler inequalities.

We have $1 - 2\sin x = 0$ when $\sin x = \frac{1}{2}$, so when $x = \frac{\pi}{6}$ or $x = \frac{5\pi}{6}$. We can either sketch the graph of $y = 1 - 2\sin x$ or try values to determine whether $1 - 2\sin x$ is positive or negative in each range. Sketching the graph seems possibly hard, as there are multiple transformations, and it would be easy to make a mistake. So on this occasion, we'll plug in values: $x = 0$ and $x = \pi$ both give $1 - 2\sin x = 1$; while $x = \frac{\pi}{2}$ gives $1 - 2\sin x = -1$. Therefore

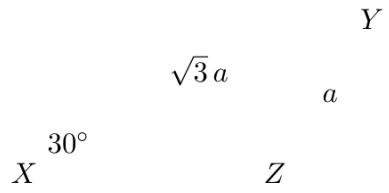
$$1 - 2\sin x \geqslant 0 \text{ when } 0 \leqslant x \leqslant \tfrac{\pi}{6} \text{ and when } \tfrac{5\pi}{6} \leqslant x \leqslant \pi$$
$$1 - 2\sin x \leqslant 0 \text{ when } \tfrac{\pi}{6} \leqslant x \leqslant \tfrac{5\pi}{6}$$

We can do similarly for $\cos x$; this time we can think about the graph in our head, and we have

$$\cos x \geqslant 0 \text{ when } 0 \leqslant x \leqslant \tfrac{\pi}{2}$$
$$\cos x \leqslant 0 \text{ when } \tfrac{\pi}{2} \leqslant x \leqslant \pi$$

We can now put everything together. Both expressions are greater than or equal to 0 when $0 \leqslant x \leqslant \frac{\pi}{6}$, and they are both less than or equal to 0 when $\frac{\pi}{2} \leqslant x \leqslant \frac{5\pi}{6}$.
Hence the answer is A.

Let us sketch a fun triangle.



The area of XYZ is given by the '$\frac{1}{2}ab\sin C$' formula: it is $\frac{1}{2}XY.XZ\sin 30° = \frac{1}{4}\sqrt{3}\,a.XZ$. We must therefore find the possible values of XZ.

We can use the cosine rule to find XZ: we have

$$YZ^2 = XY^2 + XZ^2 - 2XY.XZ\cos 30°.$$

Filling in the known values, we obtain

$$a^2 = 3a^2 + XZ^2 - 3a.XZ$$

so XZ satisfies the quadratic equation

$$XZ^2 - 3a.XZ + 2a^2 = 0.$$

This factorises to give

$$(XZ - 2a)(XZ - a) = 0$$

so either $XZ = a$ or $XZ = 2a$.

It is useful to think about what the two resulting triangles look like and whether these values of XZ make sense.

We are asked to find the ratio of the areas of these possible fun triangles. Since the area is given by $\frac{1}{4}\sqrt{3}\,a.XZ$, the ratio of areas is given by the ratio of possible XZ values. Since triangle S has the larger area, it must have the larger of the XZ values, so the ratio is $2a : a$, which equals $2 : 1$. The correct answer is thus option B.

> ***Commentary:*** *There are lots of appearances of XY, YZ and XZ here, following the language used in the question. It might have been simpler to use the standard side-length naming convention, letting x be the length of the side YZ opposite X, letting y be the length of the side XZ opposite Y and z the length of XY. Then we would have the area of XYZ being $\frac{1}{2}yz\sin 30° = \frac{1}{4}\sqrt{3}\,ay$, the cosine rule would read*
>
> $$x^2 = y^2 + z^2 - 2yz\cos 30°,$$
>
> *and we would finally deduce that $y = a$ or $y = 2a$.*

4.13 Answer E

TMUA 2017 · Paper 2 · Q8

We can start by estimating or calculating the values.

A $2 < \log_2 7 < 3$, since $\log_2 4 = 2$ and $\log_2 8 = 3$

B $(2^{-3} + 2^{-2})^{-1} = (\frac{1}{8} + \frac{1}{4})^{-1} = (\frac{3}{8})^{-1} = \frac{8}{3}$ which again lies between 2 and 3

C $\pi/3$ is just a little greater than 1, so $2^{\pi/3}$ is a little greater than 2

D We expand $(\sqrt{2} - 1)^3$ using the binomial theorem to get

$$(\sqrt{2})^3 - 3(\sqrt{2})^2 + 3\sqrt{2} - 1 = 2\sqrt{2} - 6 + 3\sqrt{2} - 1 = 5\sqrt{2} - 7$$

Thus

$$\frac{1}{4(\sqrt{2}-1)^3} = \frac{1}{4(5\sqrt{2}-7)} = \frac{5\sqrt{2}+7}{4((5\sqrt{2})^2 - 7^2)} = \frac{5\sqrt{2}+7}{4}$$

This is greater than $\frac{5+7}{4} = 3$, so is certainly not the smallest in value.

E $\sin(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}$, so $4\sin^2(\frac{\pi}{4}) = 4\left(\frac{1}{\sqrt{2}}\right)^2 = 4 \times \frac{1}{2} = 2$

Since A, B and C are each greater than 2, the correct answer is E.

4.14 Answer E

TMUA 2022 · Paper 1 · Q9

It may well require reading this question a couple of times to understand what it is asking! It seems like we need to find all possible functions $\mathrm{f}(x)$ *and* $\mathrm{g}(x)$*, then for each one, to find the minimum value that* $\mathrm{f}(x)$ *takes, and then to find the minimum of all the minima. There might, of course, be some shortcuts!*

Let us try to solve the equations to find $\mathrm{f}(x)$. The first equation rearranges to give $\mathrm{g}(x) = \mathrm{f}(x) - 2\sin x$, so we can substitute this into the second equation to give

$$\mathrm{f}(x)(\mathrm{f}(x) - 2\sin x) = \cos^2 x.$$

This is a quadratic in $\mathrm{f}(x)$, so we can rearrange it into the standard form and use the quadratic formula:

$$(\mathrm{f}(x))^2 - (2\sin x)\mathrm{f}(x) - \cos^2 x = 0,$$

hence

$$\begin{aligned}\mathrm{f}(x) &= \frac{2\sin x \pm \sqrt{4\sin^2 x + 4\cos^2 x}}{2} \\ &= \sin x \pm 1.\end{aligned}$$

So there are two possible functions $\mathrm{f}(x)$, namely $\mathrm{f}(x) = \sin x + 1$ and $\mathrm{f}(x) = \sin x - 1$. The minimum value taken by these for any x is 0 for the first function and -2 for the second, so the minimum possible value is -2, which is option E.

4.15 Answer E

TMUA 2023 · Paper 1 · Q9

We have $(1 + 3\cos 3\theta)^2 = 4$ if and only if $1 + 3\cos 3\theta = \pm 2$. We consider each possibility separately.

We have

$$\begin{aligned} & 1 + 3\cos 3\theta = 2 \\ \text{if and only if} \quad & 3\cos 3\theta = 1 \\ \text{if and only if} \quad & \cos 3\theta = \tfrac{1}{3}. \end{aligned}$$

Since $0° \leq \theta \leq 180°$, we have $0° \leq 3\theta \leq 540°$, and there are 3 values of 3θ which have $\cos 3\theta = \frac{1}{3}$ in this interval. (One is between $0°$ and $90°$, one is between $270°$ and $360°$, and one is between $360°$ and $450°$, by considering the graph of $y = \cos x$.)

Now considering the other possibility, we have

$$\begin{aligned} & 1 + 3\cos 3\theta = -2 \\ \text{if and only if} \quad & 3\cos 3\theta = -3 \\ \text{if and only if} \quad & \cos 3\theta = -1. \end{aligned}$$

Again, $0° \leq 3\theta \leq 540°$, but this time there are only 2 values of 3θ which satisfy the equation: $3\theta = 180°$ and $3\theta = 540°$.

Neither of these values of 3θ overlap with the values of 3θ found earlier, so in total there are $3 + 2 = 5$ values of 3θ in the interval $0° \leq 3\theta \leq 540°$, and hence 5 solutions to the original equation in the given interval. The correct answer is option E.

4.16 Answer D

TMUA Specimen · Paper 2 · Q11

We work out or approximate all five numbers.

A $\tan\left(\frac{3\pi}{4}\right) = -1$

B $\log_{10} 100 = 2$

C $\sin^{10}\left(\frac{\pi}{2}\right) = 1^{10} = 1$

D $\log_2 10 = 3.\cdots$, or more formally, $3 < \log_2 10 < 4$, as $2^3 = 8$ and $2^4 = 16$

E $0 < \sqrt{2} - 1 < 1$, so $0 < (\sqrt{2} - 1)^{10} < 1$

Only one of these numbers is greater than 3, so the answer is D.

Let us start by multiplying both sides by $4^{\sin^2 x}$ to remove the fraction:

$$2^{\tan^2 x} = 4^{\sin^2 x}.$$

We note that $4^a = (2^2)^a = 2^{2a}$, so we can rewrite this equation as

$$2^{\tan^2 x} = 2^{2\sin^2 x},$$

and therefore the equation can be simplified to

$$\tan^2 x = 2\sin^2 x.$$

Next, we know that $\tan x = \frac{\sin x}{\cos x}$, so we can rewrite this as

$$\frac{\sin^2 x}{\cos^2 x} = 2\sin^2 x.$$

Multiplying by $\cos^2 x$ to eliminate fractions once more gives

$$\sin^2 x = 2\sin^2 x \cos^2 x,$$

which we can rearrange and factorise to give

$$\sin^2 x(1 - 2\cos^2 x) = 0.$$

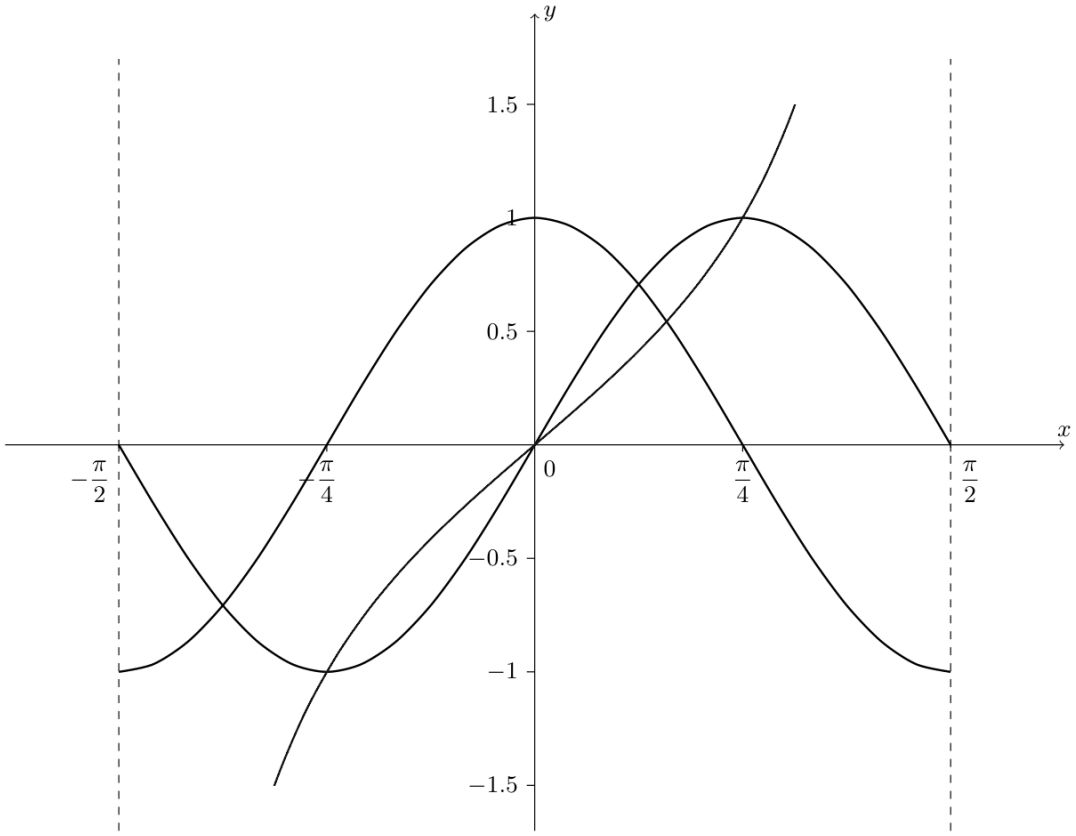
Therefore either $\sin x = 0$ or $\cos^2 x = \frac{1}{2}$. The first has solutions $x = 0$, π, 2π. The second gives $\cos x = \pm\frac{1}{\sqrt{2}}$, so has 4 solutions in the range $0 \leq x \leq 2\pi$, none of which are 0, π or 2π. (We could list them explicitly: they are $\frac{\pi}{4}$, $\frac{3\pi}{4}$, $\frac{5\pi}{4}$, $\frac{7\pi}{4}$.)

In total, there are 7 solutions, which is option F.

We clearly need to sketch the graph of $y = \tan x$ onto the given sketch.

This graph has asymptotes at $x = \pm\frac{\pi}{2}$, and passes through the points $(\frac{\pi}{4}, 1)$ and $(-\frac{\pi}{4}, -1)$. Since the function is increasing as it passes through these points, it must lie below the graph of $y = \sin 2x$ between 0 and $\frac{\pi}{4}$, and above it between $-\frac{\pi}{4}$ and 0.

This results in the following graph:



We now pick various values for x and see which order the functions lie in. For example, when x is close to $-\frac{\pi}{2}$, we have $\tan x < \cos 2x < \sin 2x$. Here are some cases as we move from $x = -\frac{\pi}{2}$ to $x = \frac{\pi}{2}$, each together with the option it eliminates:

x value	Function order	Option
x close to $-\frac{\pi}{2}$	$\tan x < \cos 2x < \sin 2x$	F
x just less than $-\frac{\pi}{4}$	$\tan x < \sin 2x < \cos 2x$	E
x just more than $-\frac{\pi}{4}$	$\sin 2x < \tan x < \cos 2x$	D
x just more than 0	$\tan x < \sin 2x < \cos 2x$	E
x just more than $\frac{\pi}{8}$	$\tan x < \cos 2x < \sin 2x$	F
x just less than $\frac{\pi}{4}$	$\cos 2x < \tan x < \sin 2x$	B
x just more than $\frac{\pi}{4}$	$\cos 2x < \sin 2x < \tan x$	A

The only option which is not eliminated is C: $\sin 2x < \cos 2x < \tan x$ for some real number x with $-\frac{\pi}{2} < x < \frac{\pi}{2}$

It is also clear from the graph that this is impossible: the only time that $\tan x$ is greater than

4.19 Answer C

TMUA 2023 · Paper 2 · Q12

We can rewrite the two statements in if/then language as

I if $n = 3$, then $p > 1$

II if $n = 7$, then $-1 < p < 1$

We will determine whether these statements are true or false using this if/then formulation.

The equation can be rearranged as $\sin x(\cos^2 x - p^2) = 0$, so this is true if and only if $\sin x = 0$ or $\cos x = \pm p$. The first of these has solutions $x = 0$, π, 2π, while the solutions to the second depend on the value of p:

- If $p = 0$, then it has solutions $x = \frac{\pi}{2}$, $\frac{3\pi}{2}$.
- If $-1 < p < 0$ or $0 < p < 1$, then it has four solutions, one with $0 < x < \frac{\pi}{2}$, one with $\frac{\pi}{2} < x < \pi$, one with $\pi < x < \frac{3\pi}{2}$ and one with $\frac{3\pi}{2} < x < 2\pi$; these are all different from the solutions to $\sin x = 0$.
- If $p = \pm 1$, then it has three solutions, $x = 0$, π, 2π, which are exactly the same as the solutions of $\sin x = 0$.
- If $p < -1$ or $p > 1$, then it has no solutions.

Therefore the original equation has:

- $n = 5$ distinct solutions if (and only if) $p = 0$
- $n = 7$ distinct solutions if (and only if) $-1 < p < 0$ or $0 < p < 1$
- $n = 3$ distinct solutions if (and only if) $p = \pm 1$ or $p < -1$ or $p > 1$

We can now return to the statements I and II in our rewritten form:

I This is false: we could have $p = 1$ or $p \leq -1$

II This is true, even though $p = 0$ is included here: if $-1 < p < 0$ or $0 < p < 1$, then certainly $-1 < p < 1$ is true

The correct option is therefore C, II only.

4.20 Answer B

TMUA 2019 · Paper 1 · Q13

We will not use calculus for this question, as the specification does not expect knowledge of differentiating exponentials or trigonometric functions. (And it also turns out that using calculus would make the solution far more complicated.)

Instead, we note that this is a quadratic expression in $u = 2^{\sin x}$, and we can write the function as

$$u^2 - 4u + \tfrac{17}{4}.$$

We also note that as $-1 \leq \sin x \leq 1$, $\frac{1}{2} \leq u \leq 2$ for all real x (and u can take every value in this interval).

Now we can complete the square on the quadratic to understand its behaviour; we get

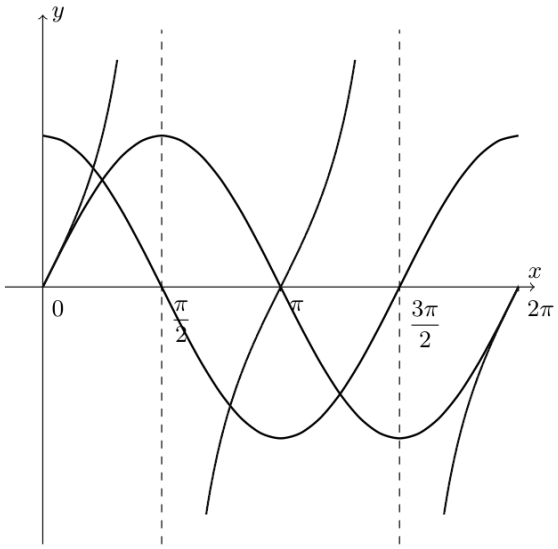
$$(u - 2)^2 - 2^2 + \tfrac{17}{4} = (u - 2)^2 + \tfrac{1}{4}$$

Thus the minimum value of the function occurs when $u = 2$, and for u in the interval $\frac{1}{2} \leq u \leq 2$, the maximum value occurs when $u = \frac{1}{2}$. In this case, we get $(-\frac{3}{2})^2 + \frac{1}{4} = \frac{10}{4} = \frac{5}{2}$, so the correct answer is B.

Commentary: *One thing we can do when faced with a question like this it to first try different values of θ to see whether the conjecture appears to be true or false. If it is false, as shown by some counterexample, then substituting this value of θ into every line might indicate where the error arises. If, on the other hand, we cannot find a counterexample, then the conjecture and argument may well be true.*

In this case, given that we are dealing with trigonometric functions, we would do well to think about their graphs and try values of θ in different quadrants.

If we sketch (or think about) the graphs of sin, cos and tan, we have the following:



If the conjecture is false, we need a value of θ where $\tan\theta > 0$ but $\sin\theta + \cos\theta < 1$

We see that $\tan\theta > 0$ for $0 < \theta < \frac{\pi}{2}$ but there $\sin\theta + \cos\theta > 1$ looks like it might be true. We also have $\tan\theta > 0$ when $\pi < \theta < \frac{3\pi}{2}$ and there $\sin\theta$ and $\cos\theta$ are both negative, so $\sin\theta + \cos\theta > 1$ cannot be true. So if we take $\theta = \frac{5\pi}{4}$ we may find the source of the error.

On line (I), we get $\sin\theta\cos\theta = (-\frac{1}{\sqrt{2}})(-\frac{1}{\sqrt{2}} = \frac{1}{2}$ and $\tan\theta\cos^2\theta = 1 \cdot (-\frac{1}{\sqrt{2}})^2 = \frac{1}{2}$ so this line looks reasonable.

On line (II), we get $1 + 2 \cdot \frac{1}{2} = 1 + 1 > 1$ which is correct.

On line (III), we get $\frac{1}{2} + 2 \cdot \frac{1}{2} + \frac{1}{2} = 2 > 1$ so this is correct.

On line (IV), we get $(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}})^2 = 2 > 1$ so this line is correct for this value of θ.

On line (V), we have $-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\sqrt{2}$ and $-\sqrt{2} > 1$ is false. So it appears that the first error is likely to occur here: indeed, we have taken the square root of $x^2 > 1$ and obtained $x > 1$, which is not necessarily the case, as we could have $x < -1$

We should now go back and check that the earlier steps are actually valid, as there could have been an earlier error. But with some thinking, we can see that all of them are correct:

(II) This is obtained by doubling the previous inequality and then adding 1.

(III) Here, the identity $\sin^2\theta + \cos^2\theta = 1$ has been used to rewrite the left hand side.

(IV) This factorisation is correct.

(V) As we saw, an error occurs here.

Hence the answer is F.

4.22 Answer B

We write $S = \sin 2x$ and $C = \cos 2x$, so the simultaneous equations become

$$S + \sqrt{3}C = -1$$
$$\sqrt{3}S - \quad C = \sqrt{3}$$

Multiplying the second equation by $\sqrt{3}$ and then adding to the first equation gives

$$4S = 2$$

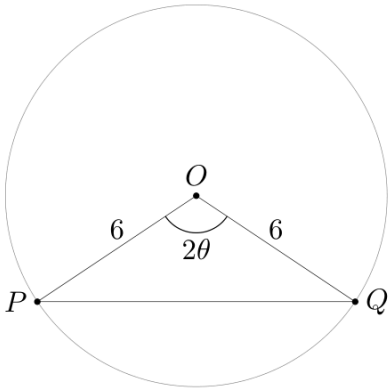
so $S = \frac{1}{2}$ and hence $C = -\frac{\sqrt{3}}{2}$.

The solutions to $\sin 2x = \frac{1}{2}$ for $0° \leq x \leq 360°$, or $0° \leq 2x \leq 720°$, are given by $2x = 30°$, $150°$, $390°$ and $510°$.

The solutions to $\cos 2x = -\frac{\sqrt{3}}{2}$ in the same range are given by $2x = 150°$, $210°$, $510°$ and $570°$, so the values of $2x$ for which both are true are $150°$ and $510°$.

Thus the solutions for x in this range are $75°$ and $255°$, and their sum is $330°$. Hence the correct answer is option B.

Let us start by drawing the triangle POQ in the circle:



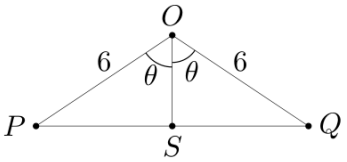
(We have called the angle 2θ as we will shortly want to consider half of the angle.)

We are told that the area of POQ is $9\sqrt{3}$, so we can use the formula for the area of a triangle given two sides and the included angle to determine 2θ: the area formula is $A = \frac{1}{2}ab\sin C$, which in this case gives

$$\tfrac{1}{2} \times 6 \times 6 \times \sin 2\theta = 9\sqrt{3}$$

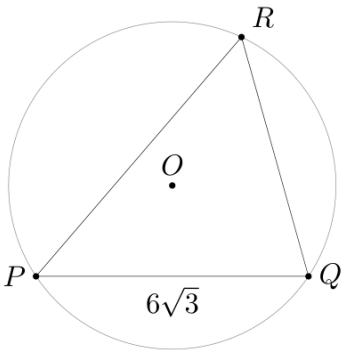
so $\sin 2\theta = \frac{\sqrt{3}}{2}$. Since $2\theta \geq \frac{\pi}{2}$, we must have $2\theta = \frac{2}{3}\pi$ (or $120°$).

We can now work out the length PQ by dropping a perpendicular from O to PQ:

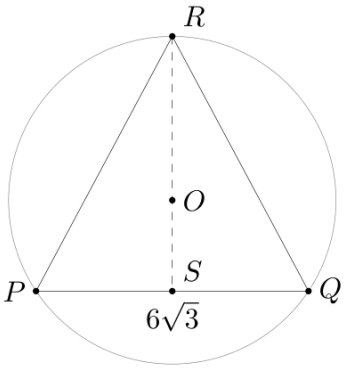


We have $PS = SQ = 6\sin\theta = 6\sin\frac{1}{3}\pi = 3\sqrt{3}$, so $PQ = 6\sqrt{3}$, and we also have $OS = 6\cos\theta = 6\cos\frac{1}{3}\pi = 3$.

We can now work out the maximum possible area of the triangle PQR. The triangle PQR looks like this:



Fixing PQ and allowing R to move, we see that the largest possible area of PQR occurs when



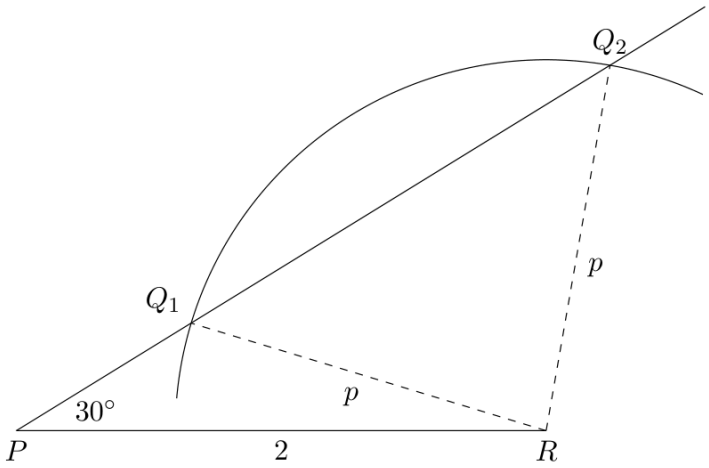
We have $RS = RO + OS$. Now $RO = 6$ (it is a radius of the circle) and $OS = 3$ as we calculated earlier, so $RS = 9$. Therefore the maximum possible area of PQR is

$$\tfrac{1}{2} \times PQ \times RS = \tfrac{1}{2} \times 6\sqrt{3} \times 9 = 27\sqrt{3},$$

which is option D.

4.24 Answer E

We sketch the situation using the given information. This sketch shows a case when there is not a unique solution. The length of PQ is uniquely determined when there is a unique triangle satisfying the conditions.



If p is such that the circle centred at R with radius p touches the line from P at angle $30°$, there will be a unique solution. If p is smaller than this, then there will be no intersections, and hence no solutions. If p is larger than this there will be two solutions, which suggests that the answer is A or B.

We can work out the value of p which makes the circle tangent to the line through P: RQP will be a right-angled triangle with the right angle at Q in this case, and so $p = 2\sin 30° = 1$

This means that $p = 1$ gives a unique solution, and the solution is either A or possibly E.

What happens when $p = 2$? Then the circle centred on R passes through P, and so there is only one possible location for Q. And then we see that if $p > 2$, the point Q_1 will no longer exist: it will be beyond P, on the extension of the line "below" PR. So for any $p \geqslant 2$, there is also only one possible triangle.

Hence the answer is option E.

4.25 Answer F

TMUA 2023 · Paper 1 · Q15

The minimum and maximum values occur when $\cos x = \pm 1$ (though which corresponds to the maximum and which to the minimum depends on whether $a < 1$ or $a > 1$). Therefore the minimum and maximum values are a and $1/a$ in some order.

Assuming $a > 1$, so that $a > 1/a$, we require $a - \frac{1}{a} = 3$. Multiplying by a and rearranging gives $a^2 - 3a - 1 = 0$, which has roots $a = \frac{3 \pm \sqrt{13}}{2}$. The minus sign would give $a < 0$, so the root we want is $a = \frac{3+\sqrt{13}}{2}$, and this clearly satisfies $a > 1$.

Now assuming $a < 1$, so that $1/a > a$, we require $\frac{1}{a} - a = 3$. Multiplying by a and rearranging gives $a^2 + 3a - 1 = 0$, which has roots $a = \frac{-3 \pm \sqrt{13}}{2}$. The minus sign would give $a < 0$, so the root we want is $a = \frac{-3+\sqrt{13}}{2}$ (and as $3 < \sqrt{13} < 4$, this lies between 0 and $\frac{1}{2}$, hence $a < 1$).

The sum of these two values is $\frac{3+\sqrt{13}}{2} + \frac{-3+\sqrt{13}}{2} = \sqrt{13}$, which is option F.

4.26 Answer B

TMUA 2022 · Paper 1 · Q16

The given quartic can be thought of as a quadratic in x^2, and as such it has roots

$$x^2 = \frac{6 \pm \sqrt{6^2 - 28}}{14} = \frac{3 \pm \sqrt{2}}{7}.$$

Since the roots of the quartic are $\pm\cos\theta$ and $\pm\cos\beta$, these x^2 values are the values of $\cos^2\theta$ and $\cos^2\beta$ (in some order).

Now $\sin^2\theta = 1 - \cos^2\theta$ and $\sin^2\beta = 1 - \cos^2\beta$, so

$$\sin^2\theta, \sin^2\beta = 1 - \frac{3 \pm \sqrt{2}}{7} = \frac{4 \mp \sqrt{2}}{7}.$$

Therefore the equations $x^2 - \frac{4+\sqrt{2}}{7} = 0$ and $x^2 - \frac{4-\sqrt{2}}{7} = 0$ have solutions $\pm\sin\theta$ and $\pm\sin\beta$ (again, in some order, and x is not the same as earlier). It follows that the product of these two equations will have the four solutions $\pm\sin\theta$ and $\pm\sin\beta$; this equation is

$$\left(x^2 - \frac{4+\sqrt{2}}{7}\right)\left(x^2 - \frac{4-\sqrt{2}}{7}\right) = 0.$$

To obtain the form given in the question, we first expand the brackets:

$$x^4 - \left(\frac{4+\sqrt{2}}{7} + \frac{4-\sqrt{2}}{7}\right)x^2 + \frac{4+\sqrt{2}}{7} \cdot \frac{4-\sqrt{2}}{7} = 0$$

then perform the surd calculations to give

$$x^4 - \frac{8}{7}x^2 + \frac{2}{7} = 0.$$

Finally, multiplying by 7 gives $7x^2 - 8x + 2 = 0$, which is option B.

4.27 Answer D

TMUA 2016 · Paper 1 · Q17

The formula for y is the product of two factors, namely $1 + 2\cos x$ and $\cos 2x$. The whole expression is negative when one of the two factors is positive and the other is negative. The factors change sign when they cross a point where the factor is zero, and we can find these points:

- $1 + 2\cos x = 0$ when $\cos x = -\frac{1}{2}$, which is when $x = \frac{2\pi}{3}$ (within the range $0 < x < \pi$).
- $\cos 2x = 0$ when $x = \frac{\pi}{4}$ and when $x = \frac{3\pi}{4}$.

We can now make a table showing the signs of the two factors in the different parts of the interval $0 < x < \pi$. (This is a useful technique in general.)

	$0 < x < \frac{\pi}{4}$	$x = \frac{\pi}{4}$	$\frac{\pi}{4} < x < \frac{2\pi}{3}$	$x = \frac{2\pi}{3}$	$\frac{2\pi}{3} < x < \frac{3\pi}{4}$	$x = \frac{3\pi}{4}$	$\frac{3\pi}{4} < x < \pi$
$1 + 2\cos x$	$+$	$+$	$+$	0	$-$	$-$	$-$
$\cos 2x$	$+$	0	$-$	$-$	$-$	0	$+$
y	$+$	0	$-$	0	$+$	0	$-$

Therefore y is negative when $\frac{\pi}{4} < x < \frac{2\pi}{3}$ and when $\frac{3\pi}{4} < x < \pi$, and so the answer is D.

4.28 Answer C

TMUA 2019 · Paper 1 · Q17

A product of two terms is at least 0 if both factors are positive or both terms are negative. So we look at the signs of the two factors.

We start by finding the values of θ for which they are zero.

We have $\sin 2\theta = \frac{1}{2}$ when $2\theta = \frac{\pi}{6}$ or $\frac{5\pi}{6}$, so for $\theta = \frac{\pi}{12}$ or $\frac{5\pi}{12}$. Thinking about a sketch of the graph of $y = \sin 2\theta$, we see that $\sin 2\theta - \frac{1}{2}$ is positive for θ between these values and negative elsewhere.

For the second factor, we have $\sin\theta = \cos\theta$ when $\tan\theta = 1$, so $\theta = \frac{\pi}{4}$. Thinking about the graphs of $y = \sin\theta$ and $y = \cos\theta$, we see that $\sin\theta - \cos\theta$ is negative for $\theta < \frac{\pi}{4}$ and positive for $\theta > \frac{\pi}{4}$. For consistency, we write $\frac{\pi}{4} = \frac{3\pi}{12}$ in what follows.

We thus have the following table:

	$0 \leq \theta < \frac{\pi}{12}$	$\theta = \frac{\pi}{12}$	$\frac{\pi}{12} < \theta < \frac{3\pi}{12}$	$\theta = \frac{3\pi}{12}$	$\frac{3\pi}{12} < \theta < \frac{5\pi}{12}$	$\theta = \frac{5\pi}{12}$	$\frac{5\pi}{12} < \theta \leq \pi$
$\sin 2\theta - \frac{1}{2}$	$-$	0	$+$	$+$	$+$	0	$-$
$\sin\theta - \cos\theta$	$-$	$-$	$-$	0	$+$	$+$	$+$
product	$+$	0	$-$	0	$+$	0	$-$

The lengths of the subintervals where the product is positive (or zero) is

$$(\tfrac{\pi}{12} - 0) + (\tfrac{5\pi}{12} - \tfrac{3\pi}{12}) = \tfrac{3\pi}{12} = \tfrac{\pi}{4}$$

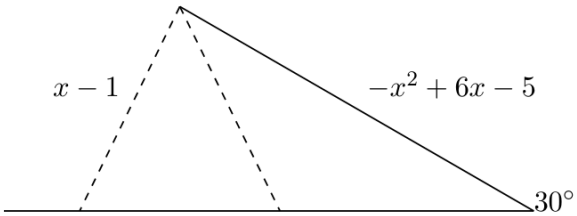
The whole interval has length π, so the fraction of the interval for which the inequality is satisfied is $\frac{1}{4}$, and the answer is option C.

4.29 Answer D

TMUA 2022 · Paper 1 · Q17

We first require that the lengths shown are both positive; $x-1>0$ if and only if $x>1$, and $-x^2+6x-5=-(x-5)(x-1)>0$ if and only if $1<x<5$. Therefore we require $1<x<5$ to ensure both lengths are positive.

There are two non-congruent triangles when the side marked $x-1$ can be drawn in two configurations:



This is possible when the following conditions are satisfied:

- $x-1<-x^2+6x-5$ (otherwise one of the possibilities will lie to the right of the 30° angle in the above diagram)
- $x-1>(-x^2+6x-5)\sin 30°$ (so that the distance from the end of the $-x^2+6x-5$ side to the base is at least $x-1$, making it possible to form a triangle; if there is equality here, then the dashed circle will be tangent to the base of the triangle, and there will only be one possibility)

The first inequality rearranges as $x^2-5x+4<0$, which factorises as $(x-1)(x-4)<0$, giving $1<x<4$.

The second equality, using $\sin 30°=\frac{1}{2}$, becomes $2x-2>-x^2+6x-5$, so $x^2-4x+3>0$. This factorises as $(x-1)(x-3)>0$, so $x<1$ or $x>3$.

Requiring both of these inequalities to hold thus gives $3<x<4$; this also satifies the positivity requirement we started with $(1<x<5)$, and so the answer is option D.

4.30 Answer B

TMUA 2018 · Paper 1 · Q18

The lines of symmetry of a sine or cosine graph (that is, something of the form $\sin(kx+\theta)+c$ or $\cos(kx+\theta)+c$) are at the x-values where the sine or cosine equals ± 1. So in this case, if we solve $\sin(2x-\frac{4\pi}{3})=\pm 1$ we will find the lines of symmetry.

We solve each of them separately.

We have $\sin\theta=1$ when $\theta=\frac{\pi}{2}+2n\pi$ (for any integer n), so in this case, this gives

$$\begin{aligned} & 2x-\tfrac{4\pi}{3}=\tfrac{\pi}{2}+2n\pi \\ \iff \quad & 2x=\tfrac{11\pi}{6}+2n\pi \\ \iff \quad & x=\tfrac{11\pi}{12}+n\pi \end{aligned}$$

So the smallest positive value of x with $\sin(2x-\frac{4\pi}{3})=1$ is $x=\frac{11\pi}{12}$

In the second case, we have $\sin\theta=-1$ when $\theta=-\frac{\pi}{2}+2n\pi$ (for any integer n), so in this case, this gives

$$\begin{aligned} & 2x-\tfrac{4\pi}{3}=-\tfrac{\pi}{2}+2n\pi \\ \iff \quad & 2x=\tfrac{5\pi}{6}+2n\pi \\ \iff \quad & x=\tfrac{5\pi}{12}+n\pi \end{aligned}$$

So the smallest positive value of x with $\sin(2x-\frac{4\pi}{3})=-1$ is $x=\frac{5\pi}{12}$

This is smaller than $\frac{11\pi}{12}$ so the answer is $\frac{5\pi}{12}$, which is option B.

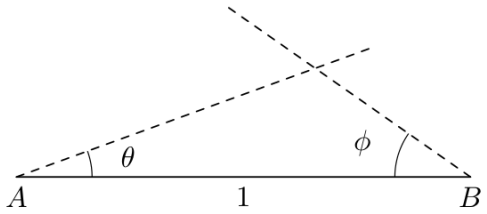
Since we are only given a condition on $\sin A$, the angle A could be acute or obtuse, and similarly for angle B.

Let us write $\theta = \sin^{-1} x$ and $\phi = \sin^{-1} y$ for the acute angles; the respective obtuse angles are then $180° - \theta$ and $180° - \phi$.

As we are given $x < y$, we have $\theta < \phi$.

We can now consider the four possible cases.

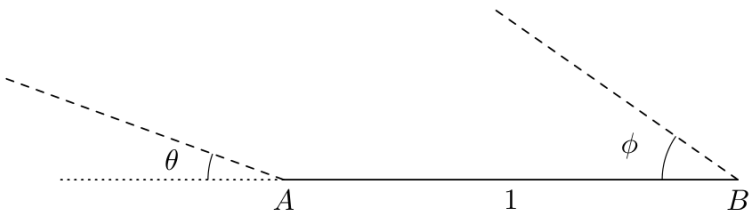
- Case 1: both A and B are acute. Then we have this situation:



  We therefore know the three angles of the triangle ABC (as given two, we automatically know the third) and one side, and this determines the triangle uniquely (regarding congruent triangles as the same).

  So we always obtain exactly one triangle from this case.

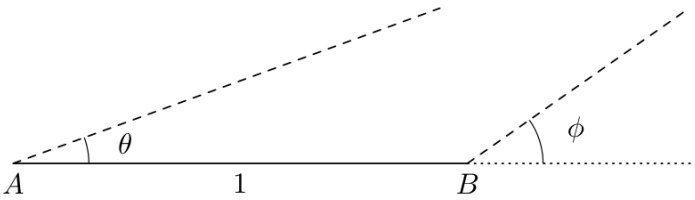
- Case 2: A is obtuse and B is acute. Then $A = 180° - \theta$ and we have this situation:



  Since $\theta < \phi$, the two lines (rays) from A and B shown in this diagram will never meet. Alternatively, we could note that $A + B = 180° - \theta + \phi$, and since $\phi - \theta > 0$, this means that the angles A and B sum to more than $180°$.

  Therefore this case can never produce a triangle.

- Case 3: A is acute and B is obtuse. Then $B = 180° - \phi$ and we have this situation:



  Since $\theta < \phi$, the two lines (rays) from A and B shown will eventually meet, and so we get a unique triangle. Alternatively, we have $A + B = \theta + 180° - \phi = 180° - (\phi - \theta)$, and since $\phi - \theta > 0$, this means that the angles A and B sum to less than $180°$, so they can be two angles of a triangle.

- Case 4: A and B are both obtuse. This is impossible in a triangle.

  Therefore this case can never produce a triangle.

In summary, cases 1 and 3 always produce a unique triangle, and cases 2 and 4 never produce a triangle, so the student can always draw exactly two different triangles, regardless of the values of x and y.

Therefore the correct answer is C.

Let us substitute values into the four equations to begin with.

When $x = 0$, we have

$$\begin{aligned}(\cos x)^{\cos x} &= 1^1 = 1\\ (\sin x)^{\sin x} &= 0^0 = ?\\ (\cos x)^{\sin x} &= 1^0 = 1\\ (\sin x)^{\cos x} &= 0^1 = 0\end{aligned}$$

This shows that graph Q is $y = (\sin x)^{\cos x}$. (We do not know what $(\sin x)^{\sin x}$ is when $x = 0$, but from the graphs given, since three of the functions have value 1 at $x = 0$ and only one is 0, it must be 1.)

Next, let us consider the value at $x = \frac{\pi}{2}$:

$$\begin{aligned}(\cos x)^{\cos x} &= 0^0 = ?\\ (\sin x)^{\sin x} &= 1^1 = 1\\ (\cos x)^{\sin x} &= 0^1 = 0\\ (\sin x)^{\cos x} &= 1^0 = 1\end{aligned}$$

Again, as only one graph has value 0 at $x = \frac{\pi}{2}$, graph P depicts $(\cos x)^{\sin x}$. (We did not actually need to calculate the final line of this list, as we have already identified this function as being graph Q.)

So we are left with $(\cos x)^{\cos x}$ and $(\sin x)^{\sin x}$ as graphs R and S in some order. Let us put in the value $x = \frac{\pi}{6}$, as that is clearly different between the two functions. We have

$$\begin{aligned}(\cos x)^{\cos x} &= \left(\frac{\sqrt{3}}{2}\right)^{\frac{\sqrt{3}}{2}}\\ (\sin x)^{\sin x} &= \left(\frac{1}{2}\right)^{\frac{1}{2}}\end{aligned}$$

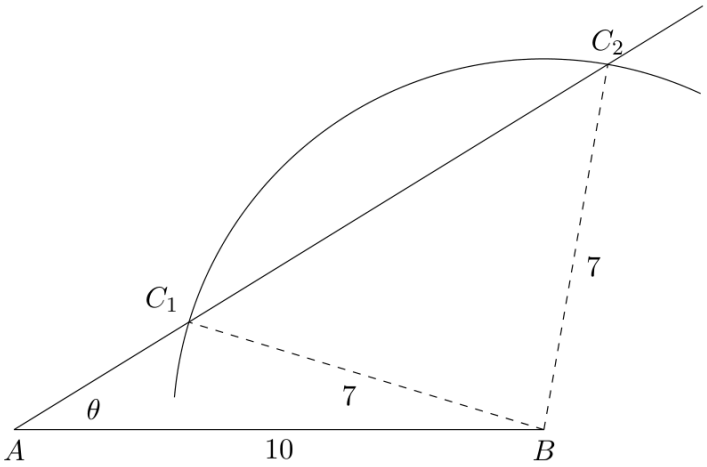
The first of these looks horrible, but we can approximate the value of the second of these; it is just $\frac{1}{\sqrt{2}} \approx 0.7$, so this must be graph S. Similarly, $(\cos x)^{\cos x} \approx 0.7$ at $x = \frac{\pi}{3}$, which is graph R.

Therefore the graphs are:

$$\begin{aligned}(\cos x)^{\cos x} &: \text{ graph R}\\ (\sin x)^{\sin x} &: \text{ graph S}\\ (\cos x)^{\sin x} &: \text{ graph P}\\ (\sin x)^{\cos x} &: \text{ graph Q}\end{aligned}$$

which is option E.

This questions calls for a sketch. We draw $AB = 10$, a line from A at an angle θ to AB (though we do not know what θ is), and a circular arc centred on B with radius 7.



The sketch shows the two triangles which could be formed subject to these conditions, with the third vertex being at C_1 or C_2.

One approach, hinted at by the instruction to find $\cos\theta$, is to use the cosine rule.

In the triangle ABC_1, we have

$$7^2 = (AC_1)^2 + 10^2 - 2(AC_1)10\cos\theta \tag{1}$$

while in the triangle ABC_2, we have

$$7^2 = (AC_2)^2 + 10^2 - 2(AC_2)10\cos\theta \tag{2}$$

Unfortunately, we do not know either AC_1 or AC_2. We do, however, know that the larger triangle has three times the area of the smaller one. Taking AC_1 or AC_2 as the base of the triangle, the perpendicular height is from B to the line AC_2, and is the same for both triangles. So $AC_2 = 3(AC_1)$

We can now substitute this into equation (2) to get

$$7^2 = 9(AC_1)^2 + 10^2 - 6(AC_1)10\cos\theta$$

Subtracting equation (1) then gives

$$0 = 8(AC_1)^2 - 40(AC_1)\cos\theta$$

so dividing by $8AC_1$, we have $AC_1 = 5\cos\theta$.

We can now substitute this back into equation (1) to get

$$7^2 = (5\cos\theta)^2 + 10^2 - 20(5\cos\theta)\cos\theta$$

so

$$49 = 25\cos^2\theta + 100 - 100\cos^2\theta$$

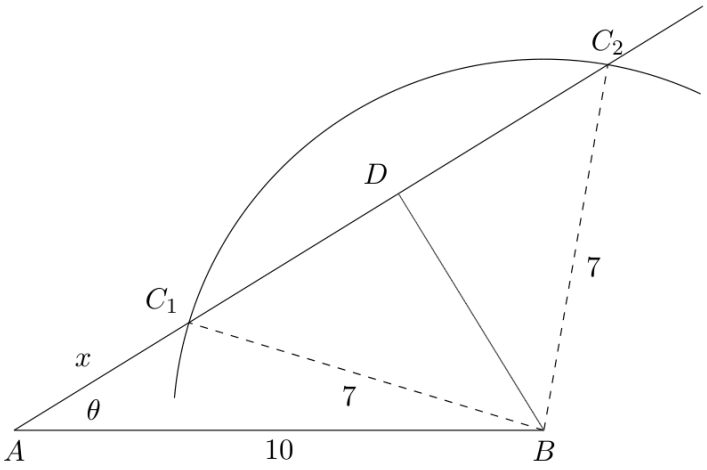
which simplifies to

$$75\cos^2\theta = 51$$

so $\cos^2\theta = \dfrac{51}{75} = \dfrac{17}{25}$ giving $\cos\theta = \dfrac{\sqrt{17}}{5}$ (as $\cos\theta > 0$)

There is an alternative approach which does not use the cosine rule.

If we drop a perpendicular from B to the line AC_2, and let $x = AC_1$, our diagram becomes:



Now D is midway between C_1 and C_2 by symmetry (or by using the "RHS" congruency rule on the triangles BDC_1 and BDC_2), so $C_1D = DC_2$.

Also, the area of the triangle ABC_1 is $\frac{1}{2}(AC_1)(BD)$ and the area of the triangle ABC_2 is $\frac{1}{2}(AC_2)(BD)$

Since the larger triangle has three times the area of the smaller one, $AC_2 = 3(AC_1)$ (as before). So $AC_2 = 3x$, hence $C_1C_2 = 2x$, and $C_1D = DC_2 = x$

We can now apply Pythagoras to BDC_1 and BDA. We obtain, respectively,

$$x^2 + (BD)^2 = 7^2$$
$$(2x)^2 + (BD)^2 = 10^2$$

Subtracting gives $3x^2 = 51$, so $x^2 = 17$

Since $\cos\theta = \dfrac{AD}{AB} = \dfrac{2x}{10}$ we get $\cos\theta = \dfrac{\sqrt{17}}{5}$ as before.

This also easily shows that $BD = \sqrt{32} = 4\sqrt{2}$ and that the area of the two triangles ABC_1 and ABC_2 are $2\sqrt{34}$ and $6\sqrt{34}$.

We can start by solving the equation before considering the possible values of x.

We have

$$\sin\left(4^{\cos\theta} \times 60^\circ\right) = \pm\frac{\sqrt{3}}{2}$$

so that

$$4^{\cos\theta} \times 60^\circ = 60^\circ, 120^\circ, 240^\circ, 300^\circ, \ldots$$

(noting that the left hand side is always positive, so we do not need to consider negative values for the right hand side).

Dividing by 60° gives

$$4^{\cos\theta} = 1, 2, 4, 5, \ldots$$

and so

$$\cos\theta = 0, \tfrac{1}{2} \text{ or } 1;$$

there are no values of $\cos\theta$ for which $4^{\cos\theta} > 4$.

We can now find the possible values of θ:

- $\cos\theta = 0$ gives $\theta = 90^\circ$, 270°, 450°, ...
- $\cos\theta = \frac{1}{2}$ gives $\theta = 60^\circ$, 300°, 420°, ...
- $\cos\theta = 1$ gives $\theta = 0^\circ$, 360°, ...

Collecting these solutions and arranging them in increasing order gives $\theta = 0^\circ$, 60°, 90°, 270°, 300°, ...

We are told that there are exactly three solutions in the range $0^\circ \leq \theta \leq x^\circ$. Therefore $x \geq 90$ to get the third solution, and $x < 270$ to avoid getting the fourth solution. This gives the answer $90 \leq x < 270$, which is option B.

We can rewrite the given equation as

$$\sin\theta\sqrt{(1+\sin\theta)(1-\sin\theta)}+\cos\theta\sqrt{(1+\cos\theta)(1-\cos\theta)}=0$$

and then multiply out the brackets to give

$$\sin\theta\sqrt{1-\sin^2\theta}+\cos\theta\sqrt{1-\cos^2\theta}=0$$

which simplifies to

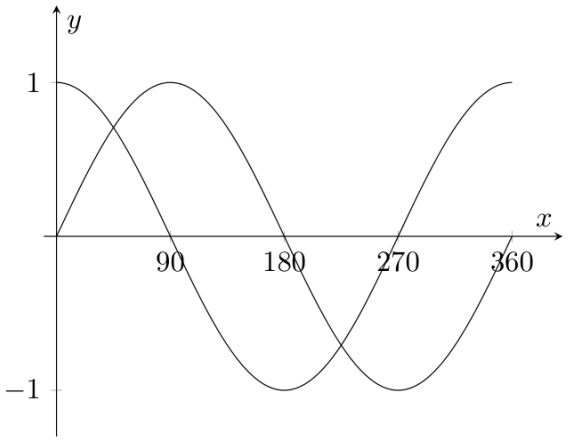
$$\sin\theta\sqrt{\cos^2\theta}+\cos\theta\sqrt{\sin^2\theta}=0.$$

It would be easy at this point to make the mistake of simplifying $\sqrt{\cos^2\theta}$ to $\cos\theta$, but this would only work in the case that $\cos\theta\geq 0$ as square roots are always positive. We can, though, write $|\cos\theta|$ for this square root: we always have $\sqrt{x^2}=|x|$. Therefore we can rewrite the equation as

$$\sin\theta\,|\cos\theta|+\cos\theta\,|\sin\theta|=0.$$

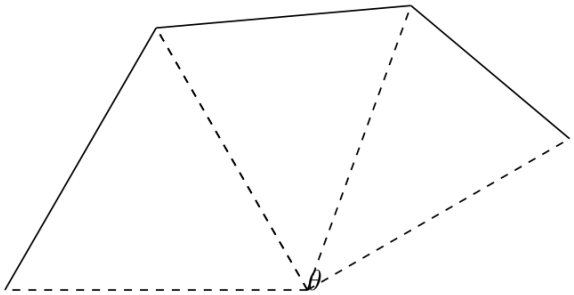
If both $\sin\theta$ and $\cos\theta$ are positive, then this becomes $2\sin\theta\cos\theta=0$, which cannot be true. Likewise, if both $\sin\theta$ and $\cos\theta$ are negative, this becomes $-2\sin\theta\cos\theta=0$, which again cannot be true. However, if one is positive and one is negative, say $\sin\theta>0$ and $\cos\theta<0$, this becomes $-\sin\theta\cos\theta+\sin\theta\cos\theta=0$, which is certainly true. The same holds if $\sin\theta<0$ and $\cos\theta>0$. Finally, if either $\sin\theta=0$ or $\cos\theta=0$, the equation is also satisfied.

We therefore need to count the number of given angles for which $\sin\theta$ and $\cos\theta$ have opposite signs or one of them is zero. A quick sketch helps here:



We see that the relevant ranges are $90^\circ\leq\theta\leq 180^\circ$ and $270^\circ\leq\theta\leq 360^\circ$; $x=0^\circ$ would also work, but is not one of the angles in the given list. There are 91 angles in each of the two ranges (remembering to include both endpoints), so there are 182 angles in total, which is option F.

We sketch a part of the polygon with the triangles described in the question:



Suppose the angle of one of the triangles is θ, as shown. Then the area of the triangle is given by $\frac{1}{2}r^2\sin\theta$ where r is the radius of the circle (using the formula $A = \frac{1}{2}ab\sin C$).

Since r is the same for each of the triangles, all the triangles will have equal area if and only if $\sin\theta$ is the same for each triangle. This will certainly be the case if the polygon is regular, but we have to determine if this can be true in any other situation.

If two of the triangles have angles at the centre of the circle of θ and ϕ, then $\sin\phi = \sin\theta$ if and only if $\phi = \theta$ or $\phi = \pi - \theta$. Can we have the latter possibility? If so, it would mean we have some angles being θ and some being $\pi - \theta$.

Suppose then that θ is acute and $\pi - \theta$ is obtuse. (We cannot have a reflex angle as the centre of the circle lies inside the polygon, and if θ is a right angle, then $\pi - \theta = \theta$.) Suppose further that k of the angles of the polygon triangles equal $\pi - \theta$ and the remaining $n - k$ equal θ. Then the sum of these triangle angles is given by

$$(n-k)\theta + k(\pi - \theta) = 2\pi$$

since they form a whole circle. We can rearrange this to $(n - 2k)\theta = (2 - k)\pi$ and hence

$$\theta = \frac{(2-k)\pi}{n-2k}.$$

There can be at most three obtuse angled triangles (as four would be more than a whole circle), so let's try $k = 1$, $k = 2$ and $k = 3$ in turn.

If $k = 1$, we have $\theta = \frac{\pi}{n-2}$. It looks as though this will work for every n, but we have to be a little careful as we need θ to be acute. When $n = 3$, we get $\theta = \pi$, which is not allowed. When $n = 4$, we get $\theta = \frac{\pi}{2}$, which is again not allowed. When $n > 4$, $\theta < \frac{\pi}{2}$, so this does work. Therefore when $n \geq 5$, it is possible for the conditions to be satisfied but for the polygon not to be regular.

Now consider $k = 2$, giving $\theta = \frac{0}{n-4}$. We only need to consider $n = 3$ and $n = 4$, as we have dealt with $n \geq 5$ above. This is zero when $n = 3$, so it does not work for a triangle. When $n = 4$, this formula is not meaningful (it gives $\frac{0}{0}$). We can instead go back to the original equation $(n - 2k)\theta = (2 - k)\pi$, which becomes $0 = 0$ in this case, meaning that it works for any value of θ. And indeed, this gives two angles of θ and two of $\pi - \theta$, so we get a non-regular polygon with equal area triangles. The simplest example is a non-square rectangle.

obtuse angles of $\frac{\pi}{3}$ is an equilateral triangle satisfying the conditions. Therefore the only triangle ($n = 3$) satisfying the conditions is regular.

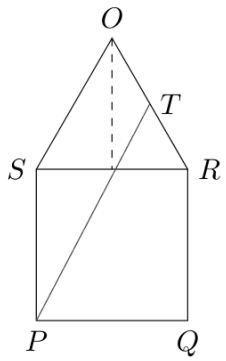
We have thus been able to construct non-regular polygons satisfying the conditions for every $n \geq 4$, but not for $n = 3$. Hence the correct answer is option B.

Let us label the midpoint of OR as T, so that we can refer to it.

We first observe that since all the edges of the pyramid are of length 20m, the triangles are all equilateral.

There are two ways to get from P to the midpoint of OR in a short distance: one can either travel along the base to RS and then along the triangular face to T, or one could travel along the triangular face OPS to the edge OS and from there to T. It is not obvious which route is shorter, so we will calculate both. (There are also routes which are mirror-reflections of these going via the edge QR or OQ.)

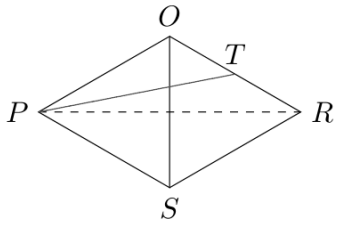
To find these distances, we can "unfold" the tetrahedron, that is, make a net for the tetrahedron. For the first route, we draw just the faces $PQRS$ and ORS:



The height of the triangle OSR is $20\sin 60° = 20 \times \frac{\sqrt{3}}{2} = 10\sqrt{3}$, so T lies $5\sqrt{3}$ above the line SR; T also lies $\frac{3}{4} \times 20 = 15$ to the right of S (as it is at the midpoint of OR). We can therefore find the length PT using Pythagoras's theorem:

$$\begin{aligned} PT^2 &= 15^2 + (20 + 5\sqrt{3})^2 \\ &= 225 + 400 + 200\sqrt{3} + 75 \\ &= 700 + 200\sqrt{3} \end{aligned}$$

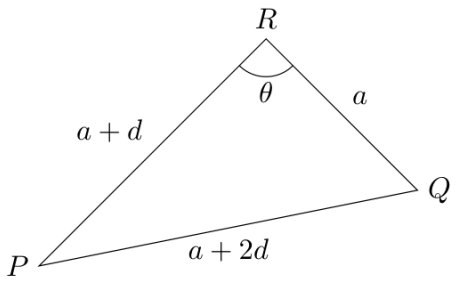
We now look at the other route, which we can do by drawing the triangles OPS and OSR:



Using the height of the triangle as calculated above, P lies $10\sqrt{3}$ to the left of OS and T lies $5\sqrt{3}$ to its right, while T lies 5 above P (being a quarter of OS). Pythagoras's theorem therefore gives

$$PT^2 = (15\sqrt{3})^2 + 5^2$$

It would be really helpful to sketch this triangle so that we don't get confused! We will write θ for the angle $\angle PRQ$.



It is clear that we want to use the cosine rule to relate the three sides and the angle θ. Using '$c^2 = a^2 + b^2 - 2ab\cos C$', we have

$$(a+2d)^2 = a^2 + (a+d)^2 - 2a(a+d)\cos\theta$$

which we can expand and simplify to give

$$a^2 + 4ad + 4d^2 = 2a^2 + 2ad + d^2 - 2a(a+d)\cos\theta$$

Rearranging gives

$$\cos\theta = \frac{a^2 - 2ad - 3d^2}{2a(a+d)}$$

We can simplify this by noticing that the numerator factorises:

$$\cos\theta = \frac{(a+d)(a-3d)}{2a(a+d)} = \frac{a-3d}{2a}$$

Since $3d > 2a$, $a - 3d < -a$, and so $\cos\theta < \frac{-a}{2a} = -\frac{1}{2}$. This gives $\theta > 120°$, and so the answer must be option E.

We can go further and show that $\theta < 180°$. Since PQR is a triangle, the length PQ must be less than the sum of PR and RQ, that is $a + 2d < a + d + a$, so $d < a$. This gives $\cos\theta > \frac{-2a}{2a} = -1$ so $\theta < 180°$.

An alternative approach is to just try values for d. We are given that $d > \frac{2}{3}a$, so we could try $d = \frac{2}{3}a$. This gives side lengths of a, $\frac{5}{3}a$, $\frac{7}{3}a$, which we can take to be 3, 5, 7 if we choose $a = 3$. Plugging these lengths into the cosine rule gives $\theta = 120°$, so this is one limit.

We are allowed larger values of d. If we take d too large, then we will no longer have a triangle, as $PQ > PR + RQ$. So as d increases, the angle θ must increase too. Therefore the answer must be E: $120° < \theta < 180°$.

This reminds us of adding the terms of an arithmetic series. One way to do so is to write out the series twice, once forwards and once backwards, and then add corresponding pairs of terms together. If we do that here, we obtain

$$S = \sin^2 0° + \sin^2 1° + \sin^2 2° + \cdots + \sin^2 88° + \sin^2 89° + \sin^2 90°$$
$$S = \sin^2 90° + \sin^2 89° + \sin^2 88° + \cdots + \sin^2 2° + \sin^2 1° + \sin^2 0°$$

so that

$$2S = (\sin^2 0° + \sin^2 90°) + (\sin^2 1° + \sin^2 89°) + \cdots + (\sin^2 90° + \sin^2 0°)$$

Now we know that $\sin^2 x + \cos^2 x = 1$ for all x, and since $\sin(90° - x) = \cos x$ (by, for example, thinking about a right-angled triangle or by looking at the graphs of the functions), we can rewrite this sum as

$$\begin{aligned} 2S &= (\sin^2 0° + \cos^2 0°) + (\sin^2 1° + \cos^2 1°) + \cdots + (\sin^2 90° + \cos^2 90°) \\ &= 1 + 1 + \cdots + 1 \\ &= 91 \end{aligned}$$

(since there are 91 terms), therefore $S = 45.5$, which is option E.

Algebra and Functions 2

5.1 Answer B

TMUA 2020 · Paper 1 · Q3

We first work on each inequality separately. Each one is a product of factors, so we consider when each factor is positive or negative.

For the first inequality, $(x+4)(x+3)(1-x) > 0$, we have the following table:

	$(x+4)$	$(x+3)$	$(1-x)$	product
$x < -4$	$-$	$-$	$+$	$+$
$-4 < x < -3$	$+$	$-$	$+$	$-$
$-3 < x < 1$	$+$	$+$	$+$	$+$
$x > 1$	$+$	$+$	$-$	$-$

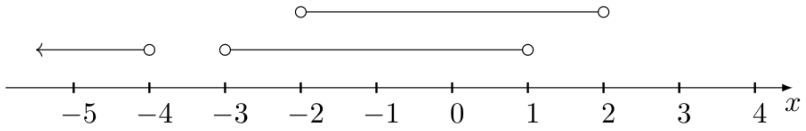
Thus the first inequality holds when $x < -4$ and when $-3 < x < 1$.

Likewise for the second inequality, $(x+2)(x-2) < 0$:

	$(x+2)$	$(x-2)$	product
$x < -2$	$-$	$-$	$+$
$-2 < x < 2$	$+$	$-$	$-$
$x > 2$	$+$	$+$	$+$

Therefore the second inequality holds when $-2 < x < 2$.

We can combine these by illustrating them on a number line:



Combining these two conditions, they both hold when $-2 < x < 1$, which is option B.

5.2 Answer E

TMUA Specimen · Paper 1 · Q4

We can factorise the given inequality to give $(x+1)(x-1)(x-2) > 0$. The function $f(x) = (x+1)(x-1)(x-2)$ is zero when $x = -1$, $x = 1$ and $x = 2$.

We can now make a table showing the signs of the three factors for different values of x. (This is a useful technique in general.)

	$x < -1$	$x = -1$	$-1 < x < 1$	$x = 1$	$1 < x < 2$	$x = 2$	$x > 2$
$x+1$	$-$	0	$+$	$+$	$+$	$+$	$+$
$x-1$	$-$	$-$	$-$	0	$+$	$+$	$+$
$x-2$	$-$	$-$	$-$	$-$	$-$	0	$+$
$f(x)$	$-$	0	$+$	0	$-$	0	$+$

Therefore $f(x)$ is positive when $-1 < x < 1$ and when $x > 2$, so the answer is E.

5.3 Answer F

I If we take the logarithm to base a of the equation $(*)$, we get $x = \log_a x$.

This suggests that this equation must have the same number of real solutions as $(*)$, but we must be careful and check that we are allowed to take the logarithm of both sides of $(*)$ without adding or losing solutions. We can only take the logarithm of a positive number. The left hand side of $(*)$ is always positive, and if $x \leq 0$, then this value of x does not give a solution to $(*)$. So the solutions to $(*)$ (if there are any) all have $x > 0$. So we can take the logarithm of both sides without losing any solutions. And by exponentiating the resulting equation, we get back to $(*)$, so we do not introduce any new solutions. Therefore this equation must have the same number of real solutions as $(*)$.

II If we square both sides of $(*)$, we obtain $a^{2x} = x^2$. Squaring can introduce new solutions, though; does it on this occasion? If we have a solution to $a^{2x} = x^2$, we can deduce that $a^x = \pm x$. We always have $a^x > 0$, so it might be that we get a new solution with $x < 0$. Let us try a numerically simple example to see if this does happen. As $x < 0$, we have $0 < a^x < 1$, so we need $-1 < x < 0$. Taking $x = -\frac{1}{2}$, $(*)$ becomes $a^{-1/2} = \frac{1}{2}$; we then get $a = 4$. So in this case, $4^{2x} = x^2$ has an extra solution, $x = -\frac{1}{2}$, which is not a solution to $4^x = x$. Therefore this equation may not have the same number of real solutions as $(*)$.

Another way to think about it is graphically. The graphs of $y = a^x$ and $y = a^{2x}$ are both exponential graphs with $y < 1$ for all $x < 0$. The graph of $y = x$ is negative for all $x < 0$ so any solutions to $(*)$ have $x > 0$. On the other hand, $y = x^2$ is positive for all $x \neq 0$ and it reaches 1 at $x = -1$. Therefore there will always be a solution to $a^{2x} = x^2$ with $-1 < x < 0$.

III If we replace $2x$ with u in this equation, we get $a^u = u$, so there are the same number of real solutions for u as $(*)$ has. Every (real) value of u corresponds to a unique value of x, namely $x = u/2$, so there must be the same number of real solutions to $a^{2x} = 2x$ as $(*)$.

Thus the answer is option F, I and III only.

5.4 Answer C

A simple way to decide between the four given options is to determine which of the points $(1, 0)$, $(-1, 0)$, $(0, 1)$ and $(0, -1)$ lie on the graph.

We start with $(1, 0)$: this gives $1 \times 1 = 1$, so $(1, 0)$ lies on the graph. We now look at $(0, 1)$: this gives $1 \times 1 = 1$, so this also lies on the graph. On the other hand, $(-1, 0)$ gives $-1 \times 1 = -1$ and $(0, -1)$ gives the same, so neither of these points lies on the graph.

Therefore the graph must be that shown in option C.

Alternatively, we can multiply out the brackets, giving $x^3 + y^3 = 1$; therefore for large values of x, we must have $y^3 \approx -x^3$ or $y \approx -x$, giving A or C. Now when $0 < x < 1$, we must have $0 < y < 1$, and hence the correct graph is C.

5.5 Answer E

TMUA 2020 · Paper 2 · Q8

To prove that a 'for all' statement is true, it is necessary to show that the formula holds for every choice of the variables. On the other hand, to show that it is false, it is sufficient to just find a single counterexample, because that shows that it is not true for *all* choices of the variables.

This student has provided a correct counterexample (and it is easy to check that their arithmetic is correct). Therefore the student has correctly shown that $(*)$ is false, and the correct option is E.

One can also observe that 'for all real numbers a and b, $|a+b| \leq |a| + |b|$' is a true statement, but noting that is different from showing that $(*)$ is false.

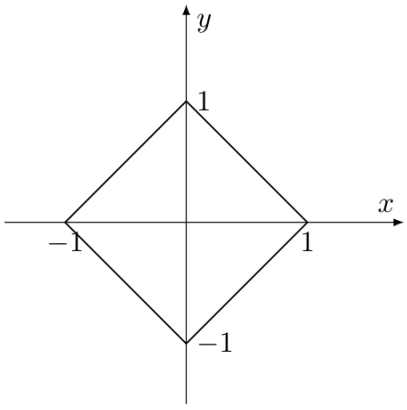
5.6 Answer C

TMUA 2021 · Paper 1 · Q9

We first need to work out what the graph looks like. There are four cases for x and y depending on whether there are positive or negative. We list them in a table:

quadrant	x	y	$\lvert x\rvert$	$\lvert y\rvert$	equation	rearranged to $y = \cdots$
1st	$+$	$+$	x	y	$x+y=1$	$y=-x+1$
2nd	$-$	$+$	$-x$	y	$-x+y=1$	$y=x+1$
3rd	$-$	$-$	$-x$	$-y$	$-x-y=1$	$y=-x-1$
4th	$+$	$-$	x	$-y$	$x-y=1$	$y=x-1$

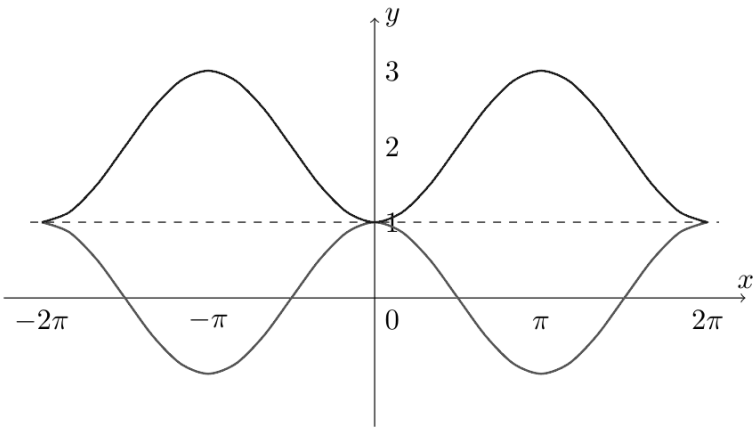
We therefore have (at most) four lines making up this graph, and we can sketch these. We need to recall that we only draw the part of each line that appears within the relevant quadrant; for example, $y = x + 1$ only appears in the 2nd quadrant.



This shape is a square. We can think of it as made up of 4 right angled triangles, each with base and height equal to 1, so each has area $\frac{1}{2} \times 1 \times 1 = \frac{1}{2}$, and hence the square has area 2. Thus the correct option is C.

5.7 Answer D

When the graph is reflected in the line $y = 1$, we obtain the following:



We can find the equation of the reflected curve in a variety of ways. One is to observe that the reflection is centred about $y = 2$, as the values go from $y = 1$ to $y = 3$. The cosine curve is 'upside-down', so the equation must be $y = -(\cos x) + 2 = 2 - \cos x$.

Another way to see this is as follows. The refletion of $y = \mathrm{f}(x)$ in the x-axis ($y = 0$) is $y = -\mathrm{f}(x)$. When the line of reflection is translated, the whole reflection is translated. Since $y = 0$ is transformed to $y = 2$, the reflection of the function must be translated to give $y = 2 - \mathrm{f}(x)$.

The result is then translated by $\frac{\pi}{4}$ in the positive x-direction, so that x is replaced by $x - \frac{\pi}{4}$. (We can see this because the new $x = \frac{\pi}{4}$ corresponds to the old $x = 0$.)

Therefore, the equation of the resulting graph is

$$y = 2 - \cos\left(x - \frac{\pi}{4}\right).$$

Thus the correct answer is option D.

5.8 Answer E

We notice that the first inequality involves x only and the second involves y only, so we can solve each inequality separately.

For the first one, $|2 - x| \leqslant 6$, this means that x is at distance at most 6 from 2, so $-4 \leqslant x \leqslant 8$

Likewise, the second inequality, $|y + 2| \leqslant 4$ means that y is at most 4 from -2 (as $|y + 2| = |y - (-2)|$, so it measures the distance of y from -2). So $-6 \leqslant y \leqslant 2$

We can now obtain the greatest possible value of $|xy| = |x|.|y|$ by taking the greatest possible value of $|x|$, which is 8, and the greatest possible value of $|y|$, which is 6, so the greatest possible value of $|xy|$ is $8 \times 6 = 48$, which is option E.

5.9 Answer A

TMUA 2020 · Paper 1 · Q10

The most straightforward way to answer this question is by performing the transformations to the formula in the standard way. An alternative approach is to sketch the graphs, but then it is not immediately clear how we would determine the resulting formula.

We start with $y = 4x^2$.

1. Translating it right by 3 transforms the curve into $y = 4(x-3)^2$. Then translating it down by 5 transforms it into $y = 4(x-3)^2 - 5$.

2. Reflecting it in the x-axis (the horizontal axis!) replaces y by $-y$, or equivalently replaces $y = \mathrm{f}(x)$ by $y = -\mathrm{f}(x)$, giving $y = -4(x-3)^2 + 5$.

3. Stretching it parallel to the x-axis with scale factor 2 replaces x by $\frac{x}{2}$, giving $y = -4(\frac{x}{2}-3)^2+5$.

We can now expand the resulting formula, to give $y = -4(\frac{x^2}{4} - 3x + 9) + 5 = -x^2 + 12x - 31$, which is option A.

5.10 Answer C

TMUA 2022 · Paper 1 · Q10

Any sequence of translations is equivalent to a translation in the x-direction followed by a translation in the y-direction (where the translations could be zero).

Translating $y = x^3$ by a in the x-direction gives the graph

$$y = (x-a)^3 = x^3 - 3ax^2 + 3a^2x - a^3.$$

Then translating by b in the y-direction gives the graph

$$y = x^3 - 3ax^2 + 3a^2x - a^3 + b.$$

Now considering the three graphs:

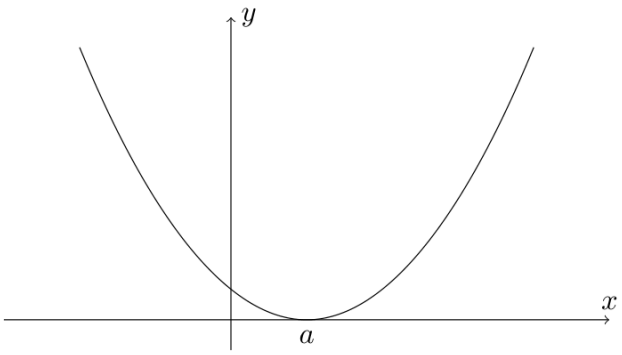
I The x^2 term is $-3x^2$, which means we require $a = 1$. Substituting this into the above general equation gives $y = x^3 - 3x^2 + 3x - 1 + b$, which does not match the given equation. So this is not a translation of $y = x^3$.

II The x^2 term is $-9x^2$, so we require $a = 3$. Substituting gives $y = x^3 - 9x^2 + 27x - 27 + b$, which matches the equation with $b = 24$, so this is a translation of $y = x^3$.

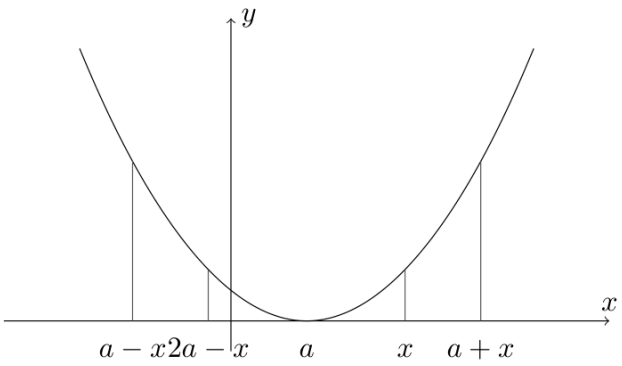
III The x^3 term does not match here, so this is not a translation of $y = x^3$.

Therefore the answer is option C, II only.

It would help to sketch a graph which has reflection symmetry in $x = a$. We should pick a value of a which is not zero. Let us pick a positive value of a. We can draw a quadratic with vertex at $x = a$, for example.



We now consider the three conditions and see which are certainly not **necessary** for the reflection symmetry by looking at this example. We mark on x, $a \pm x$ and $2a - x$ for a value of x which is not equal to a, where we have chosen $x \approx 1.5a$:



It seems from this sketch that $f(x)$ and $f(2a - x)$ are likely to be equal, as are $f(a + x)$ and $f(a - x)$. So we reject condition III, leaving us with conditions I and II to consider.

For a graph $y = f(x)$ to be symmetrical about $x = a$, we need the value of the function to be equal whenever we go the same distance to the right or left of $x = a$. Looking at this sketch, let's take the distance to be x. Then we need $f(a + x) = f(a - x)$ for every choice of x if the graph is symmetrical about $x = a$, so condition I is **necessary**. But it is also **sufficient**, for how could a graph not be symmetrical? Only if for some x, the function values are different at $a + x$ and $a - x$.

For condition II, we can do the same. This time, the values we want to consider are $2a - x$ and x. Now $2a - x$ is $x - a$ to the left of a (assuming that $x > a$) and x is $x - a$ to the right of a, so these are the same distance from a, and again this condition is **necessary**. Since every distance can be written as $x - a$ for some choice of x, the condition is also **sufficient**.

So both conditions I and II are **necessary and sufficient**, and the correct option is B.

We first find the minimum points of the two graphs.

Approach 1: Direct approach

We have, for the first graph,

$$y = f(kx) = (kx)^2 - 6(kx) = k^2x^2 - 6kx.$$

We can find the minimum point using either calculus or completing the square; we will use calculus here. We have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2k^2x - 6k$$

so $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ exactly when $x = \frac{3}{k}$. At this point, $y = k^2(\frac{3}{k})^2 - 6k(\frac{3}{k}) = -9$.

For the second graph, we have

$$y = f(x-c) = (x-c)^2 - 6(x-c) = x^2 - 2cx + c^2 - 6x + 6c = x^2 - (6+2c)x + c^2 + 6c.$$

We use completing the square to find the minimum point: we have

$$y = (x - (3+c))^2 - (3+c)^2 + c^2 + 6c = (x - (3+c))^2 - 9$$

so the minimum point is at $x = 3 + c$, $y = -9$.

Approach 2: Using graph transformations

We first find the minimum point of $y = x^2 - 6x$. We can complete the square to get $y = (x-3)^2 - 9$, so the minimum point is at $x = 3$, $y = -9$.

The graph $y = f(kx)$ is obtained by stretching the original graph by a factor of $1/k$ in the x-direction, so the minimum of this graph is at $x = 3/k$, $y = -9$.

The graph $y = f(x-c)$ is obtainedd by translating the original graph by c in the x-direction, so the minimum of this graph is at $x = 3 + c$, $y = -9$.

For the two minima to be at the same place, we require $3/k = 3 + c$, so taking the reciprocal gives

$$\frac{k}{3} = \frac{1}{3+c}.$$

Multiplying by 3 then gives

$$k = \frac{3}{3+c},$$

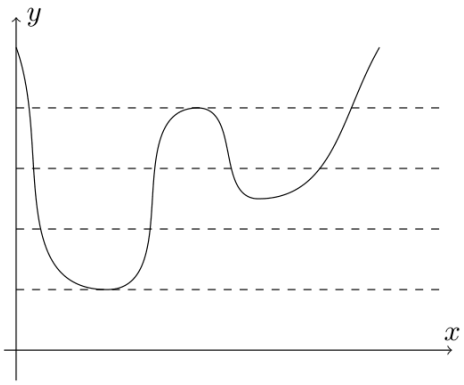
which is option B.

The number of solutions to $f(x) = 1$ is the number of times the graph of $y = f(x)$ intersects with the line $y = 1$, and similarly for the other three cases. So we need to think about possible shapes of $y = f(x)$. We assume that $a \neq 0$ for now, so that we are dealing with a quartic; this will be enough to answer the question. (Each of the given options has one of p, q, r or s equal to 4, which is impossible if the curve is a cubic, quadratic or linear.)

A quartic with $a > 0$ either has a W-shape, or it is a bit like a quadratic in that it has exactly one local minimum. If $a < 0$, the possibilities are the reflections of these.

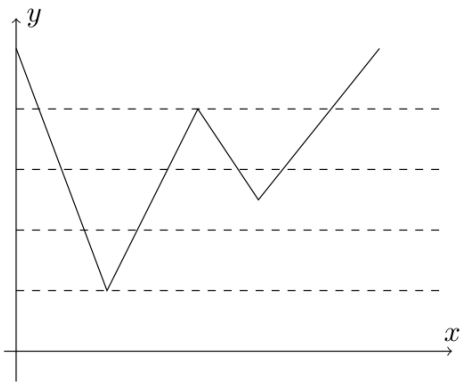
We can then try to construct curves with the given conditions.

A $p = 1$ and $q = 2$ indicates that $a > 0$ and that $y = 1$ is tangent to the curve. So something like this would work, where we have drawn dashed lines to show $y = 1$, $y = 2$, $y = 3$ and $y = 4$. (We could have left out the y-axis for clarity.)



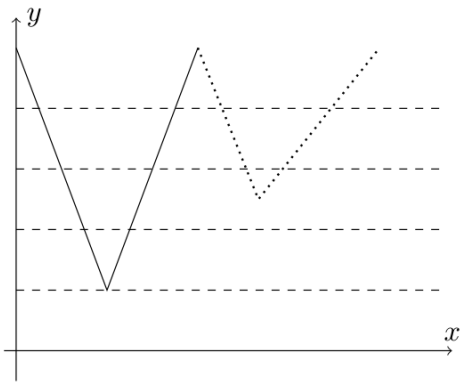
(An accurately-plotted quartic would not look quite like this, as you can see if you use graph-drawing software. Nevertheless, the general shape could be correct, with the stationary points in roughly the locations shown.)

We could also draw this schematically as:



We will use this schematic representation for the rest of the cases.

B Again, we require a minimum at $y = 1$. But as there are 4 crossings at $y = 4$, the curve must do something like this:

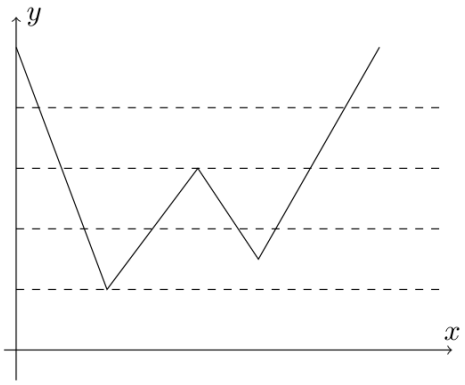


where the last part is not yet clear.

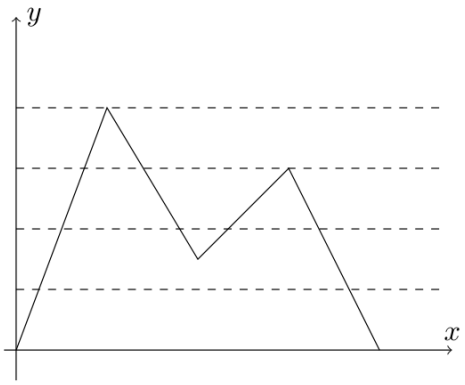
However, to get $r = 2$, the last minimum must lie above $y = 3$, but then we cannot have $q = 3$. Hence this is not possible.

We look at the other three options for the sake of completeness.

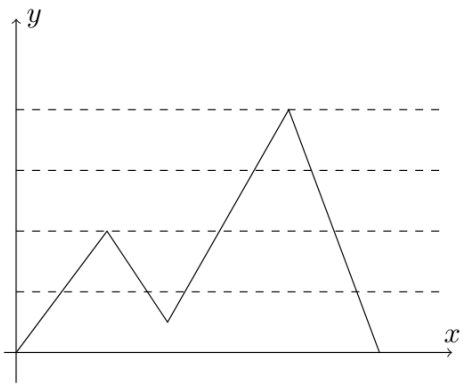
C Again, there is a minimum at $y = 1$, and $r = 3$ gives a stationary point at $y = 3$. This sketch does the job:



D Since $s = 1$, we must have $a < 0$ and $y = 4$ is a maximum. Likewise, $y = 3$ must also be a stationary point. The following sketch achieves this:



E This is similar to the previous option: $a < 0$ and there is a maximum at $y = 4$. As $q = 3$, there is a stationary point at $y = 2$, which gives the following sketch:



The correct option is B.

5.14 Answer D

TMUA 2022 · Paper 2 · Q14

To solve this question, we use the fact that $|x-a|$ can be understood as the distance of x from a.

The first inequality is true if and only if x is closer to -5 than to -11, i.e., if and only if $x > -8$.

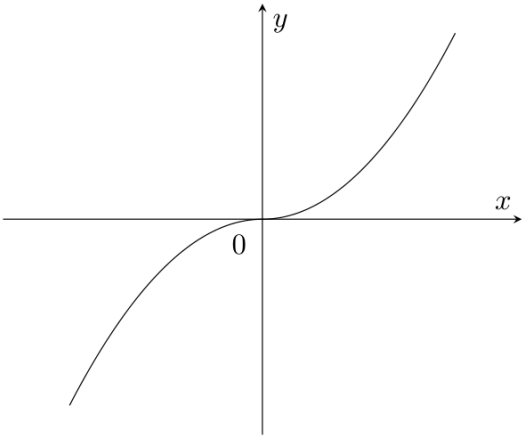
The second inequality is true if and only if x is closer to -11 than to -1, i.e., if and only if $x < -6$.

Therefore both inequalities are true if and only if $-8 < x < -6$, which is an interval of length 2, which is option D.

One could also do this question by sketching the graphs of $y = |x + 5|$ and $y = |x + 11|$ and working out where they intersect, and so on. But that requires a lot more work than than the approach presented here.

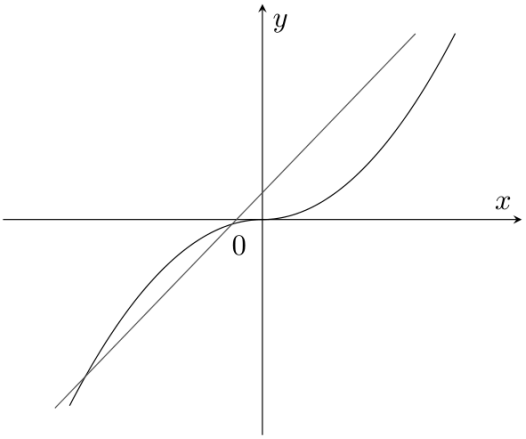
Approach 1: graph sketching

We are looking for the intersections of the graphs of $y = x|x|$ and $y = px + q$. The latter is just a straight line graph, while the former is $y = x^2$ when $x \geq 0$ and $y = -x^2$ when $x < 0$:



This graph looks quite similar to the graph of $y = x^3$: it has a point of inflexion at the origin. Any straight line will meet this graph at least once: if $q > 0$, then it will meet it for some $x > 0$, and if $q < 0$, then it will meet it for some $x < 0$. A straight line can meet the graph twice if it is tangent to the curve somewhere, and it can meet it three times, for example $y = x$.

But a straight line cannot meet the curve four times. It can meet the curve at most twice for $x \geq 0$, and at most twice for $x < 0$ (as each half of the curve is a quadratic, and a straight line and quadratic can meet at most twice). But if a straight line meets the quadratic twice in the region $x < 0$, then it must have $q > 0$, for example:



It can then meet the curve only once in the region $x > 0$. The same applies if it meets the curve twice in the region $x > 0$. And if the line passes through the origin, then it can meet the curve in at most two other locations.

So the correct answer is option E.

If x is a solution, then either $x \leq 0$ and $-x^2 = px + q$ or $x > 0$ and $x^2 = px + q$; we can rewrite these as $x^2 + px + q = 0$ and $x^2 - px - q = 0$ respectively.

If $q < 0$, then the quadratic equations both have positive discriminant. The first quadratic has roots of opposite signs (as $\sqrt{p^2 - 4q} > p$), so there is exactly one negative solution. The second quadratic (corresponding to $x > 0$) may have 0, 1 or 2 positive solutions, giving 1, 2 or 3 solutions in total.

If $q > 0$, exactly the opposite is true, and we still end up with 1, 2 or 3 solutions.

If $q = 0$, then $x = 0$ is a root of both quadratics, and there are 0, 1 or 2 other roots.

Thus the possible values are 1, 2 and 3.

Examples with these numbers are: $p = q = 0$; $p = -1$, $q = 0$, and $p = -2$, $q = 1$ respectively.

5.16 Answer A

TMUA 2017 · Paper 1 · Q18

We can do this algebraically.

A translation of the graph of $y = \mathrm{f}(x)$ by 2 units in the positive y-direction gives the graph of $y = \mathrm{f}(x) + 2$

A stretch of factor k parallel to the x-axis gives the graph of $y = \mathrm{f}(\frac{1}{k}x) = \mathrm{f}(\frac{x}{k})$

For these to be the same, we need $\mathrm{f}(x) + 2 = \mathrm{f}(\frac{1}{k}x)$ for all (positive) x.

In our case, $\mathrm{f}(x) = \log_{10} x$, so we require

$$\log_{10} x + 2 = \log_{10}(\tfrac{x}{k})$$

But $\log_{10}(\frac{x}{k}) = \log_{10} x - \log_{10} k$, so we require $\log_{10} k = -2$, giving $k = 10^{-2} = 0.01$. Hence the answer is A.

We have done more than required to answer this question: we have actually shown that these transformations are equivalent. An alternative approach is to assume that the question is correct and that these transformations are equivalent, and to use a single point to find the value of k.

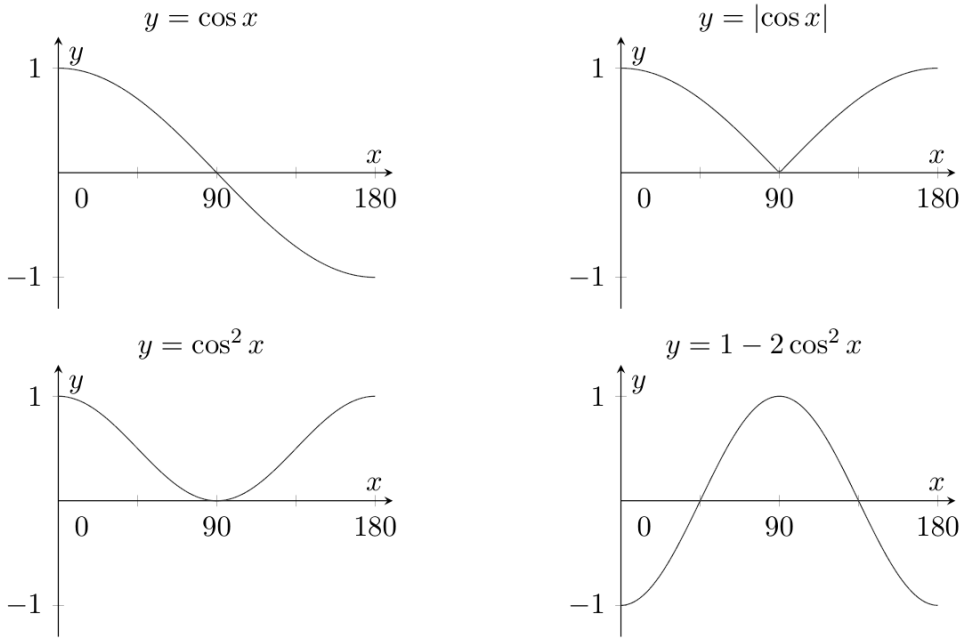
We could see which point ends up on the x-axis after the translation: it was at $(a, -2)$, so it must have been at $(0.01, -2)$. Therefore the translated graph crosses the x-axis at 0.01, while the original graph crossed the x-axis at 1. Therefore the stretch needs to be by a factor of 0.01 parallel to the x-axis.

We could have done a similar calculation with any other point.

Approach 1: graph sketching

One way to start this question is to sketch the two graphs $y = 1 - 2\cos^2 x$ and $y = |\cos x|$. It may not be obvious how to sketch the first of these graphs (unless you happen to know about double-angle formulae), but we can sketch something adequate for this question.

We can sketch the graph of $y = \cos x$, then use that to sketch both $y = |\cos x|$ and $y = \cos^2 x$, and then the latter one can be used to sketch $y = 1 - 2\cos^2 x$. Squaring $\cos x$ gives a graph which is never negative, and is zero when $\cos x = 0$. So the graphs look like this:



Imagining overlaying the graphs of $y = |\cos x|$ and $y = 1 - 2\cos^2 x$, we see that there are two points of intersection symmetrically placed around 90°, so their sum is 180°, which is option A.

Approach 2: using algebra

We solve the two equations $1 - 2\cos^2 x = \cos x$ when $\cos x \geq 0$ and $1 - 2\cos^2 x = -\cos x$ when $\cos x < 0$.

First, $1 - 2\cos^2 x = \cos x$ when $\cos x \geq 0$ gives $2\cos^2 x + \cos x - 1 = 0$, which factorises as $(2\cos x - 1)(\cos x + 1) = 0$. This has solutions $\cos x = \frac{1}{2}$ and $\cos x = -1$. We must reject the second solution as we require $\cos x \geq 0$, so we have the solution $x = 60°$ (in the range $0° \leq x \leq 180°$).

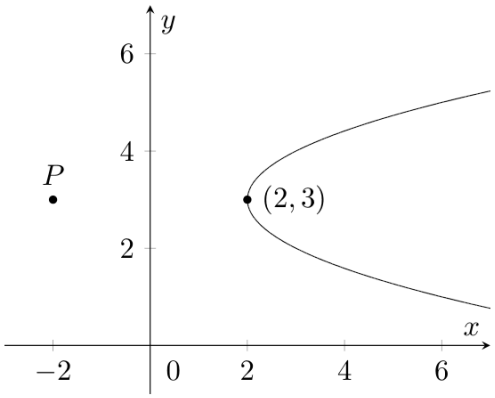
Next, $1 - 2\cos^2 x = -\cos x$ when $\cos x < 0$ gives $2\cos^2 x - \cos x - 1 = 0$, which factorises as $(2\cos x + 1)(\cos x - 1) = 0$. This has solutions $\cos x = -\frac{1}{2}$ and $\cos x = 1$, of which only the first is valid. This gives the solution $x = 120°$.

Therefore there are two solutions and their sum is $60° + 120° = 180°$.

We should start by sketching the original curve. This is a quadratic in y. It does not factorise, but we can complete the square:

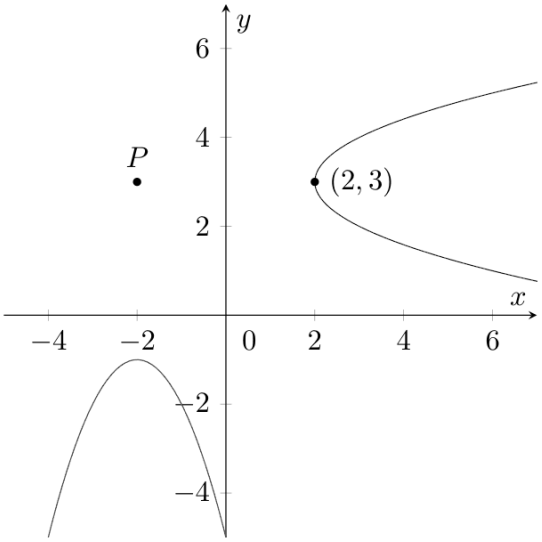
$$x = (y-3)^2 + 2$$

It therefore has its vertex at $y = 3$, $x = 2$, and $x \geq 2$ for all values of y. We can then sketch it:



The vertex of the parabola is 4 units to the right of P.

Now rotating this by 90° clockwise about P moves the vertex to 4 units below P, that is $(-2, -1)$, giving the following:



This curve then has equation $y = -(x+2)^2 - 1$, which we can expand to $y = -x^2 - 4x - 5$ giving option B.

The equation $f(x) = g(x)$ is a polynomial equation with degree 5, so has at most 5 distinct real solutions; we wish there to be 5 such solutions for all values of p.

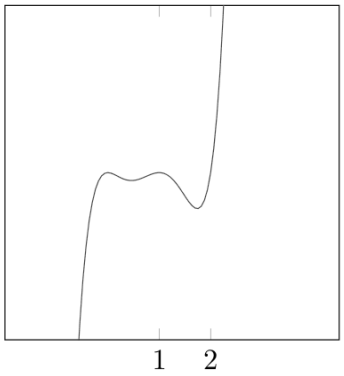
We would have fewer than 5 distinct real solutions if we either have a square factor of the form $(x-a)^2$ for some a, or if the graph of $y = f(x) - g(x)$ has fewer than four turning (stationary) points, or if this graph has a minimum point above the x-axis or a maximum point below it, for then it would not cross the x-axis 5 times.

Let us write

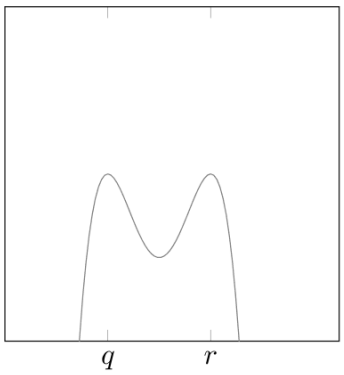
$$h(x) = f(x) - g(x) = x^2(x-1)^2(x-2) + p(x-q)^2(x-r)^2$$

We can try expanding and differentiating this, but that looks horrible. Let us think about the above description, though, remembering that there need to be 5 distinct real solutions for *all* values of p.

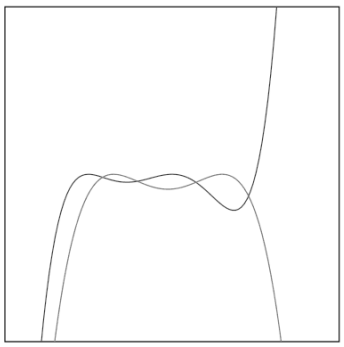
The graph of $y = f(x) = x^2(x-1)^2(x-2)$ intersects the x-axis at $x = 0$ (where it is tangent to the axis, because it is a double root), $x = 1$ (likewise) and $x = 2$, and it is positive when $x > 2$ and negative or zero when $x < 2$, so the graph looks like this:



The graph of $y = g(x) = -p(x-q)^2(x-r)^2$ has two points of intersection with the x-axis, one at $x = q$ and one at $x = r$, and at both of these points, the graph is tangent to the axis. (In the case $q = r$, there is just one point of tangency.) The graph is entirely below or touching the x-axis as we are given that $p > 0$. It looks like this:



large p. So we must have $1 < r < 2$. If $1 < q < r$, then it will miss the left valley for large enough p, so we must have $0 < q < 1$ and $1 < r < 2$. In this case, the graph of $y = g(x)$ must intersect both valleys twice as $g(1) < 0$:



This sketch shows four points of intersection, and we are looking for five. But the graph of $y = \mathrm{f}(x)$ tends to $-\infty$ as x tends to $-\infty$ far faster that $y = \mathrm{g}(x)$, regardless of the value of p, as $y = \mathrm{f}(x)$ is a quintic but $y = \mathrm{g}(x)$ is only a quartic. So there will always be another point of intersection with $x < 0$.

Hence the correct answer is option B.

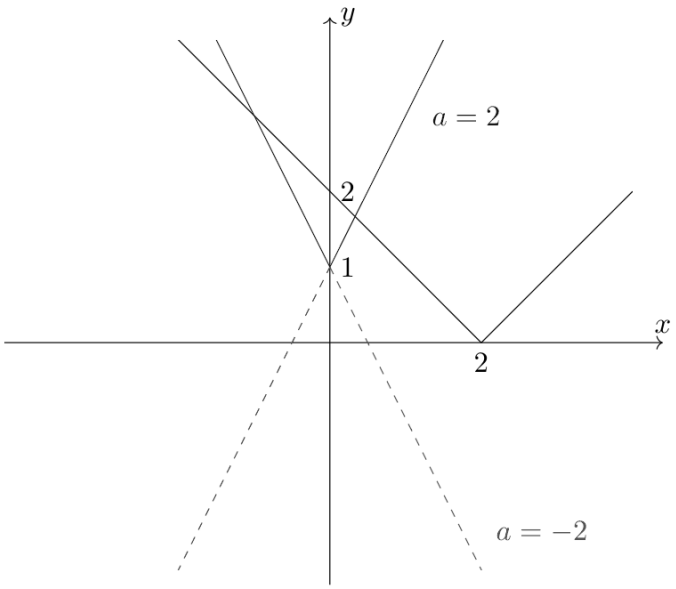
Note that there is actually a small simplification in the above argument. We showed that we cannot have $1 < q < r$ *and deduced that we must have* $0 < q < 1$*. But what about the possibility* $q = 1$*? In that case, there would be a double root at* $x = 1$*, and so there would not be five distinct roots, hence we cannot have* $q = 1$*. But what about* r*? We showed that we cannot have* $r > 2$ *or* $r < 1$*. If* $r = 1$*, we have the same problem, but what about* $r = 2$ *and* $0 < q < 1$*? In this case, it is still true that the equation has five distinct real solutions for all positive* p*, as* $x = 2$ *is only a single root of* $\mathrm{f}(x) = 0$*. So the full set is actually* $q < 1$ *and* $1 < r \leq 2$*.*

We would do well to sketch a graph for this question! There are two functions involved, $y = a|x| + 1$ and $y = |x - 2|$, so we sketch these for different values of a.

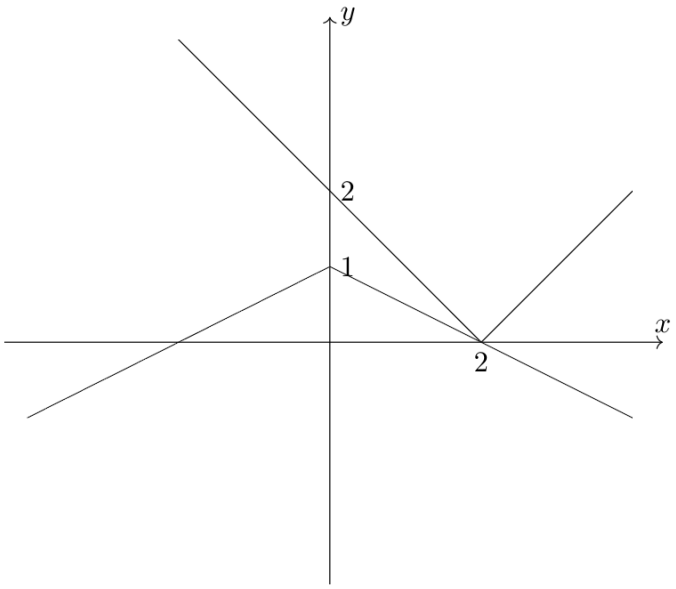
The graph of $y = a|x| + 1$ is the graph of $y = |x|$ first stretched by a factor of a in the y-direction, and then translated by 1 in the positive y-direction.

The graph of $y = |x - 2|$ is the graph of $y = |x|$ translated by 2 in the positive x-direction.

This gives us the following sketch:



What we can see from this sketch is that $y = a|x| + 1$ will be below $y = |x - 2|$ when a is sufficiently negative; when a is positive, it will certainly intersect the graph of $y = |x - 2|$. We can find the value of a at which the behaviour changes by sketching the graph at this point and then doing some algebra:



We can work out the value of a by noting that the graph passes through $(2, 0)$:

$$a|2| + 1 = 0$$

therefore $a = -\frac{1}{2}$, and hence the correct answer is $a \leq -\frac{1}{2}$, which is option E.

The first equation can be rewritten as

$$\mathrm{f}(x) = x^3 + ax^2 - bx - c = 0 \tag{1}$$

and the second equation as

$$\mathrm{g}(x) = x^3 - ax^2 - bx + c = 0. \tag{2}$$

Comparing these two equations, we see that the coefficient of x^2 and the constant term have been negated. We can also negate equation (2) to give

$$-\mathrm{g}(x) = -x^3 + ax^2 + bx - c = 0. \tag{3}$$

This is similar to equation (1), but now the coefficients of x^3 and x have been negated. This indicates that we can relate this to equation (1) by substituting in $-x$. We find that

$$\mathrm{f}(-x) = (-x)^3 + a(-x)^2 - b(-x) - c = -x^3 + ax^2 + bx - c = -\mathrm{g}(x)$$

and so equation (3) becomes

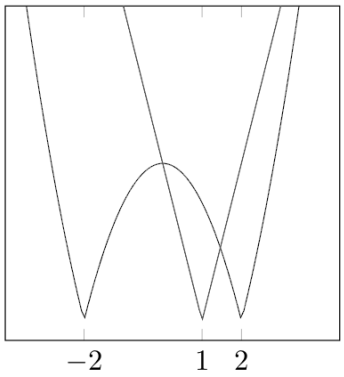
$$\mathrm{f}(-x) = 0.$$

Since we are told that $\mathrm{f}(x) = 0$ has three real roots, one positive and two negative, and $\mathrm{f}(-x) = 0$ must have three real roots, one negative and two positive. (It might be helpful to note here that $y = \mathrm{f}(-x)$ is the reflection of $y = \mathrm{f}(x)$ in the y-axis.)

Hence the answer is B.

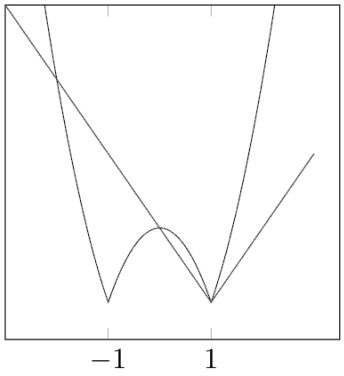
The graph $y = |x^2 - a^2|$ intersects the x-axis at $\pm a$. The graph $y = a^2\,|x - 1|$ intersects the x-axis at $x = 1$. Replacing a with $-a$ does not change the equations, so we may as well assume that $a \geq 0$.

Here is a sketch of the graphs with $a = 2$; we note that the graphs always intersect at $x = 0$ (and $y = a^2$ there):



We immediately see that when $a > 1$, there will be 4 points of intersection: the graph $y = a^2|x-1|$ is below the "hill" at $(1, 0)$, so intersects it twice as x moves away from $x = 1$, and then intersects the quadratic $y = x^2 - a^2$ for some $x < -a$ and some $x > a$ as the quadratic grows faster than the linear $y = -a^2(x - 1)$ or $y = a^2(x - 1)$ in these regions.

What happens when $a = 1$? In this case, the equations become $y = |x^2 - 1|$ and $y = x - 1$. The derivative of the quadratic $y = x^2 - 1$ is $2x$, so the gradient is just over 2 when x is just over 1. (There is no gradient at the point $x = 1$.) Therefore the graph looks like this:



We see that we have lost one point of intersection: there are now only three points of intersection.

What happens if $0 < a < 1$? Now the vertex of $y = a^2|x - 1|$ will be to the right of $x = a$, where the quadratic meets the axis, and so there will be no points of intersection there – the quadratic grows too quickly to meet the linear function for $x > 1$. (To be more precise: the gradient of the quadratic to the right of $x = a$ is $2x > 2a > a^2$, so the quadratic will be growing faster than the linear function.)

5.23 Answer F

TMUA 2023 · Paper 1 · Q20

Let us first consider the function $y = g(x)$, where $g(x) = x^2 - x$; we will then substitute $f(x)$ into this expression once we understand the behaviour of $y = g(x)$.

$g(x)$ is a quadratic and we can complete the square: $g(x) = (x - \frac{1}{2})^2 - \frac{1}{4}$. So $y = g(x)$ has a minimum value of $y = -\frac{1}{4}$ at $x = \frac{1}{2}$.

When we now substitute the values of $f(x)$ into this completed-square form for $g(x)$, obtaining $\big(f(x)\big)^2 - f(x)$. We can see from the graph that there is some value of x with $f(x) = \frac{1}{2}$, so this expression has minimum value of $-\frac{1}{4}$.

For the maximum value, we see that the value of $g(x)$ increases symmetrically either side of $x = \frac{1}{2}$. So the maximum value of $\big(f(x)\big)^2 - f(x)$ will either occur when $f(x) = 2$ or when $f(x) = -2$. Since -2 is further from $\frac{1}{2}$ than 2 is, $f(x) = -2$ will give the maximum value, which is $(-2)^2 - (-2) = 6$. (And if we hadn't made that observation, we could have calculated $2^2 - 2 = 2$ and seen that it is smaller than 6.)

Therefore the difference between the maximum and minimum values is $6 - (-\frac{1}{4}) = \frac{25}{4}$, which is option F.

5.24 Answer B

TMUA 2019 · Paper 2 · Q20

> ***Commentary:*** *It could be helpful for this question to take a piece of paper and physically move it around following the given instructions to get a feel for what's going on. This suggests that we may be looking for a periodic function.*
>
> *We could then try to see what happens with a sine or cosine function, but these turn out to have too much symmetry, and could lead us to the wrong answer. A tan graph is periodic, but does not have line symmetry, so is likely to be better for our needs.*
>
> *With all of this, though, we might still get ourselves in quite a muddle, so beginning with an algebraic approach is likely to be more successful.*

When we reflect the graph of $y = \mathrm{f}(x)$ in the y-axis, we obtain the graph of $y = \mathrm{f}(-x)$. When we translate this by 2 units in the negative x-direction, we obtain the graph of $y = \mathrm{f}(-(x + 2))$, so $\mathrm{g}(x) = \mathrm{f}(-(x + 2)) = \mathrm{f}(-x - 2)$.

Similarly, when we translate the graph of $y = \mathrm{f}(x)$ by 2 units in the negative x-direction, we obtain the graph of $y = \mathrm{f}(x + 2)$. When we reflect this in the y-axis, we obtain the graph of $y = \mathrm{f}(-x + 2)$, so $\mathrm{h}(x) = \mathrm{f}(-x + 2)$.

If the functions $\mathrm{g}(x)$ and $\mathrm{h}(x)$ are identical, then we require

$$\mathrm{f}(-x - 2) = \mathrm{f}(-x + 2) \text{ for all } x.$$

This is quite close to what we want, so let us let $u = -x - 2$, so that $-x + 2 = u + 4$. Then the above becomes

$$\mathrm{f}(u) = \mathrm{f}(u + 4) \text{ for all } u,$$

which is option B, just with x replaced by u.

This means that the function is periodic with a period of 4 (though it may have a smaller period such as 2 or $\frac{1}{7}$, as long as it repeats every 4).

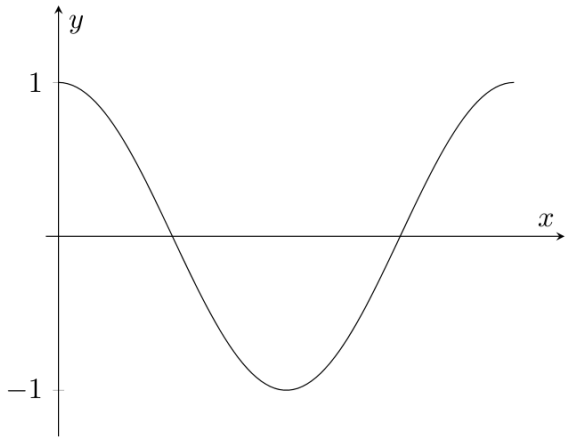
We note that we can rewrite each of the functions f_2 to f_5 in terms of the previous function:

$$\begin{aligned} f_1(x) &= \cos x \\ f_2(x) &= \sin f_1(x) \\ f_3(x) &= \cos f_2(x) \\ f_4(x) &= \sin f_3(x) \\ f_5(x) &= \cos f_4(x) \end{aligned}$$

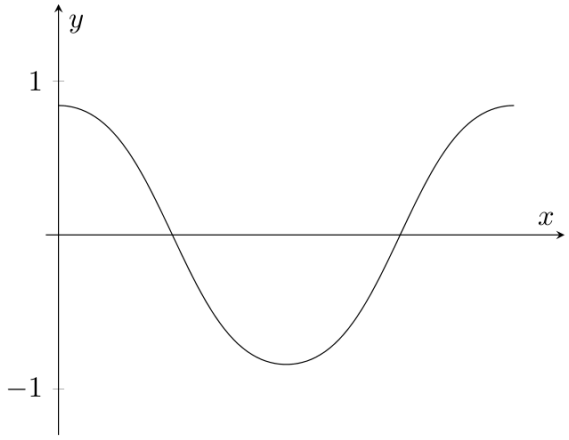
Since $f_1(x) = \cos x$ is periodic with period 2π, each of the other functions also repeats every 2π (though it may have a shortest period less than this), and so we only need to consider $0 \leq x \leq 2\pi$. (We could actually restrict ourselves to $0 \leq x \leq \pi$, as the values of $\cos x$ in the range $\pi < x \leq 2\pi$ are a repeat of the values in the range $0 \leq x \leq \pi$, but we will draw a complete period anyway.)

Sketching the graph of each of these functions is the most straightforward way to proceed. We note first a result that we will need later: $\pi \approx 3.14$ so $\frac{\pi}{2} \approx 1.6$ and hence $\sin 1 < \sin \frac{\pi}{2} = 1$.

A sketch of $f_1(x) = \cos x$ is straightforward:



We see that $m_1 = 1$, and the minimum value taken is -1. Now $f_2(x) = \sin f_1(x)$, so this goes between $-\sin 1$ and $\sin 1$, with roots where $f_1(x) = 0$; recall that $\sin 1 < 1$:



Probability

6.1 Answer C

TMUA 2021 · Paper 2 · Q3

The probability is given as a fraction in its lowest terms.

The probability that a student does not wear glasses is $\frac{11}{15}$, so the ratio of glasses to non-glasses is $4 : 11$, in its lowest terms.

Therefore the number of glasses/non-glasses wearers could be 4 and 11 (total 15), or 8 and 22 (total 30), or ...

Necessary means that it follows as a consequence.

Therefore I and III are not necessary (there could be 22 non-glasses wearers or the class could contain 15 students.
The total must be a multiple of 15, and so the number of students is divisible by 3.

The correct answer is therefore C.

6.2 Answer C

TMUA Specimen · Paper 1 · Q7

We could do this using a tree diagram, but it would have 3×3 branches, which is somewhat unwieldly in a short test.

Here are two alternative approaches.

Approach 1: use symmetry

The first ball can be anything; it does not matter what. We are left with $n-1$ balls of the same colour and $2n$ balls of different colours; there are $3n-1$ balls left in total. (This is because there are equal numbers of each colour to begin with. Were there different numbers of different colours, we would have to treat each colour separately.)

So the probability that the two balls are not the same colour is $\dfrac{2n}{3n-1}$, which is option C.

Approach 2: use negation

The probability that the two balls not the same colour is 1 minus the probability that they are the same colour. This gives

$$1-\left(\frac{n}{3n}\times\frac{n-1}{3n-1}+\frac{n}{3n}\times\frac{n-1}{3n-1}+\frac{n}{3n}\times\frac{n-1}{3n-1}\right)$$

where we have added the three colours separately. This simplifies to

$$1-\frac{n-1}{3n-1}=\frac{2n}{3n-1}.$$

We could have simplified the calculation in the same way as in approach 1: the probability that the second ball is the same colour as the first is $\dfrac{n-1}{3n-1}$, so the probability that they are different colours is 1 minus this.

6.3 Answer B

There are two factors for each member: whether the member is a man or woman, and whether the member plays tennis or cricket. We can either display this situation as a tree diagram or as a two-way table. Since the information given has both the fraction of male members who play cricket and the fraction of the cricketing members who are women, it will be simpler to use a two-way table. (In a tree diagram, we could easily display one of the two, but the other will be hard.)

We could also work with fractions or with numbers of people. In this solution, we will work with numbers. The 60% suggests that we work with a multiple of 100 people, and the presence of a fraction $\frac{2}{3}$, suggests that we work with a multiple of 3. So we start with 300 people. This is the information we obtain from the 60% of members are women and $\frac{2}{5}$ of the resulting 120 male members play cricket:

	Men	Women	Total
Tennis	72		
Cricket	48		
Total	120	180	300

We are also told that $\frac{2}{3}$ of the cricketing members are women, so the 48 cricketing men are $\frac{1}{3}$ of the cricketers, hence 96 cricketers are women, leaving 84 women to play tennis:

	Men	Women	Total
Tennis	72	84	(156)
Cricket	48	96	(144)
Total	120	180	300

(The parenthesised figures are not needed to finish the question.)

Therefore the probability that a member of the club, chosen at random, is a woman who plays tennis is $\frac{84}{300} = \frac{28}{100} = \frac{7}{25}$, and the answer is B.

6.4 Answer F

One way to answer this question is to draw a tree diagram and to work out the probabilities of the three events.

Another way is to reason as follows.

If all three students study Mathematics, then certainly the second student chosen studies Mathematics and at least one of the students chosen studies Mathematics. Therefore $T < S$ and $T < R$.

Likewise, if the second student chosen studies Mathematics, certainly at least one of the students chosen studies Mathematics. Therefore $S < R$.

Putting these together gives $T < S < R$, and the answer is F.

Calculus 1

7.1 Answer C

TMUA 2017 · Paper 1 · Q1

We are given the derivative of y, and we want to find y itself, so we need to integrate. We start by expanding the fraction:

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= 3x^2 - \frac{2-3x}{x^3} \\ &= 3x^2 - 2x^{-3} + 3x^{-2}\end{aligned}$$

so we can now integrate term-by-term to get

$$y = x^3 + x^{-2} - 3x^{-1} + c$$

Now as $y = 5$ when $x = 1$, we can substitute to give

$$5 = 1^3 + 1^{-2} - 3 \times 1^{-1} + c = -1 + c$$

so $c = 6$ and the answer is C.

Commentary: *It turns out that there is a subtlety in this question which we usually ignore at school level. The question assumes that there is a 'simple' formula for y in terms of x. Since the function is not defined at $x = 0$, it is possible that we have different constants when $x > 0$ and when $x < 0$, so we could have, for example,*

$$y = \begin{cases} x^3 + x^{-2} - 3x^{-1} + 6 & \text{when } x > 0 \\ x^3 + x^{-2} - 3x^{-1} + 8 & \text{when } x < 0 \end{cases}$$

This may seem quite strange, but it would be a valid solution to the differential equation given in the question.

7.2 Answer D

TMUA 2018 · Paper 1 · Q1

We write the integrand (the thing being integrated) in terms of powers of x and split up the fraction in order to be able to compute the integral:

$$\begin{aligned}\int_1^4 \frac{3-2x}{x\sqrt{x}}\,\mathrm{d}x &= \int_1^4 \frac{3-2x}{x^{\frac{3}{2}}}\,\mathrm{d}x \\ &= \int_1^4 3x^{-\frac{3}{2}} - 2x^{-\frac{1}{2}}\,\mathrm{d}x \\ &= \left[\frac{3x^{-\frac{1}{2}}}{(-\frac{1}{2})} - \frac{2x^{\frac{1}{2}}}{\frac{1}{2}}\right]_1^4 \\ &= \left[-6x^{-\frac{1}{2}} - 4x^{\frac{1}{2}}\right]_1^4 \\ &= (-6 \times \tfrac{1}{2} - 4 \times 2) - (-6 \times 1 - 4 \times 1) \\ &= -11 - (-10) \\ &= -1\end{aligned}$$

so the correct answer is D.

7.3 Answer C

TMUA 2020 · Paper 1 · Q1

We start by writing the expression as a sum of powers of x:

$$\frac{x^3-5x^2}{2x\sqrt{x}}=\frac{x^3-5x^2}{2x^{\frac{3}{2}}}=\frac{x^{\frac{3}{2}}-5x^{\frac{1}{2}}}{2}=\tfrac{1}{2}x^{\frac{3}{2}}-\tfrac{5}{2}x^{\frac{1}{2}}$$

We can now differentiate this expression to get

$$\tfrac{1}{2}\times\tfrac{3}{2}x^{\frac{1}{2}}-\tfrac{5}{2}\times\tfrac{1}{2}x^{-\frac{1}{2}}$$

We now need to convert this into the form given in the options, so we write it as

$$\tfrac{3}{4}\sqrt{x}-\frac{\frac{5}{4}}{\sqrt{x}}=\frac{3\sqrt{x}}{4}-\frac{5}{4\sqrt{x}}=\frac{3x}{4\sqrt{x}}-\frac{5}{4\sqrt{x}}=\frac{3x-5}{4\sqrt{x}}$$

which is option C.

7.4 Answer F

TMUA 2023 · Paper 1 · Q1

We calculate the integrals in terms of a and b:

$$\begin{aligned}\int_0^1(ax+b)\,\mathrm{d}x&=\left[\tfrac{1}{2}ax^2+bx\right]_0^1\\&=\tfrac{1}{2}a+b\\&=1\end{aligned}$$

$$\begin{aligned}\int_0^1x(ax+b)\,\mathrm{d}x&=\int_0^1(ax^2+bx)\,\mathrm{d}x\\&=\left[\tfrac{1}{3}ax^3+\tfrac{1}{2}bx^2\right]_0^1\\&=\tfrac{1}{3}a+\tfrac{1}{2}b\\&=1\end{aligned}$$

Multiplying the first equation by 2 and the second by 6 gives the simultaneous equations

$$\begin{aligned}a+2b&=2\qquad(1)\\2a+3b&=6\qquad(2)\end{aligned}$$

Then $2\times(1)-(2)$ gives $b=-2$, so $a=6$. Therefore $a+b=4$ and the correct answer is option F.

7.5 Answer A

TMUA 2016 · Paper 2 · Q1

We begin by expanding the brackets, either using the binomial theorem or by multiplying them out by hand, to get

$$\begin{aligned}\int_1^2 \left(x^2+\frac{4}{x^2}\right)^2 \mathrm{d}x &= \int_1^2 x^4 - 8 + \frac{16}{x^4}\,\mathrm{d}x \\ &= \int_1^2 x^4 - 8 + 16x^{-4}\,\mathrm{d}x \\ &= \left[\frac{x^5}{5} - 8x + \frac{16x^{-3}}{-3}\right]_1^2 \\ &= \left(\frac{32}{5} - 8\times 2 - \frac{16/8}{3}\right) - \left(\frac{1}{5} - 8 - \frac{16}{3}\right) \\ &= \frac{31}{5} - 8 + \frac{14}{3} \\ &= \frac{93-120+70}{15} \\ &= \frac{43}{15}\end{aligned}$$

and so the answer is A.

7.6 Answer A

TMUA 2017 · Paper 2 · Q1

We first expand the brackets and split up the fraction:

$$\begin{aligned}y &= \frac{(1-3x)^2}{2x^{\frac{3}{2}}} \\ &= \frac{1-6x+9x^2}{2x^{\frac{3}{2}}} \\ &= \tfrac{1}{2}x^{-\frac{3}{2}} - 3x^{-\frac{1}{2}} + \tfrac{9}{2}x^{\frac{1}{2}}\end{aligned}$$

and we can now differentiate this to get

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \tfrac{1}{2}(-\tfrac{3}{2})x^{-\frac{5}{2}} - 3(-\tfrac{1}{2})x^{-\frac{3}{2}} + \tfrac{9}{2}\times\tfrac{1}{2}x^{-\frac{1}{2}} \\ &= -\tfrac{3}{4}x^{-\frac{5}{2}} + \tfrac{3}{2}x^{-\frac{3}{2}} + \tfrac{9}{4}x^{-\frac{1}{2}}\end{aligned}$$

and so the correct answer is A.

7.7 Answer C

TMUA 2018 · Paper 2 · Q1

We first simplify the fraction using index laws:

$$\begin{aligned}\mathrm{f}(x) &= \frac{x^3-4x}{2\sqrt{x}} \\ &= \frac{x^3-4x}{2x^{\frac{1}{2}}} \\ &= \tfrac{1}{2}x^{\frac{5}{2}} - 2x^{\frac{1}{2}}\end{aligned}$$

so

$$\mathrm{f}'(x) = \tfrac{5}{4}x^{\frac{3}{2}} - x^{-\frac{1}{2}}$$

Substituting $x=4$ then gives

$$\mathrm{f}'(4) = \tfrac{5}{4}\times 8 - \tfrac{1}{2} = 10 - \tfrac{1}{2} = 9.5$$

and so the answer is C.

7.8 Answer D

TMUA 2021 · Paper 2 · Q1

We write the integral as powers of x and then integrate:

$$\begin{aligned}\int_1^4 \left(3\sqrt{x}+\frac{4}{x^2}\right)\mathrm{d}x &= \int_1^4 3x^{\frac{1}{2}}+4x^{-2}\,\mathrm{d}x\\ &= \left[\frac{3x^{\frac{3}{2}}}{(\frac{3}{2})}+\frac{4x^{-1}}{-1}\right]_1^4\\ &= \left[2x^{\frac{3}{2}}-4x^{-1}\right]_1^4\\ &= (2\times 8-4\times\tfrac{1}{4})-(2\times 1-4\times 1)\\ &= 15-(-2)\\ &= 17\end{aligned}$$

7.9 Answer B

TMUA 2022 · Paper 2 · Q1

We differentiate to obtain

$$\frac{\mathrm{d}y}{\mathrm{d}x}=12x^3+12x^2+12x.$$

We solve the equation $\frac{\mathrm{d}y}{\mathrm{d}x}=0$ to find the number of stationary points:

$$\begin{aligned} & 12x^3+12x^2+12x=0\\ \text{if and only if}\quad & 12x(x^2+x+1)=0.\end{aligned}$$

So either $x=0$ or $x^2+x+1=0$. But this quadratic has discriminant $1^2-4\times 1=-3<0$, so it has no real solutions.

Thus there is only one stationary point, at $x=0$, and the answer is option B.

7.10 Answer C

TMUA 2017 · Paper 1 · Q2

We expand the brackets, writing everything as powers of x:

$$\begin{aligned}\mathrm{f}(x) &= \left(\frac{2}{x}-\frac{1}{2x^2}\right)^2\\ &= \frac{4}{x^2}-2\frac{2}{x}\frac{1}{2x^2}+\frac{1}{4x^4}\\ &= 4x^{-2}-2x^{-3}+\frac{1}{4}x^{-4}\end{aligned}$$

(Take care with the last simplification: writing $\frac{1}{4x^4}=4x^{-4}$ is a common error!)

Then differentiating gives

$$\mathrm{f}'(x)=-8x^{-3}+6x^{-4}-x^{-5}$$

and differentiating a second time then gives

$$\mathrm{f}''(x)=24x^{-4}-24x^{-5}+5x^{-6}$$

We can now subsitute $x=1$ to give

$$\mathrm{f}''(1)=24-24+5=5$$

so the correct answer is C.

7.11 Answer F

We first find the turning points by differentiation. We have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 - 6 = 0 \text{ if and only if } x^2 = 2$$

so $\alpha = -\sqrt{2}$ and $\beta = \sqrt{2}$.

We can now compute the integral:

$$\begin{aligned}\int_{\alpha}^{\beta} x^3 - 6x + 3\,\mathrm{d}x &= \left[\tfrac{1}{4}x^4 - 3x^2 + 3x\right]_{-\sqrt{2}}^{\sqrt{2}} \\ &= \left(\tfrac{1}{4}(\sqrt{2})^4 - 3(\sqrt{2})^2 + 3(\sqrt{2})\right) - \left(\tfrac{1}{4}(-\sqrt{2})^4 - 3(-\sqrt{2})^2 + 3(-\sqrt{2})\right) \\ &= 6\sqrt{2}\end{aligned}$$

where to get the last line, we note that the first terms in each bracket cancel with each other, and similarly for the second terms, so we are left with just $3\sqrt{2} - (-3\sqrt{2})$. This is option F.

7.12 Answer B

We expand the brackets and divide through by the denominator to write everything as a single power of x:

$$\begin{aligned}y &= \frac{(3x-2)^2}{x\sqrt{x}} \\ &= \frac{9x^2 - 12x + 4}{x^{\frac{3}{2}}} \\ &= 9x^{\frac{1}{2}} - 12x^{-\frac{1}{2}} + 4x^{-\frac{3}{2}}.\end{aligned}$$

Therefore

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= 9 \times \tfrac{1}{2}x^{-\frac{1}{2}} - 12 \times (-\tfrac{1}{2})x^{-\frac{3}{2}} + 4 \times (-\tfrac{3}{2})x^{-\frac{5}{2}} \\ &= \tfrac{9}{2}x^{-\frac{1}{2}} + 6x^{-\frac{3}{2}} - 6x^{-\frac{5}{2}}.\end{aligned}$$

Substituting in $x = 2$ gives us the gradient at $x = 2$:

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{9}{2} \times \frac{1}{\sqrt{2}} + 6 \times \frac{1}{2\sqrt{2}} - 6 \times \frac{1}{4\sqrt{2}} \\ &= \frac{9 + 6 - 3}{2\sqrt{2}} \\ &= \frac{6}{\sqrt{2}} \\ &= 3\sqrt{2},\end{aligned}$$

hence the answer is option B.

7.13 Answer B

TMUA 2016 · Paper 2 · Q2

We begin by expanding the brackets and simplifying the powers of x:

$$\begin{aligned} \mathrm{f}(x) &= \frac{2x^3 + 10x}{x^{\frac{3}{4}}} \\ &= 2x^{\frac{9}{4}} + 10x^{\frac{1}{4}} \end{aligned}$$

so

$$\begin{aligned} \mathrm{f}'(x) &= 2 \times \tfrac{9}{4}x^{\frac{5}{4}} + 10 \times \tfrac{1}{4}x^{-\frac{3}{4}} \\ &= \tfrac{9}{2}x^{\frac{5}{4}} + \tfrac{5}{2}x^{-\frac{3}{4}} \end{aligned}$$

which is option B.

7.14 Answer F

TMUA 2023 · Paper 2 · Q2

We first expand the brackets to obtain

$$\int_9^{16} \frac{1}{x} + 2 + x \,\mathrm{d}x - \int_9^{16} \frac{1}{x} - 2 + x \,\mathrm{d}x.$$

Since these integrals have the same limits, we can combine them into a single integral to get

$$\int_9^{16} \left(\frac{1}{x} + 2 + x\right) - \left(\frac{1}{x} - 2 + x\right) = \int_9^{16} 4 \,\mathrm{d}x = 4 \times (16 - 9) = 28,$$

which is option F.

Note that integration of $\frac{1}{x}$ is not on the specification for this test; it is, of course, possible to answer this question by calculating both integrals explicitly and then subtracting, but it is much more work than the solution just presented.

7.15 Answer C

TMUA 2016 · Paper 1 · Q3

We need to find the gradient of the curve at $(1, 2)$. We have $y = 2x^{-2}$ and so $\frac{\mathrm{d}y}{\mathrm{d}x} = -4x^{-3}$. At $x = 1$, this gives $\frac{\mathrm{d}y}{\mathrm{d}x} = -4$ and so the normal has gradient $\frac{1}{4}$. As it passes through $(1, 2)$, it has equation

$$y - 2 = \tfrac{1}{4}(x - 1).$$

It cuts the x-axis ($y = 0$) at P, so at P, we have $-2 = \frac{1}{4}(x - 1)$, giving $x = -7$, and so P has coordinates $(-7, 0)$.

It cuts the y-axis ($x = 0$) at Q, so at Q, we have $y - 2 = \frac{1}{4}(-1)$, giving $y = \frac{7}{4}$, and so Q has coordinates $(0, \frac{7}{4})$.

Thus PQ has length

$$\sqrt{7^2 + (\tfrac{7}{4})^2} = \frac{7}{4}\sqrt{4^2 + 1} = \frac{7\sqrt{17}}{4}$$

(where we have written $7 = \frac{7}{4} \times 4$ for the first equality).

Hence the answer is C.

7.16 Answer F

TMUA 2022 · Paper 1 · Q3

Using the first statement, $\mathrm{f}''(x) = a$ for all x, we can integrate to get $\mathrm{f}'(x) = ax + b$ for some constant b, and a further integration gives $\mathrm{f}(x) = \frac{1}{2}ax^2 + bx + c$ for some constant c.

The second statement then yields $\mathrm{f}(0) = c = 1$ and $\mathrm{f}(1) = \frac{1}{2}a + b + 1 = 2$, so $a + 2b = 2$.

Finally, we can now use the third statement:

$$\begin{aligned}\int_0^1 \mathrm{f}(x)\,\mathrm{d}x &= \left[\tfrac{1}{6}ax^3 + \tfrac{1}{2}bx^2 + x\right]_0^1 \\ &= \tfrac{1}{6}a + \tfrac{1}{2}b + 1 \\ &= 1\end{aligned}$$

so that $a + 3b = 0$.

Combining the simultaneous equations

$$\begin{aligned}a + 2b &= 2 \\ a + 3b &= 0\end{aligned}$$

gives $b = -2$ and $a = 6$, so the correct answer is option F.

7.17 Answer C

TMUA 2023 · Paper 1 · Q3

We work out the values of each of the individual integrals by splitting them into unit intervals:

$$\begin{aligned}\int_0^3 f(x)\,\mathrm{d}x &= \int_0^1 f(x)\,\mathrm{d}x + \int_1^2 f(x)\,\mathrm{d}x + \int_2^3 f(x)\,\mathrm{d}x \\ &= 1 + 2 + 3 \\ &= 6 \\ \int_1^3 f(x)\,\mathrm{d}x &= \int_1^2 f(x)\,\mathrm{d}x + \int_2^3 f(x)\,\mathrm{d}x \\ &= 2 + 3 \\ &= 5 \\ \int_2^3 f(x)\,\mathrm{d}x &= 3 \\ \int_4^3 f(x)\,\mathrm{d}x &= -\int_3^4 f(x)\,\mathrm{d}x \\ &= -4 \\ \int_5^3 f(x)\,\mathrm{d}x &= -\int_3^5 f(x)\,\mathrm{d}x \\ &= -\left(\int_3^4 f(x)\,\mathrm{d}x + \int_4^5 f(x)\,\mathrm{d}x\right) \\ &= -(4 + 5) \\ &= -9\end{aligned}$$

where for the final two integrals, we have used the rule $\int_a^b f(x)\,\mathrm{d}x = -\int_b^a f(x)\,\mathrm{d}x$.

Adding these gives the required sum:

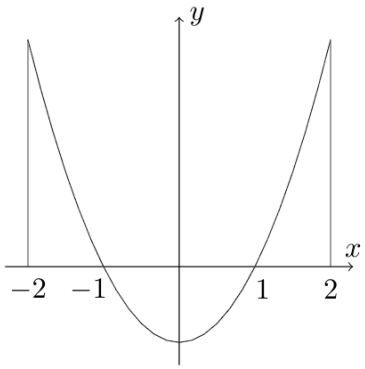
$$6 + 5 + 3 - 4 - 9 = 1$$

which is option C.

7.18 Answer C

To answer this question, it is worth drawing a sketch.

The graph is of $y = x^2 - 1 = (x+1)(x-1)$, so the parabola intersects the x-axis at $(1, 0)$ and $(-1, 0)$:



To find the area enclosed, we integrate over the three separate regions, from -2 to -1, from -1 to 1, and from 1 to 2:

$$\begin{aligned}\int_{-2}^{-1} x^2 - 1 \,\mathrm{d}x &= \left[\tfrac{1}{3}x^3 - x\right]_{-2}^{-1} \\ &= (-\tfrac{1}{3} + 1) - (-\tfrac{8}{3} + 2) \\ &= \tfrac{4}{3}\end{aligned}$$

$$\begin{aligned}\int_{-1}^{1} x^2 - 1 \,\mathrm{d}x &= \left[\tfrac{1}{3}x^3 - x\right]_{-1}^{1} \\ &= (\tfrac{1}{3} - 1) - (-\tfrac{1}{3} + 1) \\ &= -\tfrac{4}{3}\end{aligned}$$

$$\begin{aligned}\int_{1}^{2} x^2 - 1 \,\mathrm{d}x &= \left[\tfrac{1}{3}x^3 - x\right]_{1}^{2} \\ &= (\tfrac{8}{3} - 2) - (\tfrac{1}{3} - 1) \\ &= \tfrac{4}{3}\end{aligned}$$

Thus the three areas are each $\frac{4}{3}$, and the total area is 4, so the answer is C.

7.19 Answer F

We calculate the integral and use the laws of logarithms; we obtain

$$\begin{aligned}\int_{\log_2 5}^{\log_2 20} x \,\mathrm{d}x &= \left[\tfrac{1}{2}x^2\right]_{\log_2 5}^{\log_2 20} \\ &= \tfrac{1}{2}(\log_2 20)^2 - \tfrac{1}{2}(\log_2 5)^2 \\ &= \tfrac{1}{2}\left((\log_2 20)^2 - (\log_2 5)^2\right) \\ &= \tfrac{1}{2}\left((\log_2 4 + \log_2 5)^2 - (\log_2 5)^2\right) \\ &= \tfrac{1}{2}\left((2 + \log_2 5)^2 - (\log_2 5)^2\right) \\ &= \tfrac{1}{2}\left(4 + 4\log_2 5 + (\log_2 5)^2 - (\log_2 5)^2\right) \\ &= \tfrac{1}{2}(4 + 4\log_2 5) \\ &= 2 + 2\log_2 5 \\ &= \log_2 4 + \log_2 25 \\ &= \log_2 100\end{aligned}$$

so the answer is option F.

7.20 Answer C

TMUA 2018 · Paper 2 · Q6

A counterexample to an '**if** P **then** Q' statement is an example where the statement is **false**, that is where P (the antecedent) is **true** but Q (the consequent) is **false**.

We can now look through the given functions to see which of them satisfies these requirements. We give our results as a table and make comments on the working below.

	$f(x)$	$f'(x)$	$f'(x) > 0$ for all x	$f(x) > 0$ for all x
A	$x^2 + 1$	$2x$	**false**	
B	$x^2 - 1$	$2x$	**false**	
C	$x^3 + x + 1$	$3x^2 + 1$	**true**	**false**
D	$1 - x$	-1	**false**	
E	2^x	?	**true**	**true**

Once we know that $f'(x) > 0$ for all x is **false**, there is no need to go any further, so for A, B and D, we stop at this point.

For C, $3x^2 \geqslant 0$ for all x, so $3x^2 + 1 \geqslant 1 > 0$ for all x, and we know that cubics ($x^3 + x + 1$ in this case) always take both positive and negative values, so $f(x) > 0$ for all x is **false**.

For E, we don't know what the derivative of 2^x is (or at least, knowledge of it is not required for this test), but we do know that 2^x is increasing for all x, so the derivative is positive for all x.

Finally, only C has the antecedent **true** and the consequent **false**, so C provides a counterexample.

7.21 Answer A

TMUA 2020 · Paper 2 · Q6

If we think about the integrals as meaning 'the (signed) area under the graph of $y = f(x)$ between $x = -5$ and $x = 0$' and the same between $x = 0$ and $x = 5$, we realise that none of these conditions is necessary (though the first two conditions are actually sufficient).

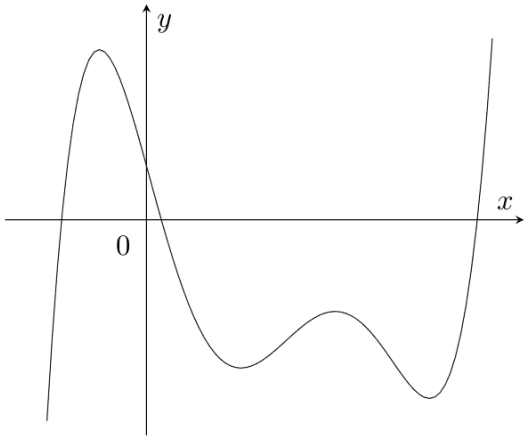
To show fully that a condition is not necessary, we need to find an example of a function for which the integral equation is satisfied but for which the condition is not.

An example of a function which satisfies the integral equation but does not satisfy conditions II or III is the function $f(x) = x^2$.

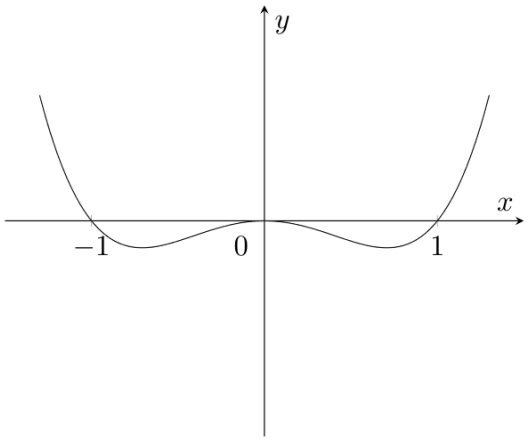
An example of a function which satisfies the integral condition but does not satisfy condition I is $f(x) = sin2\pi x$, which has 5 full periods in each of the ranges $-5 \leq x \leq 0$ and $0 \leq x \leq 5$, so each of the integrals is equal to 0, but it is not the case that $f(x) = f(-x)$ for $-5 \leq x \leq 5$.

Therefore the correct option is A.

If $f(x) = 0$ has exactly three roots then there must be a stationary point between the first and second root, and another stationary point between the second and third roots, but there may be other stationary points. For example, here is a sketch of a degree 5 polynomial $y = f(x)$ which has $f(x) = 0$ for three real values of x but $f'(x) = 0$ for five real values of x:

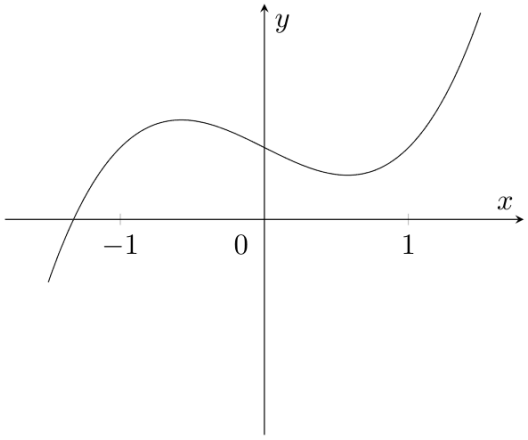


A second example is $f(x) = x^4 - x^2$. Here, the graph of $y = f(x)$ touches the x-axis at $x = 0$, so $f(x) = 0$ has exactly three real roots $x = -1$, 0, and 1, but three stationary points $x = \pm\frac{1}{\sqrt{2}}$ and $x = 0$:



So P is not sufficient for Q.

It is also not necessary, for example $f(x) = x^3 - x + 1$ has exactly two stationary points but only one real root:



So option D is correct.

We expand the given expression to get

$$y = 4q^2x - 2qx^3 + 6q - 3x^2$$

Differentiating gives

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 4q^2 - 6qx^2 - 6x$$

so the gradient of the curve at $x = -1$ is $4q^2 - 6q + 6$. We can minimise this either by using calculus or by completing the square.

Using calculus, we have $\frac{\mathrm{d}}{\mathrm{d}q}(4q^2 - 6q + 6) = 8q - 6$, and $8q - 6 = 0$ when $q = \frac{6}{8} = \frac{3}{4}$, so the answer is F.

Alternatively, completing the square, we have

$$4q^2 - 6q + 6 = 4(q^2 - \tfrac{3}{2}q) + 6 = 4\big((q - \tfrac{3}{4})^2 - (\tfrac{3}{4})^2\big) + 6$$

so the minimum occurs when $q = \frac{3}{4}$, as before.

Let us start with the condition $x\mathrm{f}(x) > 0$ for all non-zero values of x. When $x > 0$, this means that $\mathrm{f}(x) > 0$, as we can divide the inequality by x and the direction of the inequality is unchanged. On the other hand, when $x < 0$, this gives $\mathrm{f}(x) < 0$, since dividing by x now changes the direction of the inequality. In summary:

$$\begin{aligned}\mathrm{f}(x) &> 0 \text{ when } x > 0\\ \mathrm{f}(x) &= 0 \text{ when } x = 0\\ \mathrm{f}(x) &< 0 \text{ when } x < 0\end{aligned}$$

Now let us turn to the integrals. The final question asks about an integral from -2 to 0, so we would do well to split the two given integrals into the ranges -2 to 0 and 0 to 2:

$$\begin{aligned}\int_{-2}^{2} \mathrm{f}(x)\,\mathrm{d}x &= \int_{-2}^{0} \mathrm{f}(x)\,\mathrm{d}x + \int_{0}^{2} \mathrm{f}(x)\,\mathrm{d}x\\ &= 4\\ \int_{-2}^{2} |\mathrm{f}(x)|\,\mathrm{d}x &= \int_{-2}^{0} |\mathrm{f}(x)|\,\mathrm{d}x + \int_{0}^{2} |\mathrm{f}(x)|\,\mathrm{d}x\\ &= \int_{-2}^{0} -\mathrm{f}(x)\,\mathrm{d}x + \int_{0}^{2} \mathrm{f}(x)\,\mathrm{d}x\\ &= -\int_{-2}^{0} \mathrm{f}(x)\,\mathrm{d}x + \int_{0}^{2} \mathrm{f}(x)\,\mathrm{d}x\\ &= 8\end{aligned}$$

Therefore

$$\int_{-2}^{0} \mathrm{f}(x)\,\mathrm{d}x + \int_{0}^{2} \mathrm{f}(x)\,\mathrm{d}x = 4$$

$$-\int_{-2}^{0} \mathrm{f}(x)\,\mathrm{d}x + \int_{0}^{2} \mathrm{f}(x)\,\mathrm{d}x = 8$$

If we now add these two simultaneous equations together, the $\int_{-2}^{0} \mathrm{f}(x)\,\mathrm{d}x$ terms cancel and we are left with $2\int_{0}^{2} \mathrm{f}(x)\,\mathrm{d}x = 12$, so $\int_{0}^{2} \mathrm{f}(x)\,\mathrm{d}x = 6$ and $\int_{-2}^{0} \mathrm{f}(x)\,\mathrm{d}x = -2$.

The question asks us to evaluate $\int_{-2}^{0} \mathrm{f}(|x|)\,\mathrm{d}x$. Thinking about graph transformations, the graph of $y = \mathrm{f}(x)$ is transformed into the graph of $y = \mathrm{f}(|x|)$ by removing the left half of the graph (left of the y-axis) and reflecting the right half of the graph in the y-axis (the line $x = 0$). The integral of this transformed function from -2 to 0 is therefore equal to the integral of the original function from 0 to 2, thus

$$\int_{-2}^{0} \mathrm{f}(|x|)\,\mathrm{d}x = \int_{0}^{2} \mathrm{f}(x)\,\mathrm{d}x = 6$$

and the correct option is G.

7.25 Answer E

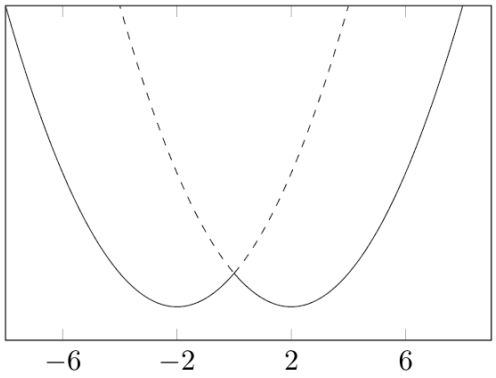
TMUA 2022 · Paper 1 · Q7

The line $y = 0$ is the x-axis; the curve $y = x^2 - 4|x| - 12$ is $y = x^2 - 4x - 12$ when $x \geq 0$ and $y = x^2 + 4x - 12$ when $x < 0$, so we sketch these two graphs.

The curve $y = x^2 - 4x - 12 = (x - 6)(x + 2)$ intersects the x-axis at $x = 6$ and $x = -2$ and has a minimum half way between them, at $x = 2$. Since we are only considering $x \geq 0$, we only observe one point of intersection with the x-axis.

Similarly, the curve $y = x^2 + 4x - 12 = (x + 6)(x - 2)$ intersects the x-axis at $x = -6$ and $x = 2$ and has a minimum half way between them, at $x = -2$. Since we are only considering $x < 0$, we only observe one point of intersection with the x-axis.

We can now sketch the graph, where we have shown the parts of the quadratics we are not using as dashed lines:



The finite area enclosed between the x-axis and the graph $y = x^2 - 4|x| - 12$ is therefore between $x = -6$ and $x = 6$. We could calculate the integrals of the two quadratics, one between $x = -6$ and $x = 0$ and the other between $x = 0$ and $x = 6$. But we could also notice that the graph is symmetrical about the y-axis, so we only need to find one of these and double the answer.

We have

$$\begin{aligned}\int_0^6 x^2 - 4x - 12\,\mathrm{d}x &= \left[\frac{x^3}{3} - 2x^2 - 12x\right]_0^6 \\ &= \frac{6^3}{3} - 2 \times 6^2 - 12 \times 6 - 0 \\ &= -72\end{aligned}$$

so $\int_{-6}^6 x^2 - 4|x| - 12\,\mathrm{d}x = -144$ and the (unsigned) area required is 144, which is option E.

7.26 Answer C

TMUA 2021 · Paper 2 · Q8

If $\mathrm{p}(a) = \mathrm{p}(b) = c$, say, then if the graph of $y = \mathrm{p}(x)$ (strictly) increases as it leaves (a, c) towards $x = b$, but then it must decrease again to get back to (b, c), so there must be a stationary point between $x = a$ and $x = b$. Similarly, if the graph (strictly) decreases as it leaves (a, c), it must then increase again, which also results in a stationary point. The only other possibility is that the graph of $y = \mathrm{p}(x)$ is constant, in which case $\mathrm{p}'(x) = 0$ throughout. In all cases, $(*)$ holds, so $\mathrm{p}(a) = \mathrm{p}(b)$ is sufficient for $(*)$.

However, it is not necessary for $(*)$: if $\mathrm{p}(x) = x^2$ and $a = -1$, $b = 2$, then this satisfies $(*)$ (with $c = 0$) but $\mathrm{p}(a) \neq \mathrm{p}(b)$.

Thus the correct option is C.

Commentary: *We want the cubic equation $2x^3-3x^2-12x+c=0$ to have three distinct real solutions. We don't know any techniques for finding the roots of a cubic equation, so we will have to be creative. (Actually, there is a formula, often known as Cardano's formula, for solving a cubic equation, but you are not expected to know this.)*

What we do *know how to do is to sketch a cubic graph by finding its turning points and y-axis intercept, even if we don't know the x-axis intercepts, so we will do this.*

We can find the stationary points on the graph of $y = 2x^3 - 3x^2 - 12x + c$ by differentiating. We have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 6x^2 - 6x - 12 = 6(x^2 - x - 2) = 6(x-2)(x+1)$$

so there are stationary points at $x = 2$ and $x = -1$ (where $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$)

These have coordinates

$$(-1, 7+c) \quad \text{and} \quad (2, -20+c)$$

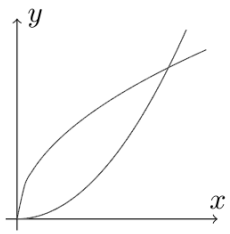
For the equation $2x^3 - 3x^2 - 12x + c = 0$ to have three distinct real solutions, we need the two turning points of the cubic to be on opposite sides of the x-axis, one above it and one below it. It is clear that $-20 + c < 7 + c$, so we need $-20 + c < 0$ and $7 + c > 0$. The first inequality becomes $c < 20$ and the second $c > -7$.

Combining these, there are three distinct real solutions if and only if $-7 < c < 20$, which is option B.

7.28 Answer D

TMUA 2019 · Paper 1 · Q9

We can rearrange $x = p\sqrt{y}$ to give $\sqrt{y} = \dfrac{x}{p}$, so $y = \dfrac{x^2}{p^2}$ (for $x \geq 0$). We can then sketch the two curves to see what is intended:



We need to find where these two curves intersect. We have

$$p\sqrt{x} = \frac{x^2}{p^2}$$

so $x^{\frac{3}{2}} = p^3$, giving $x^{\frac{1}{2}} = p$, so $x = p^2$ and thus $y = p^2$ as well; the point of intersection thus lies on the line $y = x$ and the graph is symmetrical about this line.

We can now find the area between the curves by integrating the difference from $x = 0$ to $x = p^2$:

$$\begin{aligned}
\text{area} &= \int_0^{p^2} p\sqrt{x} - \frac{x^2}{p^2}\,\mathrm{d}x \\
&= \left[\frac{px^{\frac{3}{2}}}{(\frac{3}{2})} - \frac{x^3}{3p^2}\right]_0^{p^2} \\
&= \frac{p.p^3}{(\frac{3}{2})} - \frac{p^6}{3p^2} - 0 \\
&= \tfrac{2}{3}p^4 - \tfrac{1}{3}p^4 \\
&= \tfrac{1}{3}p^4
\end{aligned}$$

so the answer is D.

7.29 Answer C

TMUA 2021 · Paper 2 · Q9

I Every cubic crosses the x-axis, so this condition is sufficient for $\mathrm{f}(x) = 0$ to have a real root. This also follows from III below if we take u to be a very large positive number and v to be a very large negative number.

II Consider, for example, $\mathrm{f}(x) = x^2 + 1$. This quadratic has $\mathrm{f}'(x) = 2x$ so $\mathrm{f}'(0) = 0$, but there is no real x for which $x^2 + 1 = 0$. So this condition is not sufficient.

III As $\mathrm{f}(u)\,\mathrm{f}(v) < 0$, we must have $\mathrm{f}(u) < 0$ and $\mathrm{f}(v) > 0$ or the other way round; either way, the graph of $y = \mathrm{f}(x)$ has values above and below the x-axis. Since $\mathrm{f}(x)$ is a polynomial, it must therefore cross the x-axis, and so the equation $\mathrm{f}(x) = 0$ has a real root. (If we did not know that $\mathrm{f}(x)$ was a polynomial, this would not be sufficient; consider $\mathrm{f}(x) = \frac{1}{x}$ for example.)

The correct option is therefore C.

7.30 Answer E

TMUA 2017 · Paper 1 · Q10

The gradient of the tangent at x is $f'(x) = -6p^2 + 6px - 3x^2$. Note that p is a constant, and so p^3 differentiates to 0.

The gradient of the tangent where $x = -1$ is therefore $-6p^2 - 6p - 3$, and so the gradient of the normal is the negative reciprocal of this, namely

$$M = \frac{-1}{-6p^2 - 6p - 3} = \frac{1}{3(2p^2 + 2p + 1)}$$

The greatest possible value of this occurs when the denominator of this fraction is as small as possible and positive. We can complete the square to find this:

$$2p^2 + 2p + 1 = 2(p + \tfrac{1}{2})^2 + \tfrac{1}{2}$$

so the minimum of this is $\frac{1}{2}$ (occurring when $p = -\frac{1}{2}$).

We could also have found this using calculus: $\frac{d}{dp}(2p^2 + 2p + 1) = 4p + 2$ which is zero when $p = -\frac{1}{2}$, giving a minimum value of $\frac{1}{2}$.

Therefore the greatest possible value of M is $\dfrac{1}{3 \times \frac{1}{2}} = \dfrac{2}{3}$ which is option E.

> ***Commentary:*** *Note that, in this case, the denominator of the fraction expression for M is always positive. If it was instead something like $p^2 - 1$, then there would not be a greatest possible value of M, as $p^2 - 1$ could get as close to zero as we like, so M could be as large as we like.*

7.31 Answer F

TMUA 2019 · Paper 1 · Q10

We do not have a rule for integrating $|x|$, so we will split this into two separate integrals, one from $x = -1$ to $x = 0$, where $|x| = -x$, and one from $x = 0$ to $x = 3$, where $|x| = x$. This gives

$$\begin{aligned}
\int_{-1}^{3} |x|(1-x)\,dx &= \int_{-1}^{0} |x|(1-x)\,dx + \int_{0}^{3} |x|(1-x)\,dx \\
&= \int_{-1}^{0} (-x)(1-x)\,dx + \int_{0}^{3} x(1-x)\,dx \\
&= \int_{-1}^{0} -x + x^2\,dx + \int_{0}^{3} x - x^2\,dx \\
&= \left[-\frac{x^2}{2} + \frac{x^3}{3}\right]_{-1}^{0} + \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_{0}^{3} \\
&= -(-\tfrac{1}{2} - \tfrac{1}{3}) + (\tfrac{9}{2} - 9) \\
&= \tfrac{5}{6} - \tfrac{9}{2} \\
&= -\tfrac{22}{6} \\
&= -\tfrac{11}{3}
\end{aligned}$$

so the correct answer is F.

To show that a condition is **sufficient**, we need to show that if the condition is satisfied, then $\int_1^3 \mathrm{f}(x)\,\mathrm{d}x = 0$

Note that it is not enough for us to find an example of a function which satisfies $\int_1^3 \mathrm{f}(x)\,\mathrm{d}x = 0$ and then to check whether the function satisfies the suggested conditions, as this would be checking whether the conditions are **necessary**, not whether they are **sufficient**.

A Knowing the value of $\mathrm{f}(x)$ at a single value of x cannot hope to force the integral to be 0; consider, for example, the quadratic $\mathrm{f}(x) = (x-2)^2$

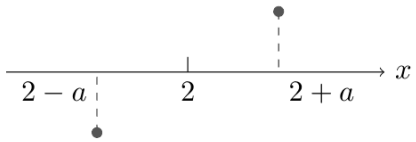
B Likewise, knowing the value of $\mathrm{f}(x)$ at the endpoints is not **sufficient**; consider, for example, the quadratic $\mathrm{f}(x) = (x-1)(x-3)$

C This says that the function is odd: the values of $\mathrm{f}(x)$ for negative values of x match those for positive values of x. But it gives no control over the values of $\mathrm{f}(x)$ between $x = 1$ and $x = 3$, so it cannot be a **sufficient** condition. For example, $\mathrm{f}(x) = x$ satisfies $\mathrm{f}(-x) = -\mathrm{f}(x)$, but $\int_1^3 x\,\mathrm{d}x > 0$

So it is one of D and E, both of which look complicated. E looks marginally easier to deal with, as $x - 2 = -(2-x)$. But that means that if we write $y = 2 - x$, condition E becomes $\mathrm{f}(-y) = -\mathrm{f}(y)$, which is exactly the same as condition C. So condition E is not **sufficient**.

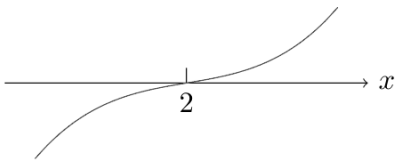
By elimination, therefore, the correct answer must be condition D.

Let us show that D is, indeed, **sufficient**. It would help to see what is going on if we write $x+2$ as $2+x$, so the condition becomes $\mathrm{f}(2+x) = -\mathrm{f}(2-x)$. If we take $0 < x \leqslant 1$, this shows that there is rotational symmetry about the point $(2, 0)$:



And when $x = 0$, we get $\mathrm{f}(2) = -\mathrm{f}(2)$ so $\mathrm{f}(2) = 0$

So the graph of $y = \mathrm{f}(x)$ itself might look like this:



Since the graph of $y = \mathrm{f}(x)$ has rotational symmetry about $(2, 0)$, the integrals $\int_1^2 \mathrm{f}(x)\,\mathrm{d}x$ and $\int_2^3 \mathrm{f}(x)\,\mathrm{d}x$ will be negative of each other, and so condition D is **sufficient** for $\int_1^3 \mathrm{f}(x)\,\mathrm{d}x = 0$.

We can find the equation of the normal to the curve $y = 10 - x^2$ at the point (p, q) as usual.

The derivative is $\dfrac{\mathrm{d}y}{\mathrm{d}x} = -2x$ so at $x = p$ the tangent has gradient $-2p$. Therefore the normal has gradient $\dfrac{1}{2p}$.

The tangent and normal both pass through the point $(p, 10 - p^2)$, which shows that $q = 10 - p^2$.
The normal thus has equation

$$y - (10 - p^2) = \frac{1}{2p}(x - p)$$

so rearranging into the form $y = mx + c$ gives

$$y = \frac{1}{2p}x - \frac{1}{2} + 10 - p^2$$

We want this to have the form $y = mx + 5$ where $m > 0$, so we have

$$-\frac{1}{2} + 10 - p^2 = 5.$$

Rearranging thus gives

$$p^2 = \frac{9}{2}$$

and hence $p = \pm\dfrac{3}{\sqrt{2}} = \pm\dfrac{3\sqrt{2}}{2}$

Since $m = \dfrac{1}{2p} > 0$ we must have $p > 0$ so $p = \dfrac{3\sqrt{2}}{2}$

The correct answer is C.

At first sight, it seems that there may not be enough information to answer the question, as we only know two points on the quadratic. But if we stretch the graph parallel to the y-axis, though we modify the quadratic, we will not change q, and the areas of R and S will remain equal. So let us proceed. For simplicity, we will write R and S for the areas of these regions.

The quadratic has equation $y = k(x-2)(x-q)$. By the above comments, we may as well take $k = 1$; if we leave k in the equation, the areas will each be multiplied by k, and when we equate them, we can then divide by k.

The area of R is

$$\begin{aligned} R &= \int_0^2 (x-2)(x-q)\,\mathrm{d}x \\ &= \int_0^2 x^2 - (q+2)x + 2q\,\mathrm{d}x \\ &= \left[\tfrac{1}{3}x^3 - \tfrac{1}{2}(q+2)x^2 + 2qx\right]_0^2 \\ &= \tfrac{8}{3} - 2(q+2) + 4q \\ &= 2q - \tfrac{4}{3} \end{aligned}$$

The area of S is as follows; note that the integral is negative, so we have to negate the integral to obtain the area.

$$\begin{aligned} S &= -\int_2^q (x-2)(x-q)\,\mathrm{d}x \\ &= -\int_2^q x^2 - (q+2)x + 2q\,\mathrm{d}x \\ &= -\left[\tfrac{1}{3}x^3 - \tfrac{1}{2}(q+2)x^2 + 2qx\right]_2^q \\ &= -\left(\tfrac{1}{3}q^3 - \tfrac{1}{2}(q+2)q^2 + 2q^2\right) + \left(\tfrac{8}{3} - 2(q+2) + 4q\right) \\ &= -\left(-\tfrac{1}{6}q^3 + q^2\right) + \left(2q - \tfrac{4}{3}\right) \\ &= \tfrac{1}{6}q^3 - q^2 + 2q - \tfrac{4}{3} \end{aligned}$$

As $R = S$, we have

$$2q - \tfrac{4}{3} = \tfrac{1}{6}q^3 - q^2 + 2q - \tfrac{4}{3}$$

and therefore $\frac{1}{6}q^3 - q^2 = 0$. As $q > 2$, we cannot have $q = 0$, so we can divide by q^2 to give $\frac{1}{6}q - 1 = 0$. It follows that $q = 6$, which is option E.

You may have noticed that the value of the integral at $x = 2$ appeared in both expressions and then cancelled out when we equated them. Another way to think about this is that as the integral from 0 to 2 is positive and the integral from 2 to q is negative, then if the areas R and S are the same, we must have the integral from 0 to q equal to 0. So we could have calculated:

$$\begin{aligned} 0 &= \int_0^q (x-2)(x-q)\,\mathrm{d}x \\ &= \int_0^q x^2 - (q+2)x + 2q\,\mathrm{d}x \\ &= \left[\tfrac{1}{3}x^3 - \tfrac{1}{2}(q+2)x^2 + 2qx\right]_0^q \\ &= -\tfrac{1}{6}q^3 + q^2 \end{aligned}$$

The function is decreasing when $f'(x) < 0$ (and possibly also at points where $f'(x) = 0$). We have

$$f(x) = x^{15/7} - x^{8/7} + x^{1/7}$$

so

$$\begin{aligned} f'(x) &= \tfrac{15}{7}x^{8/7} - \tfrac{8}{7}x^{1/7} + \tfrac{1}{7}x^{-6/7} \\ &= \tfrac{1}{7}x^{-6/7}(15x^2 - 8x + 1) \\ &= \tfrac{1}{7}x^{-6/7}(5x-1)(3x-1) \end{aligned}$$

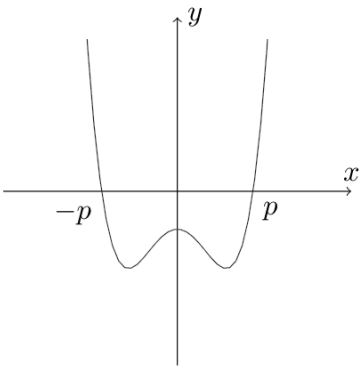
This is negative when $(5x-1)(3x-1) < 0$ (note that we are only considering the region $x > 0$ here).

We note that $(5x-1)(3x-1) = 0$ exactly when $x = \frac{1}{5}$ or $x = \frac{1}{3}$ and is therefore negative when $\frac{1}{5} < x < \frac{1}{3}$. Therefore the function $f(x)$ is decreasing when $\frac{1}{5} < x < \frac{1}{3}$ (and possibly also at the endpoints of this interval).

Thus the fraction of the interval $0 < x < 1$ for which $f(x)$ is decreasing is $\frac{1}{3} - \frac{1}{5} = \frac{2}{15}$, which is option A.

We know that the polynomial intersects the x-axis at $x = -p$ and $x = p$ only. It may touch the x-axis at those points or it may cross it – we are not told whether $x = p$ is a single root or double root, for example. We consider the three statements in turn.

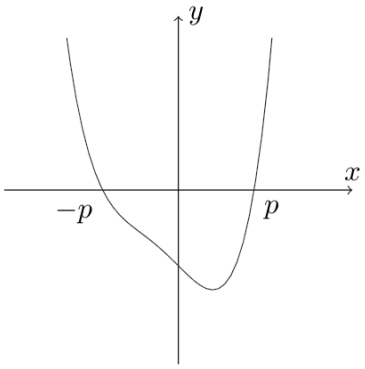
1 It is entirely possible that $y = \mathrm{f}(x)$ has multiple stationary points between $x = -p$ and $x = p$ without crossing the x-axis, for example in this sketch:



Therefore this statement need not be true.

2 This is not necessarily true for (at least) two different reasons:

- If the graph of $y = \mathrm{f}(x)$ is entirely below the x-axis between $x = p$ and $x = -p$, as in the above sketch, then the integral $\int_0^p \mathrm{f}(x)\,\mathrm{d}x$ will be negative, but the area is positive.
- If the graph of $y = \mathrm{f}(x)$ is not symmetrical about $x = 0$, then there is no reason that the integal $\int_{-p}^p \mathrm{f}(x)\,\mathrm{d}x$ will be twice $\int_0^p \mathrm{f}(x)\,\mathrm{d}x$. An example is this sketch, which is of the graph of $y = (x+1)(x-1)(x^2+x+1)$; here $p = 1$, and the graph of $y = x^2 + x + 1$ is always positive, but it is not symmetrical about $x = 0$:



3 The graph of $y = \mathrm{f}(-x)$ is obtained by reflecting the graph of $y = \mathrm{f}(x)$ in the y-axis, so this graph intersects the x-axis at $x = -p$ and $x = p$. The graph of $y = -\mathrm{f}(-x)$ is obtained from this by reflecting the graph in the x-axis, which does not change the points of intersection with the x-axis. Therefore this statement must be true.

The graph obtained by combining these two transformations is the rotation of the original graph about the origin by 180°.

We can approach this question algebraically or graphically.

We start with an algebraic approach. We first note that $\mathrm{f}(x) \geqslant 0$ for all $x > 0$ as $\mathrm{f}(x)$ is increasing. So the region described first, with area R, lies entirely above (or on) the x-axis, and

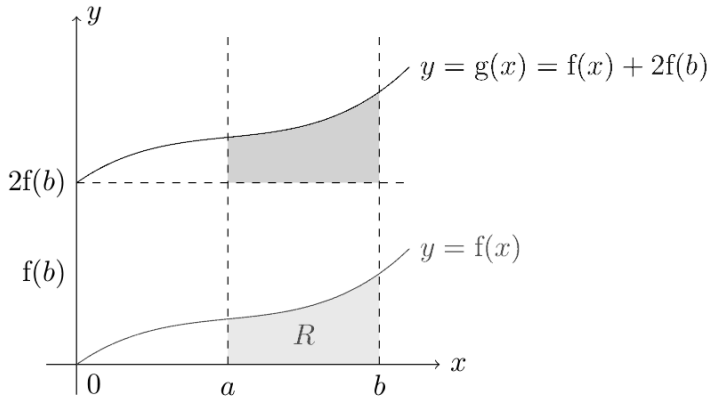
$$R = \int_a^b \mathrm{f}(x)\,\mathrm{d}x$$

We note that $\mathrm{g}(x) = \mathrm{f}(x) + 2\mathrm{f}(b) \geqslant 0$ for $x > 0$ as $\mathrm{f}(x) \geqslant 0$ and $\mathrm{f}(b) \geqslant 0$. Then the area required under the graph $\mathrm{g}(x)$ is given by

$$\begin{aligned}\int_a^b \mathrm{g}(x)\,\mathrm{d}x &= \int_a^b \mathrm{f}(x) + 2\mathrm{f}(b)\,\mathrm{d}x \\ &= \int_a^b \mathrm{f}(x)\,\mathrm{d}x + 2\mathrm{f}(b)\int_a^b 1\,\mathrm{d}x \\ &= R + 2\mathrm{f}(b)(b-a)\end{aligned}$$

On the second line we have split the integral into a sum of two integrals, and taken out the *constant* factor of $2\mathrm{f}(b)$ from the second integral. Thus the answer is option B.

We could also work graphically. We can sketch the graphs of $\mathrm{f}(x)$ and $\mathrm{g}(x)$, making up a suitable increasing shape for $\mathrm{f}(x)$:

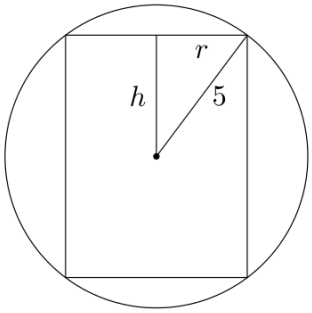


It is clear from this sketch that the area under the graph of $y = \mathrm{g}(x)$ is formed of a rectangle which is $(b-a)$ by $2\mathrm{f}(b)$ in size, and the area under the original graph, R, so that the required area is $R + (b-a) \times 2\mathrm{f}(b)$.

7.38 Answer E

TMUA 2016 · Paper 1 · Q12

Let the radius of the cylinder be r. Then the diagram is as follows (with all measurements in cm), where we have drawn in two extra lines:



The 5cm is the radius of the sphere and hence of the circle shown (as the cross section is through the centre of the sphere).

Pythagoras's theorem gives $h^2 + r^2 = 5^2$, and the volume of the cylinder is $V = \pi r^2(2h) = \pi(5^2 - h^2)(2h) = 2\pi(25h - h^3)$. We can maximise this by differentiating with respect to h. (We could write everything in terms of r instead, but that would involve square roots.)

We have $\dfrac{\mathrm{d}V}{\mathrm{d}h} = 2\pi(25 - 3h^2)$, which is zero when $25 = 3h^2$, so $h = \dfrac{5}{\sqrt{3}}$. Substituting this into the formula for V gives the largest possible V as

$$V = 2\pi(5^2 - h^2)h = 2\pi\left(25 - \frac{25}{3}\right)\frac{5}{\sqrt{3}} = \frac{500}{3\sqrt{3}}\pi = \frac{500\sqrt{3}}{9}\pi$$

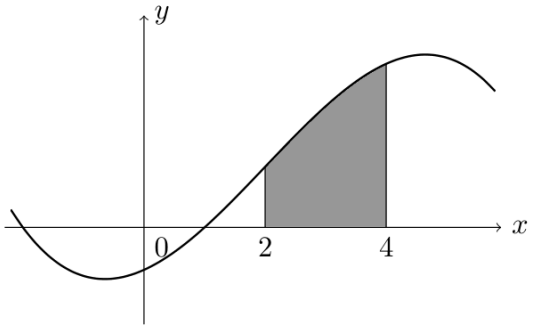
hence the answer is E.

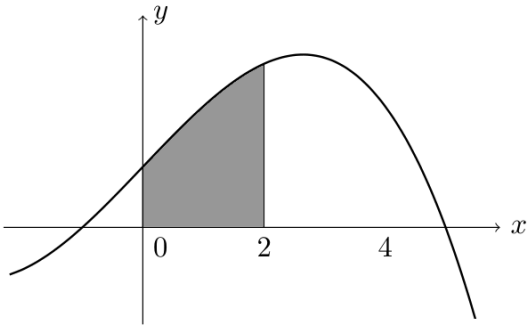
7.39 Answer B

TMUA 2017 · Paper 1 · Q12

We have $\int_2^4 \mathrm{f}(x)\,\mathrm{d}x = A$, and the question asks about $\int \mathrm{f}(x+2)\,\mathrm{d}x$ between various limits.

The graph of $y = \mathrm{f}(x+2)$ is the graph of $y = \mathrm{f}(x)$ translated left by 2 units, so we can draw a sketch to see what effect this has on the area between $x = 2$ and $x = 4$:





This means that $\int_2^4 \mathrm{f}(x)\,\mathrm{d}x = \int_0^2 \mathrm{f}(x+2)\,\mathrm{d}x = A$

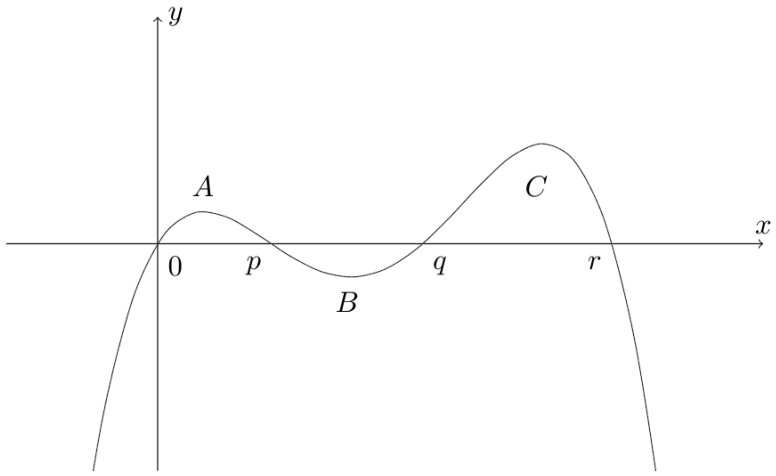
(If you have learnt about integration by substitution, you can also show this as follows: writing $x = u + 2$ gives $\mathrm{d}x = \mathrm{d}u$, and $u = x - 2$, so $\int_2^4 \mathrm{f}(x)\,\mathrm{d}x = \int_0^2 \mathrm{f}(u+2)\,\mathrm{d}u$)

Therefore the answer is either A or B. We have

$$\int_0^2 [\mathrm{f}(x+2) + 1]\,\mathrm{d}x = \int_0^2 \mathrm{f}(x+2)\,\mathrm{d}x + \int_0^2 1\,\mathrm{d}x$$

The first integral equals A, as we have shown, and $\int_0^2 1\,\mathrm{d}x = 2$, so the whole integral equals $A + 2$ and the correct answer is B.

This curve intercepts the x-axis at 0, p, q and r. As the leading power of x is $-x^4$, the curve tends to $-\infty$ as x gets very large (both positive and negative), and so the curve looks something like this:



We have labelled the three areas as A, B and C, and we want to know the value of $A + B + C$.

We can then write the three integrals we have been given in terms of these, referring to the sketch:

$$\int_0^r \mathrm{f}(x)\,\mathrm{d}x = A - B + C = 0 \tag{1}$$

$$\int_0^q \mathrm{f}(x)\,\mathrm{d}x = A - B = -2 \tag{2}$$

$$\int_p^r \mathrm{f}(x)\,\mathrm{d}x = -B + C = -3 \tag{3}$$

Subtracting (3) from (1) gives $A = 3$, from which (2) gives $B = 5$ and so $C = 2$. Therefore $A + B + C = 10$, hence the correct option is F.

> ***Commentary:*** *Although our sketch turns out to be not very accurate, it was sufficient to help us understand what was going on and to solve the problem. This is what we seek from sketches: they should be good enough to give us insight, but they do not need to be perfect.*
>
> *Incidentally, there do exist values of p, q and r which make the statements of this question true, namely $p = 1.25$, $q = 3.16$ and $r = 4.25$, correct to two decimal places.*

To find the value of V when $t = 9$, we need to integrate this expression. There is no general rule for integrating quotients such as these, but we might notice that $t - 1$ looks somewhat similar to the denominator $\sqrt{t} + 1$. In fact, we can use the difference of two squares to write $t - 1 = (\sqrt{t} + 1)(\sqrt{t} - 1)$, so we can rewrite the fraction to give

$$\frac{\mathrm{d}V}{\mathrm{d}t} = 24\pi(\sqrt{t} - 1)$$

and this is easy to integrate; we get

$$\begin{aligned} V &= \frac{24\pi t^{\frac{3}{2}}}{(\frac{3}{2})} - 24\pi t + c \\ &= 16\pi t^{\frac{3}{2}} - 24\pi t + c \end{aligned}$$

Substituting in $V = 7$ and $t = 1$ gives

$$7 = 16\pi - 24\pi + c$$

so $c = 8\pi + 7$. Then at $t = 9$, we have

$$V = 16\pi \times 9^{\frac{3}{2}} - 24\pi \times 9 + 8\pi + 7 = (16 \times 27 - 24 \times 9 + 8)\pi + 7 = 224\pi + 7$$

so the answer is option C.

Commentary: *The final calculation can be simplified somewhat by writing* $16 \times 27 = 16 \times 3 \times 9 = 48 \times 9$, *so the brackets become* $24 \times 9 + 8$. *Then* $24 \times 9 = 240 - 24 = 216$, *giving the final answer of* $224\pi + 7$.

7.42 Answer B

I We can think of the integral $\int_0^x \mathrm{f}(t)\,\mathrm{d}t$ as being the (signed) area under the graph of $y = \mathrm{f}(x)$ from 0 to x. Since $\mathrm{f}(x) \geq \mathrm{g}(x)$ for all $x \geq 0$, the area under the graph of $y = \mathrm{f}(x)$ will also be greater than that under the graph of $y = \mathrm{g}(x)$, so this statement is true.

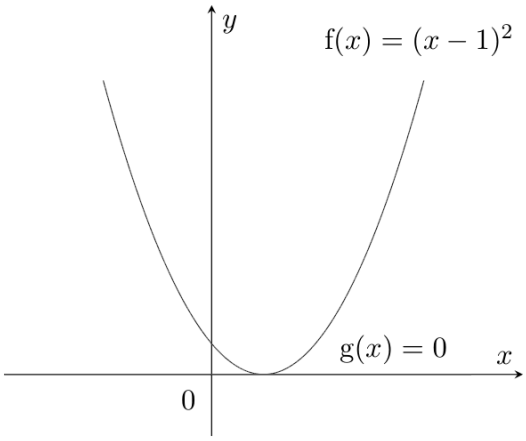
You might be concerned about values of x for which $\mathrm{g}(x)$ is negative. We could therefore demonstrate this algebraically as well.

We have

$$\int_0^x \mathrm{f}(t)\,\mathrm{d}t - \int_0^x \mathrm{g}(t)\,\mathrm{d}t = \int_0^x (\mathrm{f}(t) - \mathrm{g}(t))\,\mathrm{d}t.$$

Since $\mathrm{f}(t) - \mathrm{g}(t) \geq 0$ for all $x \geq 0$, we are integrating a non-negative function, and so the integral must be non-negative, that is to say, we must have $\int_0^x \mathrm{f}(t)\,\mathrm{d}t \geq \int_0^x \mathrm{g}(t)\,\mathrm{d}t$.

II It is not clear why $\mathrm{f}(x) \geq \mathrm{g}(x)$ should imply that the gradient (derivative) of $\mathrm{f}(x)$ is greater than that of $\mathrm{g}(x)$, and it is relatively straightforward to find a counterexample. Here is one:



In this example, $\mathrm{f}(x) \geq \mathrm{g}(x)$ for all x, but $\mathrm{f}'(0) < 0$ while $\mathrm{g}'(x) = 0$ for all x.

III This is not necessarily true, as we could add a constant to $\mathrm{g}(x)$ without changing $\mathrm{g}'(x)$. For example, we could have $\mathrm{f}'(x) = \mathrm{g}'(x) = 1$ for all x, but $\mathrm{f}(x) = x$ while $\mathrm{g}(x) = x + 1$, so $\mathrm{f}(x) < \mathrm{g}(x)$ for all x.

The correct option is therefore B.

7.43 Answer F

We do not know how to integrate any of these, so we will instead compare the integrands as the integrals are all from 0 to 1.

We have $\left(\sqrt{2}\right)^x = 2^{x/2}$, which makes R look a little simpler. Since $x/2 < x$ for $0 < x < 1$, $2^{x/2} < 2^x$ in this interval, and so $R < Q$.

Now in this interval, we also have $x < \sqrt{x}$ (as $x^2 < x$), so $2^x < 2^{\sqrt{x}}$, hence $Q < P$.

Combining these, we find that $R < Q < P$, so the correct option is F.

Commentary: *This is not something that you will have learnt how to do, and in general, this is a very difficult problem indeed. But we can use the tools at our disposal to do something helpful. We don't know how to solve a quintic (x^5) equation, but we can at least try to find the turning (stationary) points, and see where that gets us.*

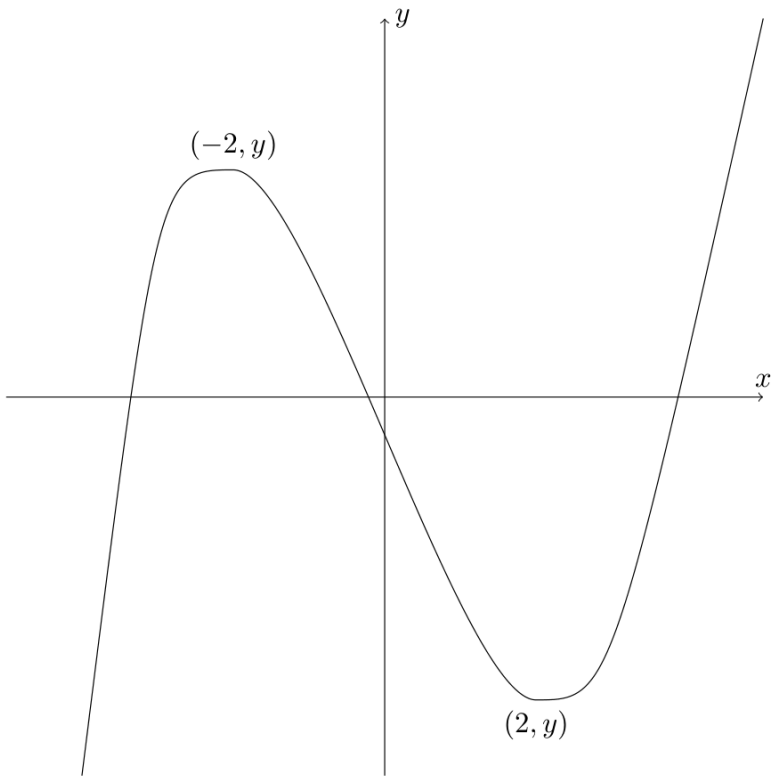
We attempt to sketch the graph of $y = 3x^5 - 10x^3 - 120x + 30$, or at least think about what a sketch might look like.

We calculate

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 15x^4 - 30x^2 - 120 = 15(x^4 - 2x^2 - 8) = 15(x^2 - 4)(x^2 + 2)$$

so there are stationary points when $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$, that is, when $x = 2$ and when $x = -2$. We can now calculate the y-coordinates: when $x = -2$, $y = -36 + 80 + 240 + 30 > 0$ (we do not need to calculate the exact value), and when $x = 2$, $y = 36 - 80 - 240 + 30 < 0$.

So the graph must look something like the following:



There are clearly 3 points of intersection of this graph with $y = 0$, so the original equation has 3 real roots, and the answer is C.

7.45 Answer C

TMUA 2018 · Paper 1 · Q13

A local minimum of $\mathrm{f}(x)$ occurs where the gradient is 0 at the point, is negative immediately to the left of the point and is positive immediately to its right.

So on the graph of $\mathrm{f}'(x)$, this will be at a point where $\mathrm{f}'(x) = 0$, and $\mathrm{f}'(x)$ is increasing from negative to positive at that point. The only such point on the graph given is C.

The behaviour of $\mathrm{f}(x)$ at all of the labelled points is as follows:

A Changing from positive to negative gradient, so a local maximum.

B A local minimum of the gradient, so a non-stationary point of inflection. If walking along the curve, one would stop turning to the right and start turning to the left.

C Changing from negative to positive gradient, so a local minimum.

D This point, where $x = 0$, has no particular relevance to the shape of $\mathrm{f}(x)$.

E A local maximum of the gradient, so a non-stationary point of inflection. If walking along the curve, one would stop turning to the left and start turning to the right.

F A local minimum of the gradient at a point of zero gradient, so a stationary point of inflection. If walking along the curve, one would stop turning to the right and start turning to the left.

7.46 Answer D

TMUA 2020 · Paper 2 · Q13

This question is about inequalities. To begin with, the only thing that we know for certain is that $\big(\mathrm{f}(x)\big)^2 \geq 0$ for all x, as it is the square of a real number. It follows that $\int_0^1 \mathrm{f}(x)\,\mathrm{d}x \geq \int_0^3 \mathrm{f}(x)\,\mathrm{d}x$, as we are adding a non-negative number to $\int_0^3 \mathrm{f}(x)\,\mathrm{d}x$ to get $\int_0^1 \mathrm{f}(x)\,\mathrm{d}x$. Therefore II is necessarily true.

Let us now take II and rewrite the integral from 0 to 3 as the sum of the integrals from 0 to 1 and from 1 to 3; this is so that we can match the integral on the right hand side. We obtain:

$$\int_0^1 \mathrm{f}(x)\,\mathrm{d}x + \int_1^3 \mathrm{f}(x)\,\mathrm{d}x \leq \int_0^1 \mathrm{f}(x)\,\mathrm{d}x$$

and we can now subtract the integral from 0 to 1 to get

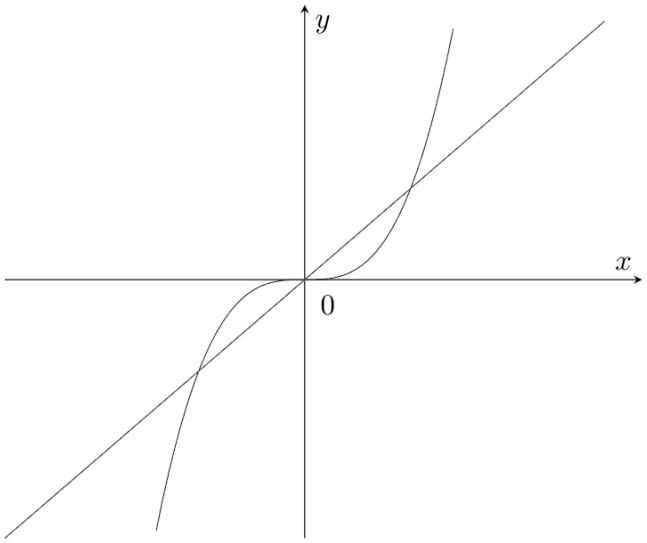
$$\int_1^3 \mathrm{f}(x)\,\mathrm{d}x \leq 0.$$

Since the integral is negative or zero, the function itself must be so somewhere between 1 and 3; if it were positive throughout the interval, the integral would be positive. Therefore condition I must also be true.

Hence the correct answer is option D.

7.47 Answer E

We make a quick sketch of $y = mx$ and $y = x^3$ so that we can identify the area enclosed. Note that if $m \leq 0$, then there is no area enclosed, so we must have $m > 0$.



The first thing we note is that there are *two* regions we need to consider: one with $x > 0$ and one with $x < 0$. By symmetry, their areas are equal, so it suffices to find m such that the area of the region with $x > 0$ is 3.

To find the area enclosed for $x > 0$, we need to find where the line and cubic intersect; this is when $x^3 = mx$, so at $x = 0$ and $x = \pm\sqrt{m}$. As we are interested in the point where $x > 0$, this is at $s = \sqrt{m}$.

The enclosed area can be calculated as the area below the line minus the area below the cubic. So as a single integral, this gives

$$\begin{aligned}\int_0^{\sqrt{m}} mx - x^3 \,\mathrm{d}x &= \left[\tfrac{1}{2}mx^2 - \tfrac{1}{4}x^4\right]_0^{\sqrt{m}} \\ &= \tfrac{1}{2}m^2 - \tfrac{1}{4}m^2 \\ &= \tfrac{1}{4}m^2\end{aligned}$$

(We could also have calculated the area under the line as the area of a triangle.)

Since this area is 3, we have $\frac{1}{4}m^2 = 3$, so $m^2 = 12$ and $m = \sqrt{12} = 2\sqrt{3}$. This is option E.

7.48 Answer A

The function will have three distinct real roots if and only if the graph $y = f(x)$ has two stationary points, one above the x-axis and one below. We first find the stationary points. We have

$$f'(x) = 2x^2 + 4mx = 2x(x + 2m)$$

so $f'(x) = 0$ when $x = 0$ or $x = -2m$.

When $x = 0$, $f(x) = n$, and when $x = -2m$,

$$f(x) = \frac{2}{3}(-8m^3) + 2m(4m^2) + n = n + \frac{8}{3}m^3.$$

Since one of the stationary points has $f(x) = n$, we cannot have $n = 0$.

Since $m > 0$, it follows that if $n > 0$, then $f(-2m) > 0$ and both turning points would be above the x-axis. We must therefore have $n < 0$ and $n + \frac{8}{3}m^3 > 0$. The second inequality rearranges to $-\frac{8}{3}m^3 < n$, so we require $-\frac{8}{3}m^3 < n < 0$, which is option A.

We can work out the locations of the local maxima and minima by solving $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$. We have

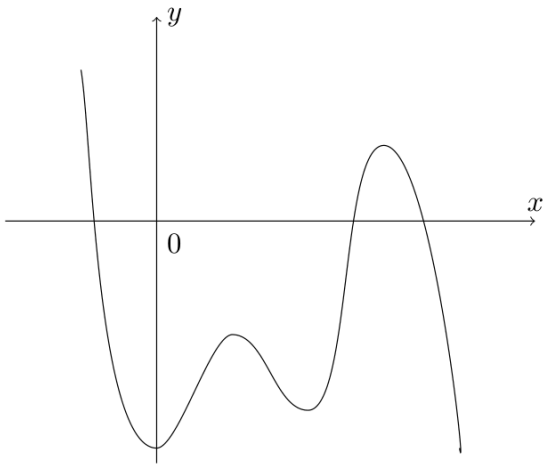
$$\frac{\mathrm{d}y}{\mathrm{d}x} = 5ax^4 + 4bx^3 + 3cx^2 + 2dx + e,$$

which is a quartic (a degree 4 polynomial). This can have at most 4 roots, so there are at most 4 stationary points.

This does not seem to help us that much, as all of the options have at most 4 stationary points!

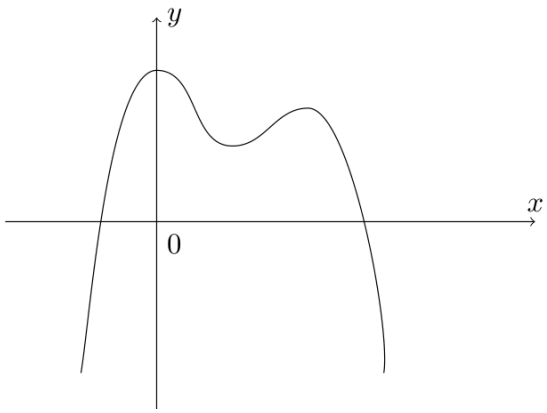
We also observe that we are starting with a degree 5 polynomial, so if $a > 0$, y tends to $+\infty$ as x tends to $+\infty$ and y tends to $-\infty$ as x tends to $-\infty$. Similarly, if $a < 0$, we have the reverse of this. If we now try sketching the four possibilities offered, we will see what happens:

A Two local minima and two local maxima might look like this:



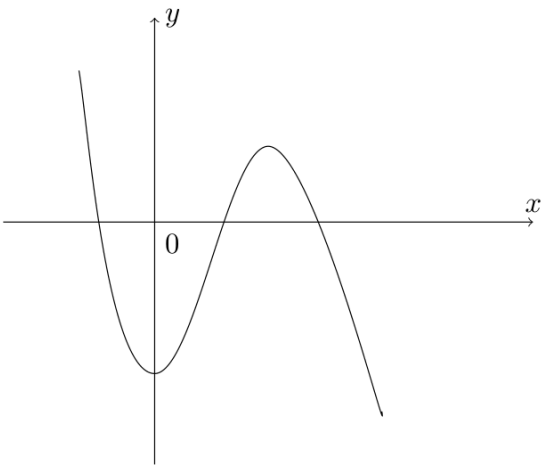
This seems plausible.

B One local minimum and two local maxima might look like this:



This is not possible, because y tends to $-\infty$ both when x tends to $+\infty$ and to $-\infty$.

C One local minimum and one local maximum might look like this:



This seems plausible; there is no requirement that the quartic $\frac{dy}{dx} = 0$ has four real roots, so there need not be exactly four stationary points.

D The curve $y = x^5$ has no local minima or local maxima (though it does have a point of inflection at the origin), so this is certainly possible.

So the only one which is not possible is B.

If we want to construct explicit examples of A and C, we could start by choosing $\frac{dy}{dx}$ as a quartic which has exactly four real roots (such as $(x-1)(x-2)(x-3)(x-4)$) or exactly two real roots (such as $(x^2+1)(x-1)(x-2)$) and then integrating it to obtain our desired function. Similarly, if we wanted to find an example for D which has no stationary points at all, we could integrate a function such as $(x^2+1)^2$.

7.50 Answer B

TMUA 2019 · Paper 2 · Q14

A necessary condition is one which must follow. To show that a condition is not necessary, we need to exhibit a counterexample, while to show that a condition is necessary, we need to give a justification.

I If $\mathrm{p}(x)$ is increasing for $a \leq x \leq b$, the value of the polynomial is not decreasing in this interval, so we must have $\mathrm{p}(a) \leq \mathrm{p}(b)$, and this is therefore a necessary condition.

II A quadratic such as $\mathrm{p}(x) = 9 - x^2$ provides a counterexample: it is increasing for $-3 \leq x \leq 0$, but $\mathrm{p}(-3) > 0$ while $\mathrm{p}(0) = 0$. Hence this is not a necessary condition.

III It is not clear why this should be the case, and we need to look beyond quadratics (as $\mathrm{p}''(x)$ is constant for every quadratic). Let us see if we can construct a counterexample. Let us take $\mathrm{p}''(x) = -x$, so that it is decreasing. Then $\mathrm{p}'(x) = -\frac{1}{2}x^2 + c$, giving $\mathrm{p}(x) = -\frac{1}{6}x^3 + cx$. (There is a further constant of integration here, but we can choose it to be zero, as it has no impact on where $\mathrm{p}(x)$ is increasing.)

If we now choose some $c > 1$, then $\mathrm{p}'(x)$ will be positive for $0 \leq x \leq 1$, so $\mathrm{p}(x)$ will be increasing there, yet $\mathrm{p}''(0) < \mathrm{p}''(1)$.

Therefore this condition is not necessary.

Another way of thinking about this is: to say that $\mathrm{p}(x)$ is increasing for $a \leq x \leq b$ only tells us about the function values; it does not tell us anything about the derivatives, so neither II nor III could possibly be necessary conditions.

Either way, the correct answer is B.

7.51 Answer A

TMUA Specimen · Paper 1 · Q15

We start by evaluating the integral to obtain

$$\begin{aligned}\int_0^1 (x-a)^2\,\mathrm{d}x &= \int_0^1 x^2 - 2ax + a^2\,\mathrm{d}x \\ &= \left[\tfrac{1}{3}x^3 - ax^2 + a^2x\right]_0^1 \\ &= \tfrac{1}{3} - a + a^2\end{aligned}$$

We can now complete the square on this final expression to obtain

$$(a-\tfrac{1}{2})^2 - (\tfrac{1}{2})^2 + \tfrac{1}{3} = (a-\tfrac{1}{2})^2 + \tfrac{1}{12}$$

and so the smallest possible value is $\frac{1}{12}$, which is option A.

We could alternatively have used calculus to find the stationary point of $\frac{1}{3} - a + a^2$ (by differentiating with respect to a), and then noting that this is a local minimum, and that the function is greater than this at all other points.

7.52 Answer C

TMUA 2016 · Paper 1 · Q15

We start by expanding the brackets to get

$$y = (2x+a)(x^2 - 4ax + 4a^2) = 2x^3 - 7ax^2 + 4a^2x + 8a^3$$

so the derivative is given by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 6x^2 - 14ax + 4a^2.$$

Therefore at $x = 1$, the gradient is $6 - 14a + 4a^2$. To find the least possible value of this as a varies, we can either differentiate this expression with respect to a or complete the square. The latter approach gives

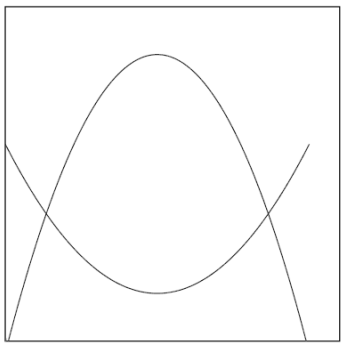
$$\begin{aligned}4a^2 - 14a + 6 &= 4(a^2 - \tfrac{7}{2}a) + 6 \\ &= 4\big((a-\tfrac{7}{4})^2 - (\tfrac{7}{4})^2\big) + 6 \\ &= 4(a-\tfrac{7}{4})^2 - \tfrac{49}{4} + 6 \\ &= 4(a-\tfrac{7}{4})^2 - \tfrac{25}{4}\end{aligned}$$

so the minimum value is $-\frac{25}{4}$ and the answer is C.

We start by sketching the two curves to understand what is happening.

They are both quadratics. The graph of p has a maximum at $(0, 8)$ and the graph of q has a minimum at $(0, -2)$. They are both symmetrical about $x = 0$ (the y-axis). It might be helpful to work out where they intersect: $8 - 2x^2 = x^2 - 2$ implies $3x^2 = 10$, so $x = \sqrt{10/3}$, which is between 1.5 and 2.

The graph of these therefore looks like this, where we have also sketched a possible rectangle. For the rectangle to have the maximum possible area, it must meet the graphs at all four of its vertices.



Let us suppose that the vertices of the rectangle are at $x = \pm a$, with $0 < a < \sqrt{10/3}$. Then the width of the rectangle is $2a$ and the height is $(8 - 2a^2) - (a^2 - 2) = 10 - 3a^2$. Therefore the area is $A = 2a(10 - 3a^2) = 20a - 6a^3$.

To maximise this area, we differentiate with respect to a to find the stationary points. We have

$$\frac{\mathrm{d}A}{\mathrm{d}a} = 20 - 18a^2$$

so there is a stationary point when this is zero, at $a = \pm\sqrt{20/18} = \pm\sqrt{10/9} = \pm\frac{1}{3}\sqrt{10}$. We are only interested in the positive value, and this is less than $\sqrt{10/3}$, so it is a valid value for our problem. We therefore obtain the maximum possible area: it is

$$\begin{aligned} A &= 2a(10 - 3a^2) \\ &= \tfrac{2}{3}\sqrt{10}(10 - \tfrac{1}{3} \times 10) \\ &= \tfrac{2}{3}\sqrt{10} \times \tfrac{2}{3} \times 10 \\ &= \frac{40\sqrt{10}}{9} \end{aligned}$$

which is option H.

This question appears to be related to graph transformations, so we will start with this approach and see where it leads.

The equation $\mathrm{f}(x) = 0$ has 3 distinct real roots, so $y = \mathrm{f}(x)$ intersects the x-axis 3 times.

I The graph of $y = \mathrm{f}(x) + 1$ is a translation of $y = \mathrm{f}(x)$ in the y-direction by 1 unit, which may or may not intersect the x-axis 3 times.

II The graph of $y = \mathrm{f}(x + 1)$ is a translation of $y = \mathrm{f}(x)$ in the negative x-direction by 1 unit, so it will still intersect the x-axis 3 times, and hence $\mathrm{f}(x + 1) = 0$ has 3 distinct roots.

III The graph of $y = \mathrm{f}(-x)$ is a reflection of $y = \mathrm{f}(x)$ in the y-axis, which still intersects the x-axis 3 times. Then $y = \mathrm{f}(-x) - 1$ is a translation in the negative y-direction by 1 unit, which may or may not intersect the x-axis 3 times.

It is still not clear whether or not I or III are necessarily true. We will have to identify where the turning points of the graph are to determine this.

We have

$$\mathrm{f}(x) = x^3 + 3qx^2 + 2$$

so

$$\mathrm{f}'(x) = 3x^2 + 6qx$$

so $\mathrm{f}'(x) = 0$ when $x = 0$ or $x == -2q$, so the stationary points are at $(0, 2)$ and $(-2q, 2 + 4q^3)$. As the graph crosses the x-axis 3 times, the stationary points must lie either side of the x-axis, so $2 + 4q^3 < 0$.

Therefore when the graph is translated up by 1 unit, we may have the new stationary point at $(-2q, 2 + 4q^3 + 1)$ with $2 + 4q^3 + 1 > 0$, but when it is translated down by 1 unit, they will be at $(0, 1)$ and $(-2q, 2 + 4q^3 - 1)$, and $2 + 4q^3 - 1 < 0$. So there will still be 3 intersections.

Therefore II and III are necessarily true, and the answer is G.

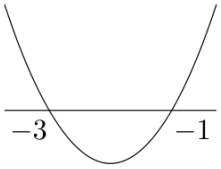
In this question, we consider a function to be increasing when $f'(x) \geq 0$, and decreasing when $f(x) \leq 0$.

> ***Commentary:*** *The classification of increasing and decreasing we use in the options for this question avoids a subtle issue: what we consider a function to be doing (i.e., increasing, decreasing, or neither) at points where $f'(x) = 0$. For this question, though, we don't need to concern ourselves with this as the options all use $\geqslant$ or $\leqslant$. However, we will set out our answer using '$\geqslant$' or '$\leqslant$'.*

We therefore calculate $f'(x)$ and $g'(x)$ and find where each is positive and negative.

$f'(x) = 6x + 12 = 6(x + 2)$, so $f'(x) > 0$ when $x > -2$ and $f'(x) < 0$ when $x < -2$.

$g'(x) = 3x^2 + 12x + 9 = 3(x^2 + 4x + 3)$. We can find where this is positive and negative by sketching the graph: $x^2 + 4x + 3 = (x + 1)(x + 3)$, giving the sketch of $g'(x)$:



so $g'(x) > 0$ when $x < -3$ or $x > -1$, and $g'(x) < 0$ when $-3 < x < -1$.

We now want one of $f'(x)$ and $g'(x)$ to be positive and the other negative.

We have $f'(x) > 0$ and $g'(x) < 0$ when $x > -2$ and $-3 < x < -1$, so when $-2 < x < -1$.

Similarly, $f'(x) < 0$ and $g'(x) > 0$ when $x < -2$ and either $x < -3$ or $x > -1$; together these give $x < -3$.

So the complete set of solutions (if we decide to include the points where $f'(x) = 0$) is $x \leq -3$ or $-2 \leq x \leq -1$, and so the answer is option E.

7.56 Answer F

TMUA 2018 · Paper 1 · Q16

We first need to find the stationary point of the curve C. We can do this either by differentiating or, as the equation is a quadratic, by completing the square.

By differentiating, we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2x + b$$

so $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$ if and only if $x = -\frac{1}{2}b$, and so

$$y = (-\tfrac{1}{2}b)^2 + b(-\tfrac{1}{2}b) + 2 = 2 - \tfrac{1}{4}b^2$$

Alternatively, we can complete the square to give $y = (x + \frac{1}{2}b)^2 - \frac{1}{4}b^2 + 2$, and so the stationary point is at $(-\frac{1}{2}b, 2 - \frac{1}{4}b^2)$.

The distance of this point from the origin is then given by

$$\sqrt{(-\tfrac{1}{2}b)^2 + (2 - \tfrac{1}{4}b^2)^2} = \sqrt{\tfrac{1}{4}b^2 + 4 - 4(\tfrac{1}{4}b^2) + \tfrac{1}{16}b^4} = \sqrt{\tfrac{1}{16}b^4 - \tfrac{3}{4}b^2 + 4}$$

To minimise this square root, we can minimise the expression inside the square root. This is a function of $B = b^2$, namely $\frac{1}{16}B^2 - \frac{3}{4}B + 4$, so we can minimise this.

> ***Commentary:*** *If a quartic is actually a quadratic in the variable squared, it is usually much easier to work with the quadratic than the quartic.*

We can minimise this by completing the square or differentiating, as before. We can start by taking out a factor of $\frac{1}{16}$ so we are trying to minimise $\frac{1}{16}(B^2 - 12B) + 4$. (There's no need to think about the 4 at the end: this additional term does not change the value of b or B at which the function is minimised.)

We therefore wish to minimise $B^2 - 12B = (B-6)^2 - 36$ so $B = 6$ gives the minimum value, so $b^2 = 6$ giving $b = \sqrt{6}$ (as $b \geqslant 0$), hence the answer is F.

> ***Commentary:*** *We could have gone ahead and differentiated the whole square root expression for distance, if we know the chain rule. But that's more effort than needed. Similarly, we could have differentiated the expression as a quartic, but again that is more complex than required. It is almost always worth spending a moment thinking how to make the problem a little easier before diving into algebra and calculations.*

7.57 Answer C

TMUA 2019 · Paper 1 · Q16

Let $A = \int_0^1 \mathrm{f}(x)\,\mathrm{d}x$, which is the (signed) area under the graph of $y = \mathrm{f}(x)$ between $x = 0$ and $x = 1$ (taking signs into account, as usual: parts of the graph under the x-axis are counted as negative areas). Likewise, let $B = \int_1^2 \mathrm{f}(x)\,\mathrm{d}x$, which is the (signed) area between $x = 1$ and $x = 2$.

The integral $\int_0^1 \mathrm{f}(x+1)\,\mathrm{d}x$ gives the (signed) area under the graph of $y = \mathrm{f}(x+1)$ between $x = 0$ and $x = 1$. This graph is the graph of $y = \mathrm{f}(x)$ translated left by 1 unit, so this equals the area under the original graph between $x = 1$ and $x = 2$, which is B. Hence $B = 6$.

The first equation gives $2A + 5B = 14$, and as $B = 6$, we get $A = -8$.

The final integral $\int_0^2 \mathrm{f}(x)\,\mathrm{d}x$ is the (signed) area between $x = 0$ and $x = 2$, which is $A + B = -8 + 6 = -2$.

Hence the answer is C.

For a function to be a counterexample to $(*)$, it must be the case that $f(x)$ is an integer for every integer x, but for which $f'(x)$ is **not** an integer for every integer x.

We can start by substituting in $x = 0$ and $x = 1$ into the given functions to see whether $f(x)$ is even potentially an integer for every integer x:

A $f(0) = \frac{1}{4}$. **This cannot be a counterexample.**

B $f(0) = 0$, $f(1) = \frac{3}{2}$. **This cannot be a counterexample.**

C $f(0) = 0$, $f(1) = 2$. **This could potentially be a counterexample.**

D $f(0) = 0$, $f(1) = 1$. **This could potentially be a counterexample.**

So we now only need to consider C and D.

Our next step is to differentiate them and then substitute $x = 0$ and $x = 1$ into the derivatives.

C $f'(x) = \dfrac{4x^3 + 3x^2 + 2x + 1}{2}$, $f'(0) = \frac{1}{2}$, so this is potentially a counterexample.

D $f'(x) = \dfrac{4x^3 + 6x^2 + 2x}{4}$, $f'(0) = 0$, $f'(1) = 4$.

It therefore seems most likely that C is a counterexample and D is not. To prove this, we need to show that for C, $f(x)$ is an integer for every integer x, and that for D, either $f(x)$ is not an integer for some integer x or that $f'(x)$ is an integer for every integer x.

For the purposes of the test, it probably makes most sense at this point to choose the answer C and move on, but we will go on here to complete the solution.

For C, we have

$$f(x) = \frac{x(x^3 + x^2 + x + 1)}{2} = \frac{x(x+1)(x^2+1)}{2}$$

We see that $x(x+1)$ is even if x is even and also if x is odd, so $x(x+1)(x^2+1)$ is even for every integer x, hence $f(x)$ is an integer for every integer x. Therefore C is a counterexample.

For D, we have

$$f(x) = \frac{x^2(x^2 + 2x + 1)}{4} = \frac{x^2(x+1)^2}{4}$$

Since $x(x+1)$ is even for every integer x, as we just showed, $x^2(x+1)^2$ is a multiple of 4 for every integer x, and thus $f(x)$ is an integer for every integer x.

For the derivative,

$$f'(x) = \frac{x(2x^2 + 3x + 1)}{2} = \frac{x(x+1)(2x+1)}{2}$$

and as before, the numerator is always even, so $f'(x)$ is an integer for every integer x. Therefore D is not a counterexample.

Considering the offered options, the focus is on the order of the steps and whether the steps prove what they should be proving; there is no requirement for us to check the algebraic calculations themselves.

The task is to prove that **if** $27b\left(b+\dfrac{4a^3}{27}\right)<0$, **then** $x^3+ax^2+b=0$ has three distinct real roots.

In step I, the student finds the stationary points of $y=x^3+ax^2+b$.

In step II, the student assumes that $27b\left(b+\dfrac{4a^3}{27}\right)<0$, which is the correct thing to do to prove an 'if … then' statement.

In step III, the student says '**if** the cubic has three distinct real roots **then** …', which is correct but is not useful: we know that one of the stationary points is above the x-axis and the other is below and we wish to deduce that the cubic has three distinct real roots.

In step IV, the student uses the result of II and the result '**if** one of the stationary points is above the x-axis and one is below, **then** the equation has three distinct real roots', which is the converse of the statement in step III.

Therefore the correct option is E: the student should have shown the converse of the result in step III.

This requires us to first differentiate the function. We therefore write

$$y = \frac{1-x}{\sqrt[3]{x^2}} = x^{-\frac{2}{3}} - x^{\frac{1}{3}}$$

which we can differentiate to get

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\tfrac{2}{3}x^{-\frac{5}{3}} - \tfrac{1}{3}x^{-\frac{2}{3}}$$

This is zero when

$$-\tfrac{2}{3}x^{-\frac{5}{3}} - \tfrac{1}{3}x^{-\frac{2}{3}} = 0,$$

so multiplying by $3x^{\frac{5}{3}}$ to clear fractions in both the coefficients and in the powers gives

$$-2 - x = 0$$

so $x = -2$. (We could also have obtained this by writing $-\frac{2}{3}x^{-\frac{5}{3}} = \frac{1}{3}x^{-\frac{2}{3}}$ and dividing one side by the other.)

We next need to determine the sign of $\frac{\mathrm{d}y}{\mathrm{d}x}$ in each region. We note that the function is not defined at $x = 0$, so we have to deal with $x < 0$ and $x > 0$ separately. It is also not clear how to find the sign of $\frac{\mathrm{d}y}{\mathrm{d}x}$ directly from its given form, so we first factorise it, giving

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\tfrac{1}{3}x^{-\frac{5}{3}}(2+x).$$

(Incidentally, this gives yet another way to see that the derivative is zero at $x = -2$.) We can now work out the signs of the two factors, and hence of the derivative, in the various ranges:

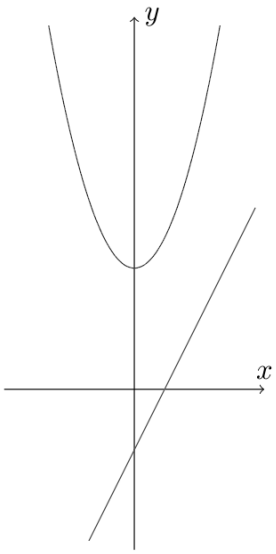
	$x < -2$	$x = -2$	$-2 < x < 0$	$x > 0$
$x^{-\frac{5}{3}}$	$-$	$-$	$-$	$+$
$2 + x$	$-$	0	$+$	$+$
$\frac{\mathrm{d}y}{\mathrm{d}x}$	$+$	0	$-$	$+$

It is therefore increasing in the region $x < -2$ and $x > 0$. (It it not increasing at $x = -2$, but rather it is stationary at that point.) The correct answer is therefore A (though the question mistakenly says $x \leqslant -2$).

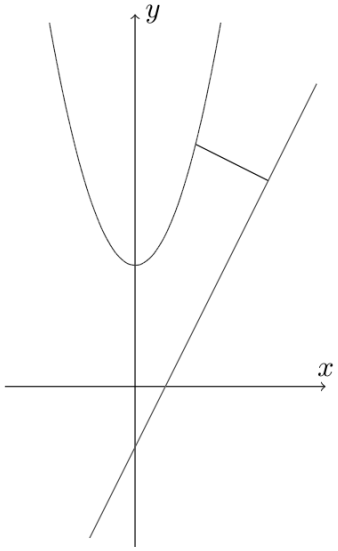
From the given options, it is clear that the curve and line do not intersect.

One way to do this would be to take a point on the curve, say (a, a^2+4), and a point on the line, say $(b, 2b-2)$, find the (square of) the distance between them, and then try to find some way of minimising it. That, though, seems quite challenging, as there are two independent variables involved.

Another way is to draw a sketch of the situation and to think about what the question means geometrically.



The shortest distance from any point on the curve to the straight line is found by drawing a perpendicular from the curve to the straight line, like this:



This may give us a few different ideas; any of the following would work:

- The smallest distance will be when the perpendicular to the line is also perpendicular to the curve; in the above sketch, it is clear that if we move the joining line closer to the origin, its length will become shorter. We can find this position by finding the gradient of the normal to the curve and finding a point at which this is perpendicular to the line.

- Taking this one step further, if we translate the line (without changing its gradient), we shorten the minimum distance of the curve from the line, but as long as we don't cross the curve, the point which is closest to the line will remain so. Therefore the closest point to the line is the point on the curve where the curve's gradient equals the line's gradient, for when the line just touches the curve, this point will be at zero distance from the curve.

We will use the final approach to solve this problem.

The gradient of the line is 2, and the curve has gradient $\frac{\mathrm{d}y}{\mathrm{d}x} = 2x$, so $\frac{\mathrm{d}y}{\mathrm{d}x} = 2$ when $x = 1$, so $(1, 5)$ is the closest point on the curve to the line.

The line through $(1, 5)$ perpendicular to the line $y = 2x - 2$ has equation $y - 5 = -\frac{1}{2}(x - 1)$. This intersects the line $y = 2x - 2$ when

$$(2x - 2) - 5 = -\tfrac{1}{2}(x - 1)$$

so $2x - 7 = -\frac{1}{2}(x - 1)$. Multiplying this by 2 gives

$$4x - 14 = -x + 1$$

so $5x = 15$, giving $x = 3$, $y = 4$.

The distance between $(1, 5)$ and $(3, 4)$ is $\sqrt{(3-1)^2 + (4-5)^2} = \sqrt{5}$, so the correct answer is B.

> ***Commentary:*** *Though we dismissed the original two-variable idea, it turns out that we could have used it in a similar way to the first idea listed above: if we fix a value for a, we can find the value of b which gives the smallest distance from that point to the line. This gives us a formula for the smallest distance from a given point to the line in terms of a; we can then find the value of a which makes this as small as possible. This changes the problem from one two-variable problem to two one-variable problems, which we know how to solve.*

For a function to be a counterexample to this 'if' statement, the first condition (the antecedent) must be true but the second condition (the consequent) must be false.

So we first check whether these functions satisfy the condition $\big(\mathrm{f}(x)\big)^2 \leqslant 1$ for all $-1 \leqslant x \leqslant 1$:

A This function goes from $-\frac{1}{2}$ to $\frac{3}{2}$ as x goes from -1 to 1. So when $x = 1$, $\big(\mathrm{f}(x)\big)^2 = (\frac{3}{2})^2 > 1$. So this function does not satisfy the antecedent.

B Similarly to **A**, when $x = -\frac{1}{2}$, $\mathrm{f}(x) = -\frac{3}{2}$, so $\big(\mathrm{f}(x)\big)^2 > 1$ and this function also does not satisfy the antecedent.

C When $x = 1$, $\mathrm{f}(x) = 2$ so $\big(\mathrm{f}(x)\big)^2 > 1$ and this function does not satisfy the antecedent.

D This function looks as though it might satisfy the antecedent, as it does not obviously fail to do so: $\mathrm{f}(-1) = \mathrm{f}(0) = \mathrm{f}(1) = 0$, so the cubic graph has two stationary points in the interval $-1 \leqslant x \leqslant 1$. We can locate these by differentiating:

$$\mathrm{f}'(x) = 1 - 3x^2$$

so $\mathrm{f}'(x) = 0$ when $x = \pm\frac{1}{\sqrt{3}}$. As

$$\mathrm{f}\big(\tfrac{1}{\sqrt{3}}bigr) = \tfrac{1}{\sqrt{3}} - \tfrac{1}{3\sqrt{3}} = \tfrac{2}{3\sqrt{3}} < 1 \quad \text{and} \quad \mathrm{f}\big(-\tfrac{1}{\sqrt{3}}\big) = -\tfrac{1}{\sqrt{3}} + \tfrac{1}{3\sqrt{3}} = -\tfrac{2}{3\sqrt{3}} > -1,$$

we see that the function lies between $-\frac{2}{3\sqrt{3}}$ and $\frac{2}{3\sqrt{3}}$ when $-1 \leqslant x \leqslant 1$, so $\big(\mathrm{f}(x)\big)^2 \leqslant 1$ in this interval, and the antecedent is satisfied.

E As in **C**, when $x = 1$, $\mathrm{f}(x) = 2$ so $\big(\mathrm{f}(x)\big)^2 > 1$ and this function does not satisfy the antecedent.

F As with **D**, $\mathrm{f}(-1) = \mathrm{f}(0) = \mathrm{f}(1) = 0$, so we look for stationary points of the function. We have

$$\mathrm{f}'(x) = 2x - 4x^3 = 2x(1 - 2x^2)$$

so $\mathrm{f}'(x) = 0$ when $x = 0$ or $x = \pm\frac{1}{\sqrt{2}}$. We have $\mathrm{f}(0) = 0$ and $\mathrm{f}\big(\pm\frac{1}{\sqrt{2}}\big) = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$, so the function lies between 0 and $\frac{1}{4}$ when $-1 \leqslant x \leqslant 1$, so the antecedent is satisfied.

We therefore only need to consider the consequent condition for **D** and **F**.

For the function in **D**, the integral on the right, $\int_{-1}^{1} \mathrm{f}(x)\,\mathrm{d}x$, is zero, since both x and x^3 are odd functions: the area between 0 and 1 and the area between -1 and 0 are equal, but the first is above the x-axis and the second below it, so they cancel each other out. We could also see this by calculating the integral explicitly. The integral on the left, $\int_{-1}^{1} \big(\mathrm{f}(x)\big)^2\,\mathrm{d}x$, is the integral of a square, which is never negative, so the integral must be greater than 0. (We could likewise find this exactly by explicit integration.) Thus $\int_{-1}^{1} \big(\mathrm{f}(x)\big)^2\,\mathrm{d}x > \int_{-1}^{1} \mathrm{f}(x)\,\mathrm{d}x$ and the consequent is false. Hence **D** provides a counterexample.

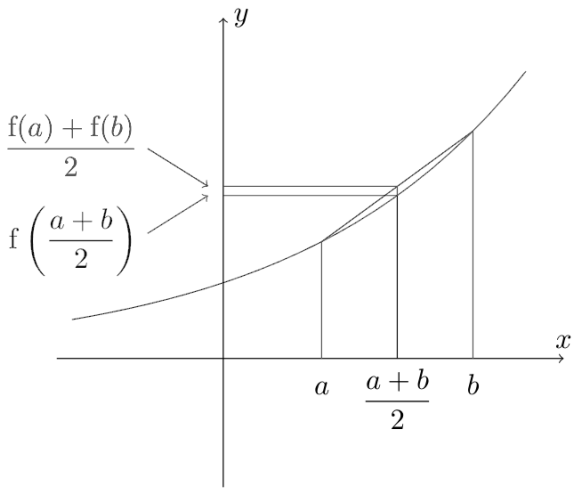
For the sake of completeness, we will show that **F** is not a counterexample. Note that $0 \leqslant \mathrm{f}(x) < 1$ for all $-1 \leqslant x \leqslant 1$. This means that $\big(\mathrm{f}(x)\big)^2 \leqslant \mathrm{f}(x)$ for all $-1 \leqslant x \leqslant 1$, and so $\int_{-1}^{1} \big(\mathrm{f}(x)\big)^2\,\mathrm{d}x \leqslant \int_{-1}^{1} \mathrm{f}(x)\,\mathrm{d}x$. Thus **F** is not a counterexample. (We could, of course, find these integrals by explicitly integrating.)

It is helpful to draw a sketch of $y = \mathrm{f}(x)$ to understand what is going on here.

Note first that $\dfrac{\mathrm{f}(a)+\mathrm{f}(b)}{2}$ is the mean of the $\mathrm{f}(a)$ and $\mathrm{f}(b)$, so if we draw a line segment from $(a, \mathrm{f}(a))$ to $(b, \mathrm{f}(b))$ on the graph of $y = \mathrm{f}(x)$, then the point $\left(\dfrac{a+b}{2}, \dfrac{\mathrm{f}(a)+\mathrm{f}(b)}{2}\right)$ is the midpoint of this line segment.

Likewise, $\left(\dfrac{a+b}{2}, \mathrm{f}\left(\dfrac{a+b}{2}\right)\right)$ is the point on the graph with this x-coordinate.

The inequality states that the line segment lies above the graph at this point, so the graph could look something like this:



We see the for the inequality to be true in this case, the curve needs to "bend up", so that it is below the chord connecting $(a, \mathrm{f}(a))$ with $(b, \mathrm{f}(b))$. This means that the second derivative, $\mathrm{f}''(x)$ must be positive. So it would seem likely that the answer is C.

Without too much difficulty we can see that the other options don't work, just by looking at the above sketch. This shows that none of D, E or F is necessary (for the curve could bend back upwards for more negative values of x, as in a quadratic). If we translate the graph downwards, then it will still have this property, but A is no longer the case, so A is not necessary. And if we continue this graph to more negative values of x, it will have $\mathrm{f}'(x) < 0$ if we go far enough, so B is not necessary either.

Therefore by elimination, C must be necessary.

We can also prove directly that C is necessary. To do so, we will prove the contrapositive: if $\mathrm{f}''(x) < 0$ for some real x, then there are real numbers a and b with $a < b$ for which

$$\frac{\mathrm{f}(a)+\mathrm{f}(b)}{2} < \mathrm{f}\left(\frac{a+b}{2}\right) \qquad (*)$$

Now, $\mathrm{f}''(x) < 0$ means that the curve is "bending downwards" at this point, which means that a sketch of the graph will look like a reflection of the above sketch in a horizontal line (at least near the point where $\mathrm{f}''(x) < 0$). And then the curve will lie above the chord, so $(*)$ is true. Hence C is necessary.

> ***Commentary:*** *An interesting question to wonder about is whether there are any examples of such polynomial functions. Clearly, constant polynomials* ($\mathrm{f}(x) = c$) *satisfy the condition, as do linear polynomials* ($\mathrm{f}(x) = ax + b$) *and quadratics with a positive coefficient of* x^2 ($\mathrm{f}(x) = ax^2 + bx + c$ *with* $a > 0$)
>
> *For higher degree polynomials, if* $\mathrm{f}(x) = ax^n + \cdots$, *then* $\mathrm{f}''(x) = an(n-1)x^{n-2} + \cdots$. *If* n *is odd, then this polynomial takes all real values, including negative ones, so this cannot satisfy the condition. If* n *is even, then it might be the case that* $\mathrm{f}''(x) \geqslant 0$ *for all* x, *which will be the case if* $f''(x) \geqslant 0$ *at all of its stationary points.*
>
> *For example, if we take* $\mathrm{f}''(x) = 3x^1 + 1$, *then the corresponding* $\mathrm{f}(x)$ *will have this condition.*
>
> *A function satisfying this condition is called a* convex function.

7.64 Answer D

TMUA 2023 · Paper 1 · Q19

One thing we might notice is that $|-6x| = |6x|$, so the minus sign in the differential equation is not actually needed. We therefore drop it.

A more important thing to realise is that we should not try to solve the whole equation in one go; we will need to treat $x \geq 0$ and $x \leq 0$ separately.

With that in mind, let us solve the equation.

When $x \geq 0$, the equation becomes $\frac{\mathrm{d}y}{\mathrm{d}x} = 6x$, so $y = 3x^2 + c$.

When $x \leq 0$, the equation becomes $\frac{\mathrm{d}y}{\mathrm{d}x} = -6x$, so $y = -3x^2 + \hat{c}$. (We use a different constant, $\hat{c}$, as we do not yet know whether it has the same value as the constant for the $x \geq 0$ region.)

Since the value of y has to be the same in both expressions when $x = 0$, we must have $\hat{c} = c$, so we have

$$y = \begin{cases} 3x^2 + c & \text{when } x \geq 0 \\ -3x^2 + c & \text{when } x < 0 \end{cases}$$

giving

$$f(x) = \begin{cases} 3x^2 & \text{when } x \geq 0 \\ -3x^2 & \text{when } x < 0 \end{cases}$$

The options offered require us to collapse this into a single expression. We can take out a common factor of $3x$ from both terms, to give

$$f(x) = \begin{cases} 3x(x) & \text{when } x \geq 0 \\ 3x(-x) & \text{when } x < 0 \end{cases}$$

and this is $3x$ times $|x|$, since

$$|x| = \begin{cases} x & \text{when } x \geq 0 \\ -x & \text{when } x < 0 \end{cases}$$

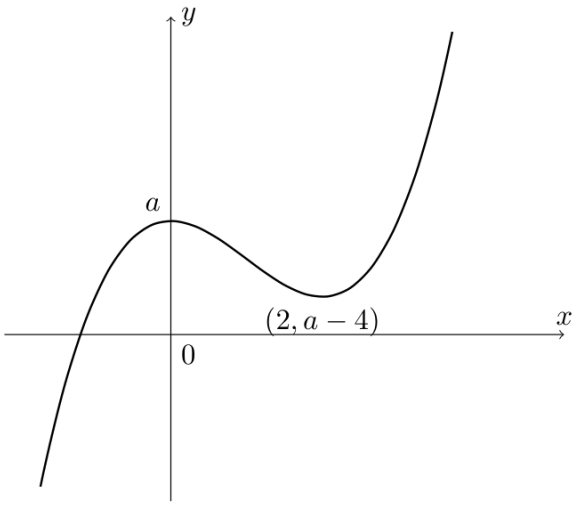
So we can write $f(x) = 3x\,|x|$, which is option D.

We recall that a '**sufficient** condition' means that if the condition holds, then the equation has exactly one real root.

We can start by sketching the graph of the cubic $y = x^3 - 3x^2 + a$, noting that the impact of changing a is just to translate the graph vertically.

We differentiate to find the stationary points: $\frac{dy}{dx} = 3x^2 - 6x$, so $\frac{dy}{dx} = 0$ when $x = 0$ and $x = 2$, so the stationary points are at $(0, a)$ and $(2, a - 4)$.

So the graph looks like this:



There will be one real root for the equation $x^3 - 3x^2 + a = 0$ if either the local maximum lies below the x-axis or if the local minimum lies above the x-axis, as in the case sketched.

The first occurs when $a < 0$, the second when $a - 4 > 0$ or equivalently, when $a > 4$.

So a **necessary and sufficient** condition is $a < 0$ or $a > 4$. This does not, however, appear as one of the options, so we need to go through them to see which one is **sufficient**. (Recall from earlier that **sufficient** means that if this condition holds, then the equation has exactly one real root, but not necessarily the other way round.)

A This includes $a = 1$, which gives more than one real root.

B This includes $a = 0$, which gives more than one real root (as the minimum lies on the x-axis).

C This includes $a = 4$, which gives more than one real root (as the maximum lies on the x-axis).

D This includes $a = 1$ again.

E This is $a > 4$ or $a < -4$, both of which are part of the valid range, so this is a **sufficient** condition.

F This includes $a = 1$ again.

G This gives more than one real root.

H $a = \frac{3}{2}$ gives more than one real root.

Since $f(x) = 0$ for exactly M real values of x, $f(x)$ has at least $M - 1$ stationary points: there must be at least one stationary point between every adjacent pair of values of x for which $f(x) = 0$. But there may also be more stationary points between these pairs of values or before the first or after the last value.

Now $g(x) = xf'(x)$ is zero whenever $f(x)$ has a stationary point. So there are at least $M - 1$ values of x for which $g(x) = 0$, but there will be more if $f(x)$ has more than $M - 1$ stationary points. In addition, $g(x) = 0$ when $x = 0$. If $x = 0$ is a stationary point of $f(x)$, then $g(x) = 0$ at every stationary point of $f(x)$ and nowhere else. If $x = 0$ is not a stationary point of $f(x)$, then $g(x) = 0$ will be true for one further value of x.

Summarising:

- the number of stationary points of $f(x)$ is at least $M - 1$
- N equals the number of stationary points of $f(x)$ or one more than the number of stationary points of $f(x)$.

Therefore N is at least $M - 1$, so $N \geq M - 1$, which rearranges to $M \leq N + 1$.

Now considering the options:

I $M < N$ is possible if $f(x)$ has extra stationary points between zeros. For example, $f(x) = x^2 + 1$ has $M = 0$ but $N = 1$. As a second example, let $f(x)$ be a cubic with both turning points above the x-axis; this has $M = 1$ and $N = 2$ or $N = 3$.

II $M = N$ is possible if the number of stationary points of $f(x)$ is $M - 1$ and $x = 0$ is not a stationary point of $f(x)$. For example, $f(x) = (x - 1)^2 - 1$ has $M = 2$ (at $x = 0$ and $x = 2$) and one stationary point at $x = 1$. So $g(x) = 0$ at $x = 0$ and $x = 1$, giving $N = 2$.

III $M > N$ is possible if the number of stationary points of $f(x)$ is $M - 1$ and $x = 0$ is a stationary point of $f(x)$. For example, $f(x) = x^2 - 1$ has $M = 2$ (at $x = \pm 1$) and one stationary point at $x = 0$. So $g(x) = 0$ at $x = 0$ only, giving $N = 1$.

The correct answer is option H, I, II and III.

Finding the values of x at which the two curves intersect is the same as finding the values of x at which the y-values are equal, that is,

$$x^3 - 12x = k - (x-2)^2$$

or

$$x^3 - 12x - k + (x-2)^2 = 0.$$

Thus we are finding out something about the roots of a cubic.

We can expand the brackets in the final equation to give

$$x^3 + x^2 - 16x - k + 4 = 0,$$

and we write $\mathrm{f}(x) = x^3 + x^2 - 16x - k + 4$.

We can determine something about the locations of the roots of $\mathrm{f}(x)$ by finding the stationary points. We have

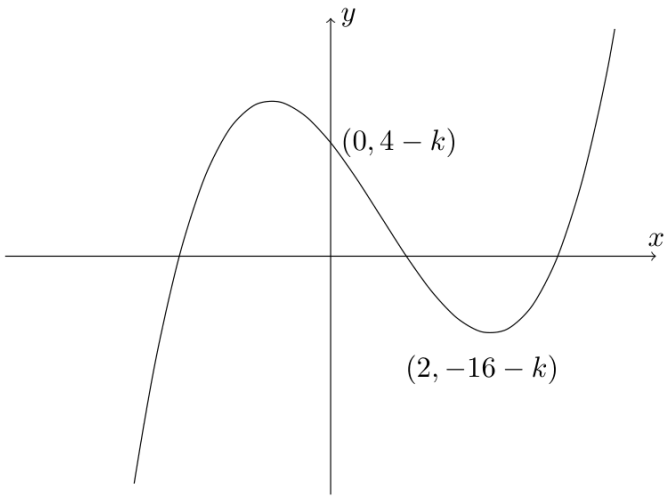
$$\mathrm{f}'(x) = 3x^2 + 2x - 16,$$

so $\mathrm{f}'(x) = 0$ when

$$x = \frac{-2 \pm \sqrt{2^2 + 4 \times 3 \times 16}}{6} = \frac{-2 \pm \sqrt{196}}{6} = \frac{-2 \pm 14}{6} = 2 \text{ or } -\frac{8}{3}.$$

(We could have found these roots by factorising, but that did not look particularly easy.) Therefore there are stationary points at $x = 2$ and $x = -\frac{8}{3}$. We have $\mathrm{f}(2) = -16 - k$, but calculating $\mathrm{f}(-\frac{8}{3})$ by hand would be painful, so let us try to avoid doing so if we can!

We note that $\mathrm{f}(0) = 4 - k$, and that $\mathrm{f}(x)$ gets big as x gets big, so if there are exactly two roots with positive x-coordinates, we need the graph to look something like this:



In particular, we require $\mathrm{f}(0) > 0$, so we must have $k < 4$.

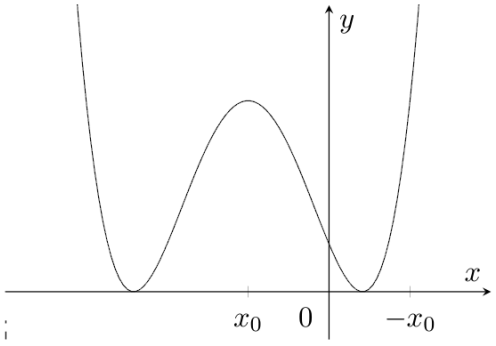
It must also be the case that the stationary point at $x = 2$ lies below the x-axis, so we need $-16 - k < 0$, that is, $-16 < k$.

Once we have met these two conditions, it becomes clear that there are three distinct roots, exactly one of which has $x < 0$. Therefore, the complete range of values of k is given by $-16 < k < 4$, which is option E.

We note first that when $x \geq 0$, $f(|x|) = f(x)$, and so for $x \geq 0$, the integrand is just $\big(f(x)\big)^2 - \big(f(x)\big)^2 = 0$. When $x < 0$, though, the integrand could take any value.

1 We present a complicated approach first.

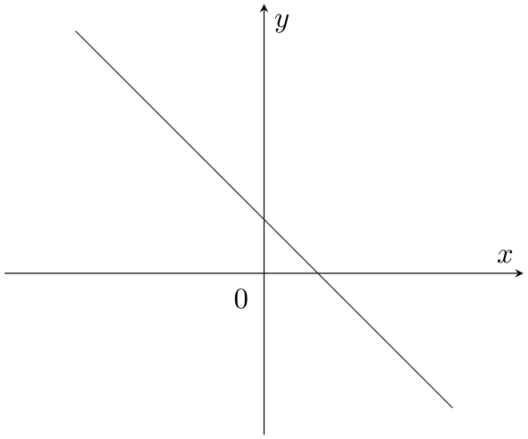
There seems to be no obvious reason why this must be true; one could imagine a polynomial for which the values of the integrand are positive for some negative values of x and negative for others, so the positive parts and negative parts cancel out making the integral zero even though $p < 0$. Here is one example: consider a quartic $f(x)$ that looks roughly like this:



For negative values of x close to $x = 0$, $\big(f(x)\big)^2 > \big(f(|x|)\big)^2$. But once $x < x_0$, we have $\big(f(x)\big)^2 < \big(f(|x|)\big)^2$ as $f(|x|) > f(x)$. We can therefore take $q = 0$ and find some $p < x_0$ for which $I_{p,q} = 0$.

A much simpler approach is that we can take $p = 0$, $q > 0$ and then $I_{p,q} = 0$, so this statement is not true. Were the question to have said $0 \leq p$ instead of $0 < p$, though, we would not have been able to use this simpler approach.

2 We could take $f(x) = 1 - x$, with $f'(x) = -1 < 0$ for all x.



Then $\big(f(x)\big)^2 > \big(f(|x|)\big)^2$ for all $x < 0$, so $I_{p,q} > 0$ for all $p < q < 0$.

3 We observed at the start of this question that the integrand is zero for all $x \geq 0$. So if $p \geq 0$, then $I_{p,q} = \int_p^q 0 \, \mathrm{d}x = 0$. The contrapositive of this statement is: if $I_{p,q} \neq 0$, then $p < 0$. The given statement ($I_{p,q} > 0$ only if $p < 0$, which is the same as saying: if $I_{p,q} > 0$, then $p < 0$) is a special case of the more general result and is therefore true.

Trigonometry 2

8.1 Answer C

TMUA 2022 · Paper 1 · Q4

Suppose that the angle of the sector is θ (in radians). Then the arc length of the smaller sector is $r\theta$ and the area is $\frac{1}{2}r^2\theta$.

For the smaller sector, this gives $r\theta = 6$, so $\theta = 6/r$. Therefore the area is $\frac{1}{2}r^2(6/r) = 3r$.

Likewise, since the larger sector has the same angle as the smaller one, its arc length is $(r+3)(6/r)$ and its area is $\frac{1}{2}(r+3)^2(6/r) = 3(r+3)^2/r$.

We are told the difference between the areas, so we have

$$\frac{3(r+3)^2}{r} - 3r = 21.$$

Multiplying this by r and rearranging gives

$$3(r+3)^2 - 3r^2 - 21r = 0$$

which we can expand and simplify to give

$$-3r + 27 = 0,$$

so $r = 9$.

We can now calculate the arc length of the larger sector; it is $(r+3)(6/r) = 8$.

Therefore, the perimeter of the first sector is $2r + 6 = 24$ and the perimeter of the second sector is $2(r+3) + 8 = 32$. The positive difference is 8, which is option C.

8.2 Answer E

This equation has trigonometric functions and an x on its own. Our best bet is to first rearrange it to bring all of the trigonometric functions together. We have a $\sin 2x$ on one side and a $\cos 2x$ on the other, which suggests that we should divide by $\cos 2x$ to obtain $\tan 2x$. This gives

$$x \tan 2x = 1$$

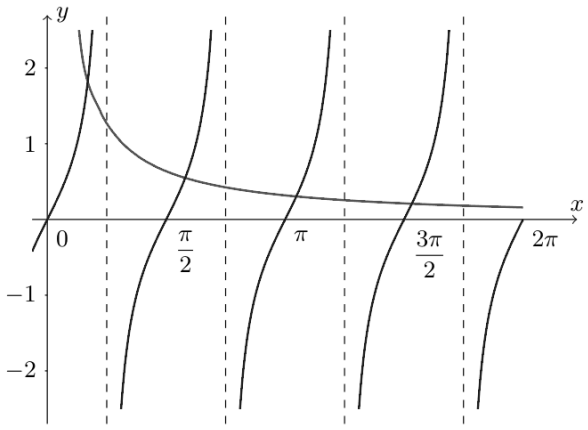
We should check, though, that we cannot have $\cos 2x = 0$, so that we are allowed to divide by it. If $\cos 2x = 0$, this would require $x \sin 2x = 0$, so either $\sin 2x = 0$, which is impossible if $\cos 2x = 0$, or $x = 0$. But if $x = 0$, then $\cos 2x = 1$, so this is also impossible. So we cannot have $\cos 2x = 0$.

Returning to $x \tan 2x = 1$, we can divide by x to get

$$\tan 2x = \frac{1}{x}$$

This equation is not possible to solve exactly. But the question (thankfully) only asks us how many solutions there are in the range $0 \leqslant x \leqslant 2\pi$

So we can sketch the graphs of $y = \tan 2x$ and $y = \frac{1}{x}$ and see how many times they intersect.



We see that the graph of $y = \frac{1}{x}$ intersects every positive branch of $y = \tan 2x$ once. Since there are four branches of $y = \tan 2x$ in the range $0 \leqslant x \leqslant 2\pi$, there are four solutions, so the correct answer is E.

8.3 Answer F

The student wishes to evaluate f(4). Writing 4 rad to explicitly indicate radians, this becomes

$$\text{f}(4) = 4\sin(4\,\text{rad}) = 4 \times \sin(4\,\text{rad}).$$

Now

$$\pi\,\text{rad} = 180^\circ,$$

so dividing by π gives

$$1\,\text{rad} = \frac{180^\circ}{\pi}.$$

Multiplying this by 4 then gives

$$4\,\text{rad} = \frac{4 \times 180^\circ}{\pi}.$$

Therefore the student could type into their calculator

$$4 \times \sin(4 \times 180 \div \pi)$$

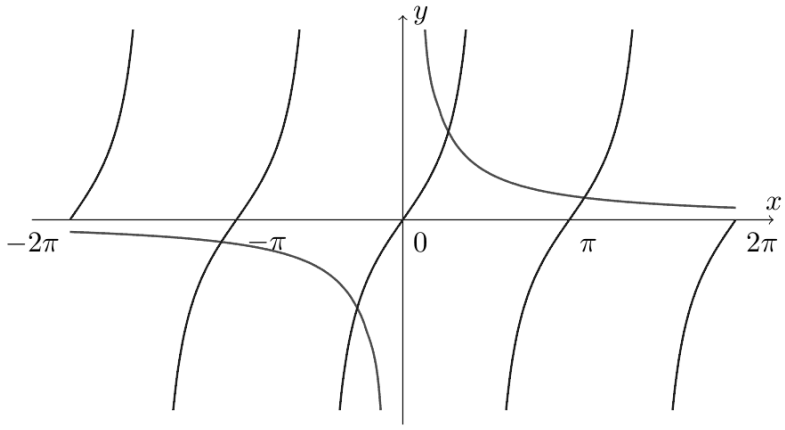
which is equivalent to the calculation of F.

8.4 Answer E

TMUA 2016 · Paper 1 · Q10

We can rearrange this equation as $\tan x = \dfrac{1}{x}$. (We note that $x = 0$ is not a solution of the original equation $x \tan x = 1$, so we can divide by x without losing any solutions.)

It is impossible to solve this equation exactly, but we are only asked for the number of solutions. Therefore we sketch the graphs of $y = \tan x$ and $y = \frac{1}{x}$ on the same axes, and count the number of intersections.



There are four points of intersection with $-2\pi \leqslant x \leqslant 2\pi$, so the answer is 4, which is option E.

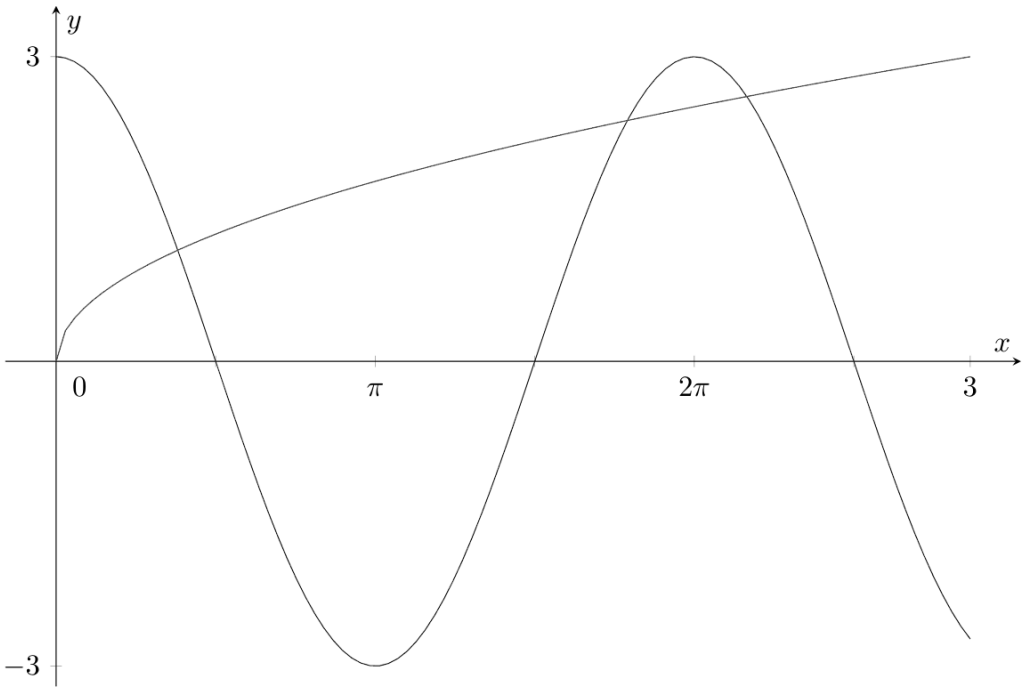
8.5 Answer D

TMUA 2020 · Paper 1 · Q12

We can make some immediate observations about this equation:

- The left hand side takes values between -3 and $+3$.
- The right hand side is only defined for $x \geq 0$ and is always non-negative, and when $x > 9$, $\sqrt{x} > 3$.

Therefore we need only consider the region $0 \leq x \leq 9$. Note that $3\pi \approx 9.5$, so we have just less than 1.5 periods of the cosine function in this range, and $3\cos 9 < 0$. We can therefore sketch the pair of graphs $y = 3\cos x$ and $y = \sqrt{x}$ and observe how many times they intersect:

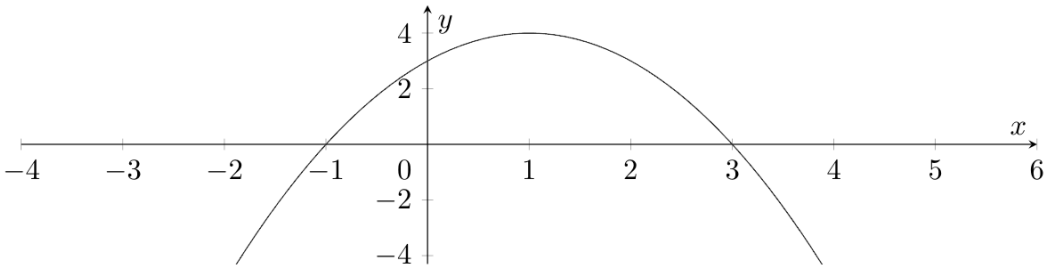


We see that there are 3 points of intersection, and hence 3 solutions to the equation, which is option D.

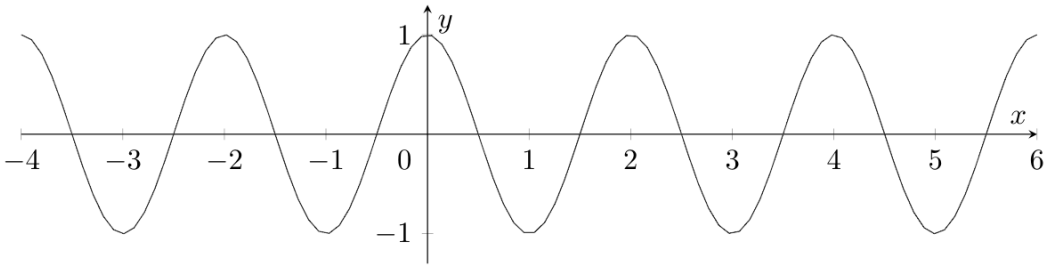
8.6 Answer B

We do not know a method of solving equations involving both trigonometric functions (on the right-hand side) and quadratics (on the left-hand side), but the question only asks us to count the number of solutions, not to solve the equation. So we will instead sketch the graphs of the left-hand and right-hand sides and see how many times they intersect.

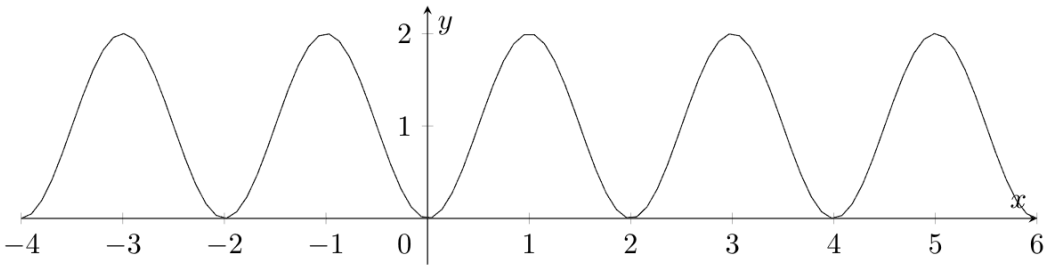
The graph of $y = (x+1)(3-x)$ is a quadratic with roots at $x = -1$ and $x = 3$. When expanded, it becomes $y = -x^2 + \cdots$, so it has a maximum point midway between the roots, at $x = 1$, $y = 4$.



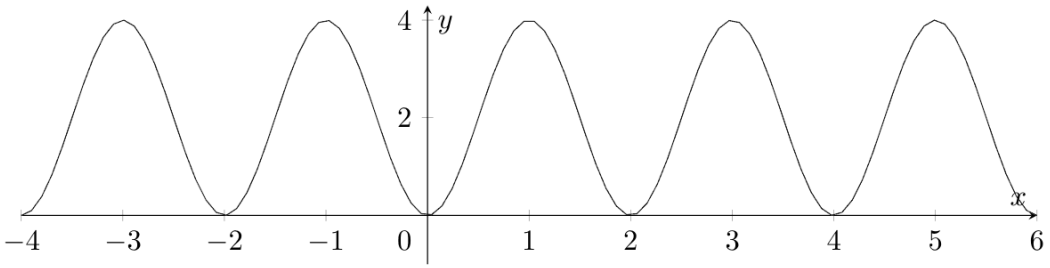
The graph of $y = 2(1 - \cos(\pi x))$ is a transformed cosine graph. $y = \cos(\pi x)$ has a period of 2: it is the graph of $y = \cos x$ scaled by a factor of $1/\pi$ in the x-direction, and looks like this:



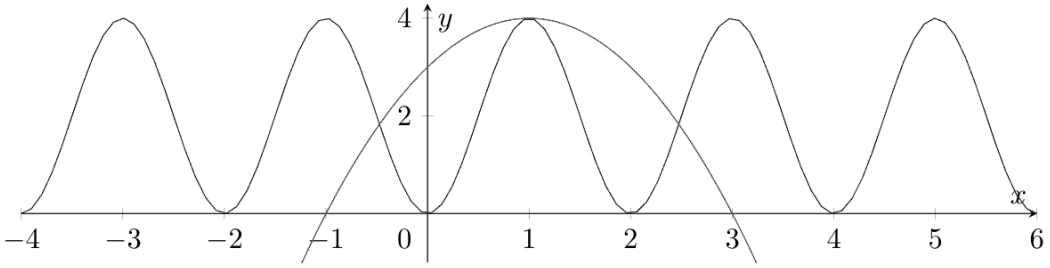
We can then work out the graph of $y = 1 - \cos(\pi x)$ by calculating the values of y at a few key points, or by observing that the graph of $y = -\cos(\pi x)$ is the reflection of $y = \cos(\pi x)$ in the x-axis, and then $y = 1 - \cos(\pi x)$ is this translated by 1 unit in the y-direction. The result is:



Finally, the graph of $y = 2(1 - \cos(\pi x))$ is the previous graph scaled by a factor of 2 in the y-direction:



We can now superimpose the graph of $y = (x + 1)(3 - x)$ on top of $y = 2(1 - \cos(\pi x))$. We see that the trigonometric graph is never below the x-axis, so the only possible points of intersection are between $x = -1$ and $x = 3$. There is one point of intersection at $(1, 4)$, one between $x = -1$ and $x = 0$ and one between $x = 2$ and $x = 3$, so there are three points of intersection in total.



Thus the correct answer is option B.

You might wonder about a subtle point: we see that the two graphs touch each other at $(1, 4)$*, but how do we know that the trigonometric graph is 'sharper' there, so does not also cross the quadratic graph close to* $x = 1$*, something like this:*



This is a plausible possibility, but it turns out not to be the case for reasons that are beyond the scope of the specification.

We note that $x^2+y^2 \geq 0$ for all x and y, so we need only consider non-negative values of x^2+y^2 that satisfy this equation. Those with $x^2+y^2 \leq 8\pi$ are

$$x^2+y^2 = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}, \frac{25\pi}{6}, \frac{29\pi}{6}, \frac{37\pi}{6}, \frac{41\pi}{6}$$

(These were obtained by noting that the solutions with $x^2+y^2 \leq 2\pi$ are $\frac{\pi}{6}$ and $\frac{5\pi}{6}$, and then adding multiples of $2\pi = \frac{12\pi}{6}$ to these.)

Consider the first value; $x^2+y^2 = \frac{\pi}{6}$; this is a circle with radius $\sqrt{\pi/6}$. Likewise, the other circles have radii $\sqrt{5\pi/6}$, …, $\sqrt{41\pi/6}$. Taking out a common factor of $\sqrt{\pi/6}$, we see that the circles have radii:

$$\rho\sqrt{\frac{\pi}{6}}, \quad \rho = 1, \sqrt{5}, \sqrt{13}, \sqrt{17}, \sqrt{25}, \sqrt{29}, \sqrt{37}, \sqrt{41}$$

We can approximate these square roots to get an idea of how they grow:

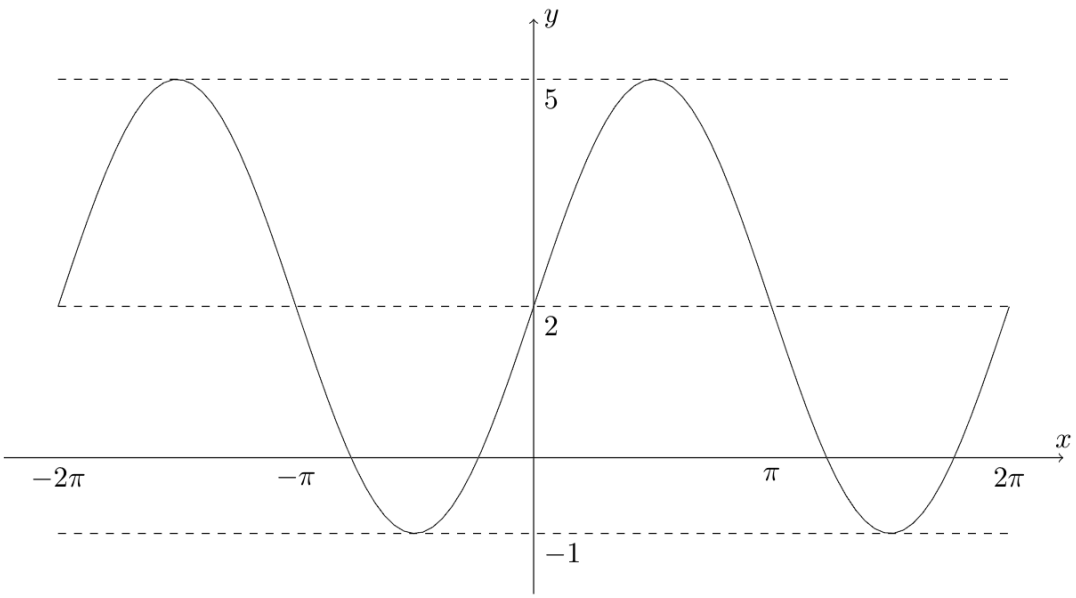
$$\rho \approx 1, 2, 3.5, 4, 5, 5.5, 6, 6.5$$

So the smallest three circles are quite far apart, but after that, they get closer and closer, which is graph A. (We can be more precise than this, and observe that the circles appear in close pairs with larger gaps between the pairs, but that is not needed to answer this question.)

The graph of $y = 3\sin x + 2$ is the standard sine graph stretched vertically by a factor of 3 and then translated in the y-direction by 2. It thus oscillates between $2-3=-1$ and $2+3=5$, and its y-axis intercept is 2.

$y = x + c$ has gradient 1 and y-intercept c.

We sketch the graph of $y = 3\sin x + 2$ to help us:



We now consider the statements in turn:

1 To have exactly one solution with $0 \leqslant x \leqslant \pi$ and at least one solution with $-\pi < x < 0$, we could take c to be just less than 2. So this statement is true.

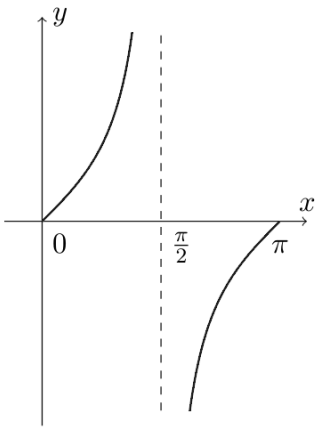
2 We want to choose c to be negative for this case; if c is such that $y = x + c$ intersects the sine curve at $x = \pi$, there will be exactly one solution with $0 \leqslant x \leqslant \pi$. This value of c must satisfy $2 = \pi + c$, so $c = 2 - \pi < -1$, so there are no solutions with $x < 0$. So this statement is true.

3 The same value of c as in statement **2** will work here: $c = 2 - \pi$. At $x = 2\pi$, $x + c = 2 + \pi > 5$, so there cannot be any solutions with $x \geqslant 2\pi$. As the sine graph is below 2 for $\pi < x < 2\pi$, there are no solutions there either. So this statement is true.

The correct option is therefore H.

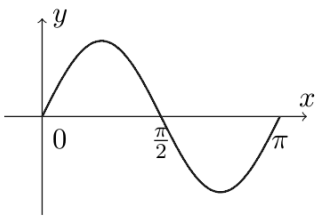
We first consider each of the inequalities separately.

A sketch of the graph of $y = \tan x$ for $0 \leqslant x \leqslant \pi$ is as follows:



Noting that $\tan \frac{pi}{4} = 1$ and $\tan \frac{3\pi}{4} = -1$, we see from the sketch that $-1 \leqslant \tan x \leqslant 1$ when $0 \leqslant x \leqslant \frac{\pi}{4}$ and when $\frac{3\pi}{4} \leqslant x \leqslant \pi$.

Next, we sketch the graph of $y = \sin 2x$ over the same interval:



Noting that $\sin \frac{\pi}{6} = \sin \frac{5\pi}{6} = 0.5$, we see from the sketch that $\sin 2x \geqslant 0.5$ when $\frac{\pi}{6} \leqslant 2x \leqslant \frac{5\pi}{6}$, that is, when $\frac{\pi}{12} \leqslant x \leqslant \frac{5\pi}{12}$.

Now comparing the intervals for the tangent and sine inequalities, we note that they are both satisfied when $\frac{\pi}{12} \leqslant x \leqslant \frac{\pi}{4}$, and the length of this interval is $\frac{\pi}{4} - \frac{\pi}{12} = \frac{\pi}{6}$, so the answer is B.

Sequences and Series

9.1 Answer H

TMUA 2016 · Paper 1 · Q1

We expand $(ax+b)^3$ and compare with the given expansion.

$$(ax+b)^3 = a^3x^3 + 3a^2bx^2 + 3ab^2x + b^3 = 8x^3 - px^2 + 18x - 3\sqrt{3}.$$

So we must have:

$$\begin{aligned} a^3 &= 8 \\ 3a^2b &= -p \\ 3ab^2 &= 18 \\ b^3 &= -3\sqrt{3} \end{aligned}$$

The final equation can be rewritten as $b^3 = -3^{\frac{3}{2}}$, so $b = -3^{\frac{1}{2}} = -\sqrt{3}$.

Then the third equation becomes $3a \times 3 = 18$, so $a = 2$.

Finally, the second equation gives $3 \times 2^2 \times (-\sqrt{3}) = -p$, so $p = 12\sqrt{3}$ and the answer is H.

9.2 Answer E

TMUA 2019 · Paper 2 · Q1

We need the coefficient of x^2 in the bracketed term to get the term in x^4 in the whole expansion.
We expand this using the binomial theorem to get

$$x^2\left(2x+\frac{1}{x}\right)^6 = x^2\left((2x)^6 + \binom{6}{1}(2x)^5\left(\frac{1}{x}\right) + \binom{6}{2}(2x)^4\left(\frac{1}{x}\right)^2 + \cdots\right).$$

We stop here as the power of x in this last term is $x^{4-2} = x^2$, as we require. This term is then

$$\begin{aligned} x^2\binom{6}{2}(2x)^4\left(\frac{1}{x}\right)^2 &= 15x^2.16x^4.x^{-2} \\ &= 240x^4 \end{aligned}$$

so the answer is E.

9.3 Answer C

TMUA 2018 · Paper 1 · Q2

One approach is to use the formulae for the sum of the first 5 and the first 8 terms (that is, $S_n = \frac{1}{2}n(2a + (n-1)d)$).

We have

$$S_5 = \tfrac{1}{2} \times 5(2a + 4d) = \tfrac{5}{2}(2a + 4d)$$
$$S_8 = \tfrac{1}{2} \times 8(2a + 7d) = 4(2a + 7d)$$

These are equal, so we have

$$\tfrac{5}{2}(2a + 4d) = 4(2a + 7d)$$

We want to rearrange this to get a formula in the form $a = \cdots$, so we multiply both sides by 2, expand the brackets and collect like terms:

$$\begin{aligned} & 5(2a + 4d) = 8(2a + 7d) \\ \iff \quad & 10a + 20d = 16a + 56d \\ \iff \quad & -6a = 36d \\ \iff \quad & a = -6d \end{aligned}$$

so the answer is C.

An alternative approach is to note that the difference between the sum of the first 5 and the sum of the first 8 terms equals the sum of the 6th, 7th and 8th terms. (This is because $S_5 = u_1 + u_2 + \cdots + u_5$ while $S_8 = u_1 + u_2 + \cdots + u_5 + u_6 + u_7 + u_8$, where we have written the sequence as $u_1, u_2, \ldots$.) Since the two sums S_5 and S_8 are equal, the sum of the 6th, 7th and 8th terms must be zero. This gives

$$(a + 5d) + (a + 6d) + (a + 7d) = 0$$

Expanding and collecting like terms gives $3a + 18d = 0$, so $a = -6d$, as before.

9.4 Answer B

TMUA 2018 · Paper 2 · Q2

The constant term is the one with x^0. The expansion is

$$(x^6)^{12} + \binom{12}{1}(x^6)^{11}\left(-\frac{1}{x^2}\right) + \cdots + \binom{12}{k}(x^6)^{12-k}\left(-\frac{1}{x^2}\right)^k + \cdots + \left(-\frac{1}{x^2}\right)^{12}$$

(This notation for binomial coefficients is much clearer than the older ${}^n\mathrm{C}_r$ notation, and is recommended.)

The kth term is therefore

$$\binom{12}{k}x^{6(12-k)}(-1)^k x^{-2k} = \binom{12}{k}(-1)^k x^{6(12-k)-2k} = \binom{12}{k}(-1)^k x^{72-8k}$$

The value of k which makes the power zero is therefore $k = 9$, so the constant term is

$$\binom{12}{9}(-1)^9 = -\binom{12}{3} = -\frac{12 \times 11 \times 10}{3 \times 2 \times 1} = -2 \times 11 \times 10 = -220$$

where we have used $\binom{n}{k} = \binom{n}{n-k}$ to simplify the calculation. Thus the answer is B.

> ***Commentary:*** *We have used k here (beginning our counting with $k = 0$). You might prefer to use another letter such as r.*

9.5 Answer E

TMUA 2022 · Paper 2 · Q2

We expand the brackets in the first term using the binomial theorem and write out the sum in full to get

$$(1+5x+10x^2+10x^3+5x^4+x^5)\times(1+x+x^2+x^3+x^4+x^5).$$

The x^5 term in the product is obtained by taking all possible pairs of a term in the first bracket and a term in the second whose product is cx^5 for some number c, and then adding these together. Therefore the x^5 term in the product is

$$1\cdot x^5+5x\cdot x^4+10x^2\cdot x^3+10x^3\cdot x^2+5x^4\cdot x+x^5\cdot 1=(1+5+10+10+5+1)x^5=32x^5.$$

Therefore the correct answer is option E.

9.6 Answer E

TMUA 2019 · Paper 1 · Q3

The coefficient of x in the expansion of $(1+x)^n$ is $\binom{n}{1}=n$, so the coefficient of x in this whole expression is

$$0+1+2+3+\cdots+79+80=\tfrac{1}{2}\times 80\times 81$$

using the formula for the sum of the first n positive integers, $\frac{1}{2}n(n+1)$. This gives $40\times 81=3240$, so the answer is E.

9.7 Answer G

TMUA 2021 · Paper 1 · Q3

Let us write the arithmetic progression as

$$\tfrac{1}{2},\quad \tfrac{1}{2}+d,\quad \tfrac{1}{2}+2d,\quad \tfrac{1}{2}+3d,\quad \ldots$$

and the geometric progression as

$$\tfrac{1}{2},\quad \tfrac{1}{2}r,\quad \tfrac{1}{2}r^2,\quad \tfrac{1}{2}r^3,\quad \ldots$$

Then the given information about the second terms and third terms can be written algebraically as

$$\tfrac{1}{2}+d+\tfrac{1}{2}r=\tfrac{1}{2}$$

$$\tfrac{1}{2}+2d+\tfrac{1}{2}r^2=\tfrac{1}{8}$$

The first equation simplifies to give $d+\frac{1}{2}r=0$, so $d=-\frac{1}{2}r$. We can then substitute this into the second equation to give

$$\tfrac{1}{2}+\tfrac{1}{2}r^2-r=\tfrac{1}{8}$$

(We could equivalently have eliminated d by subtracting twice the first equation from the second.) Multiplying by 8 and rearranging slightly gives

$$4r^2-8r+3=0.$$

We can solve this by factorising or using the formula, for example; this gives

$$r=\frac{8\pm\sqrt{(-8)^2-4\times 4\times 3}}{2\times 4}=\tfrac{1}{2}\text{ or }\tfrac{3}{2}$$

Since the geometric progression is convergent, we must have $|r|<1$, so $r=\frac{1}{2}$ and the sum to infinity is

$$S_\infty=\frac{a}{1-r}=\frac{(\frac{1}{2})}{1-\frac{1}{2}}=1$$

so the answer is option G.

9.8 Answer G

TMUA 2017 · Paper 2 · Q3

The first term is $a = 2\sqrt{3}$ and the common ratio is r. So the fourth term is $ar^3 = 2\sqrt{3}\,r^3 = \frac{9}{4}$ and hence

$$r^3 = \frac{9}{8\sqrt{3}} = \frac{9\sqrt{3}}{8\times 3} = \frac{3\sqrt{3}}{8}$$

by rationalising the denominator.

We can write $3\sqrt{3}$ as $\sqrt{27}$ or $3^{\frac{3}{2}}$ (since we want to take the cube root of this expression), and so we have

$$r^3 = \frac{3^{\frac{3}{2}}}{8} \implies r = \frac{3^{\frac{1}{2}}}{2} = \frac{\sqrt{3}}{2}.$$

(There is a unique real cube root of any real number, unlike the case for square roots.)

We can now find the sum to infinity of the geometric progression:

$$\begin{aligned}
S_\infty &= \frac{a}{1-r} \\
S_\infty &= \frac{2\sqrt{3}}{1-\frac{\sqrt{3}}{2}} \\
&= \frac{4\sqrt{3}}{2-\sqrt{3}} && \text{multiplying by } \frac{2}{2} \\
&= \frac{4\sqrt{3}(2+\sqrt{3})}{(2-\sqrt{3})(2+\sqrt{3})} && \text{rationalising the denominator} \\
&= \frac{4(2\sqrt{3}+3)}{1} && \text{paying attention to the options offered} \\
&= 4(2\sqrt{3}+3)
\end{aligned}$$

and the correct answer is G.

9.9 Answer B

TMUA 2016 · Paper 1 · Q4

> ***Commentary:*** *This looks quite scary! It is unlikely that you have ever seen a sequence looking like this, so a sensible thing to do is to work out the first few values and look for any patterns.*

We calculate the first few terms of the sequence:

$$\begin{aligned}
a_1 &= (-1)^1 - (-1)^0 + (-1)^3 = (-1) - 1 + (-1) = -3 \\
a_2 &= (-1)^2 - (-1)^1 + (-1)^4 = 1 - (-1) + 1 = 3 \\
a_3 &= (-1)^3 - (-1)^2 + (-1)^5 = (-1) - 1 + (-1) = -3 \\
a_4 &= (-1)^4 - (-1)^3 + (-1)^6 = 1 - (-1) + 1 = 3
\end{aligned}$$

The pattern is now clear (and we could prove it if we wished to): the sequence goes -3, 3, -3, 3, -3, 3 and so on. So the sum of each pair of terms is zero: $a_1 + a_2 = 0$, $a_3 + a_4 = 0$, …, $a_{37} + a_{38} = 0$. Thus

$$\begin{aligned}
\sum_{n=1}^{39} a_n &= (a_1 + a_2) + (a_3 + a_4) + \cdots + (a_{37} + a_{38}) + a_{39} \\
&= 0 + 0 + \cdots + 0 + (-3) \\
&= -3
\end{aligned}$$

and the answer is B.

9.10 Answer C

TMUA 2019 · Paper 1 · Q4

We work out the first few terms, writing them in the form of the given options:

$$x_1 = 10$$
$$x_2 = \sqrt{10} = 10^{\frac{1}{2}} = 10^{2^{-1}}$$
$$x_3 = \sqrt{10^{2^{-1}}} = \left(10^{2^{-1}}\right)^{\frac{1}{2}} = 10^{2^{-1}\times\frac{1}{2}} = 10^{2^{-2}}$$

As this pattern continues, we will end up with $x_{100} = 10^{2^{-99}}$, which is option C.

9.11 Answer D

TMUA 2020 · Paper 1 · Q4

Let the arithmetic progression have first term a and common difference d. Let the geometric progression have first term a (as the first term of the geometric progression is the same as the first term of the arithmetic progression) and common ratio r.

Then the first three terms of the geometric progression are a, ar and ar^2, while the 1st, 4th and 6th terms of the arithmetic progression are a, $a+3d$ and $a+5d$.

Therefore $ar = a+3d$ and $ar^2 = a+5d$.

The next piece of information we are given is that the sum to infinity of the geometric progression is 12, so $\dfrac{a}{1-r} = 12$, hence $a = 12(1-r)$.

So we have three equations:

$$ar = a+3d$$
$$ar^2 = a+5d$$
$$a = 12(1-r)$$

There are several ways to work out the value of a (as required for this question) from these equations; here is one approach.

We can eliminate d from the first two equations by finding 5 times the first equation minus 3 times the second equation; this gives

$$5ar - 3ar^2 = 2a$$

so dividing by a yields $5r-3r^2 = 2$, giving $3r^2-5r+2 = 0$. This factorises as $(3r-2)(r-1) = 0$, so $r = 1$ or $r = \frac{2}{3}$. Since the geometric progression has a sum to infinity, we must have $|r| < 1$, so $r = \frac{2}{3}$.

Then the final equation gives $a = 4$, which is option D.

9.12 Answer C

TMUA 2023 · Paper 1 · Q4

To help us understand what is going on here, we will work out the first few terms.

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{\sin\left(n\pi+\frac{\pi}{3}\right)}{2^n} &= \frac{\sin\left(\frac{\pi}{3}\right)}{2^0}+\frac{\sin\left(\pi+\frac{\pi}{3}\right)}{2^1}+\frac{\sin\left(2\pi+\frac{\pi}{3}\right)}{2^2}+\frac{\sin\left(3\pi+\frac{\pi}{3}\right)}{2^3}+\cdots\\ &= \frac{\sqrt{3}/2}{1}+\frac{-\sqrt{3}/2}{2}+\frac{\sqrt{3}/2}{4}+\frac{-\sqrt{3}/2}{8}+\cdots\\ &= \frac{\sqrt{3}}{2}\left(\frac{1}{1}-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\cdots\right)\end{aligned}$$

We see that the part in the brackets is a geometric series with first term $a = 1$ and common ratio $r = -\frac{1}{2}$, so its sum is

$$\frac{a}{1-r} = \frac{1}{1-\left(-\frac{1}{2}\right)} = \frac{1}{3/2} = \frac{2}{3}$$

and hence the sum of the original series is

$$\frac{\sqrt{3}}{2}\times\frac{2}{3} = \frac{\sqrt{3}}{3}$$

which is option C.

9.13 Answer E

TMUA 2019 · Paper 1 · Q5

Let the first term of S be a and the common ratio r, as usual.

The sum of the first n terms of S is $\dfrac{a(r^n-1)}{r-1}$, so the second line of the question can be written as

$$\frac{a(r^6-1)}{r-1} = \frac{9a(r^3-1)}{r-1}$$

which can be multiplied by $(r-1)$ and divided by a to give

$$r^6-1 = 9(r^3-1).$$

Writing $R = r^3$, this becomes $R^2-1 = 9R-9$, so $R^2-9R+8 = 0$. This factorises to $(R-1)(R-8) = 0$, so $R = 1$ or $R = 8$, and hence $r = 1$ or $r = 2$.

However, if $r = 1$, then the original equation is not valid; in that case, the sequence is constant, the sum of the first 6 terms is $6a$ and the sum of the first 3 terms is $3a$. We could only have $6a = 9\times 3a$ if $a = 0$, but that is impossible as the 7th term of S is 360. So we must have $r = 2$.

The 7th term of S is ar^6, so $360 = 2^6a = 64a$, and hence $a = \frac{360}{64} = \frac{45}{8}$ and so the answer is E.

9.14 Answer H

TMUA 2022 · Paper 1 · Q5

We write out the formulae for x_2, x_3 and x_4 in terms of p and q, using the values $x_1 = 3$, $x_2 = 5$ and $x_3 = 7$:

$$n = 1: \qquad 5 = \frac{3+p}{3+q}$$

$$n = 2: \qquad 7 = \frac{5+p}{5+q}$$

$$n = 3: \qquad x_4 = \frac{7+p}{7+q}$$

Multiplying the first equation by $3+q$ and the second by $5+q$ gives

$$15 + 5q = 3 + p$$
$$35 + 7q = 5 + p$$

These are simultaneous equations for p and q; subtracting gives $20 + 2q = 2$ so $q = -9$ and therefore $p = -33$.

Substituting these into the expression for x_4 gives

$$x_4 = \frac{7-33}{7-9} = 13$$

so the answer is option H.

9.15 Answer E

TMUA 2023 · Paper 1 · Q6

A coefficient is divisible by 12 if and only if it is a multiple of $2^2 = 4$ and a multiple of 3. The coefficient of x^k (for $0 \leq k \leq 12$) is given by $\binom{12}{k}2^{12-k}3^k$.

This is certainly a multiple of 3 when $k \geq 1$.

This is certainly a multiple of $2^2 = 4$ when $k \leq 10$.

We therefore only need to check $k = 0$, $k = 11$ and $k = 12$.

When $k = 0$, the coefficient is $\binom{12}{0} \times 2^{12} = 2^{12}$, which is not a multiple of 3, and so not of 12 either.

When $k = 11$, the coefficient is $\binom{12}{11} \times 2 \times 3^{11}$. But $\binom{12}{11} = \binom{12}{1} = 12$, so this is a multiple of 12.

When $k = 12$, the coefficient is $\binom{12}{12} \times 3^{12} = 3^{12}$, which is not a multiple of 2, and so not of 12 either.

Therefore the coefficient is a multiple of 12 for $1 \leq k \leq 11$, so 11 terms have such a coefficient, and the answer is option E.

We work out the first few functions in the sequence to find a pattern.

$$\begin{aligned}
\mathrm{f}_1(x) &= x^{10} \\
\mathrm{f}_2(x) &= x(10x^9) = 10x^{10} \\
\mathrm{f}_3(x) &= x(100x^9) = 100x^{10} \\
\mathrm{f}_4(x) &= x(1000x^9) = 10^3x^{10} \\
\mathrm{f}_5(x) &= x(10^4x^9) = 10^4x^{10}
\end{aligned}$$

and so the pattern is clear: $\mathrm{f}_n(x) = 10^{n-1}x^{10}$.

The sum expands to

$$\begin{aligned}
\sum_{n=1}^{20} \mathrm{f}_n(x) &= x^{10} + 10x^{10} + 10^2x^{10} + 10^3x^{10} + \cdots + 10^{19}x^{10} \\
&= (1 + 10 + 10^2 + \cdots + 10^{19})x^{10} \\
&= \left(\frac{10^{20} - 1}{10 - 1}\right) x^{10}
\end{aligned}$$

where on the final line we have summed the geometric series with $a = 1$, $r = 10$, $n = 20$ using the formula

$$S_n = \frac{a(r^n - 1)}{r - 1}.$$

The answer is therefore option C.

We calculate the first few values of u_n in the hope that this will show a clear pattern.

We have

$$\begin{aligned} u_0 &= 1 \\ u_1 &= \int_0^1 4x.1 \, \mathrm{d}x \\ &= \left[2x^2\right]_0^1 \\ &= 2 \\ u_2 &= \int_0^1 4x.2 \, \mathrm{d}x \\ &= \left[4x^2\right]_0^1 \\ &= 4 \\ u_3 &= \int_0^1 4x.4 \, \mathrm{d}x \\ &= \left[8x^2\right]_0^1 \\ &= 8 \end{aligned}$$

The pattern is now apparent: $u_n = 2^n$, and so the answer is A.

If we wanted to *prove* this to be true, we could use induction. The result is clearly true when $n = 0$, so assume that it is true when $n = k$. Then

$$\begin{aligned} u_{k+1} &= \int_0^1 4x.2^k \, \mathrm{d}x \\ &= \left[2.2^k x^2\right]_0^1 \\ &= 2.2^k \\ &= 2^{k+1} \end{aligned}$$

and so the result holds for $n = k + 1$. Hence by induction, $u_n = 2^n$ for all $n \geqslant 0$.

9.18 Answer B

TMUA 2017 · Paper 1 · Q7

Let the common difference of the arithmetic progression be d and the common ratio of the geometric progression be r. Then we can write the first three terms of each as

$$p,\ q = p + d,\ p^2 = p + 2d \qquad \text{and} \qquad p,\ p^2 = pr,\ q = pr^2.$$

The second term of the geometric progression shows that $r = p$ (as p is non-zero), so the third term gives $q = p^3$

Looking again at the arithmetic progression, using $q = p^3$ now shows that the second and third terms are

$$p^3 = p + d$$
$$p^2 = p + 2d$$

We can eliminate d from these by subtracting twice the first equation from the second, giving $p^2 - 2p^3 = -p$. This rearranges to

$$2p^3 - p^2 - p = 0$$

This factorises as $p(2p^2 - p - 1) = 0$, and then to $p(2p + 1)(p - 1) = 0$. This has the three solutions $p = 0$, $p = -\frac{1}{2}$ and $p = 1$. We were told that $p < 0$, so we must have $p = -\frac{1}{2}$.

Thus $d = p^3 - p = -\frac{1}{8} - (-\frac{1}{2}) = \frac{3}{8}$ and the sum of the first 10 terms of the arithmetic progression is given, using the formula $S_n = \frac{1}{2}n(2a + (n-1)d)$, by

$$S_{10} = \frac{1}{2} \times 10 \left(2 \times \left(-\frac{1}{2}\right) + 9 \times \frac{3}{8}\right) = 5\left(-1 + \frac{27}{8}\right) = \frac{95}{8}$$

and the correct answer is B.

9.19 Answer A

TMUA 2018 · Paper 1 · Q7

We can expand these two expressions using the binomial theorem. We look just at the term involving x^6:

$$\begin{aligned}(1 + kx^2)^7 &= \cdots + \binom{7}{3} 1^4 (kx^2)^3 + \cdots \\ &= \cdots + \frac{7 \times 6 \times 5}{3 \times 2 \times 1} k^3 x^6 + \cdots \\ &= \cdots + 35k^3 x^6 + \cdots\end{aligned}$$

and

$$\begin{aligned}(k + x)^{10} &= \cdots + \binom{10}{6} k^4 x^6 + \cdots \\ &= \cdots + \binom{10}{4} k^4 x^6 + \cdots \\ &= \cdots + \frac{10 \times 9 \times 8 \times 7}{4 \times 3 \times 2 \times 1} k^4 x^6 + \cdots \\ &= \cdots + 210k^4 x^6 + \cdots\end{aligned}$$

We therefore need

$$35k^3 = 210k^4$$

so $k = \dfrac{35}{210} = \dfrac{5}{30} = \dfrac{1}{6}$.

Thus the correct answer is A.

9.20 Answer B

TMUA 2018 · Paper 2 · Q7

We can list the terms of the two sequences to get a feeling for what is going on:

Sequence 1: 11, 14, **17**, 20, 23, 26, 29, **32**, 35, 38, ...

Sequence 2: 2, 7, 12, **17**, 22, 27, **32**, 37, 42, ...

We see that the first two common terms are 17 and 32; the next will be 47, and the common terms are forming an arithmetic progression with first term 17 and common difference 15. (Note that 15 is the lowest common multiple of 3 and 5.)

Therefore the 20th such number, N is $17 + 19 \times 15 = 17 + 285 = 302$. Finally, dividing 302 by 7 gives 43 remainder 1, so the answer is B.

9.21 Answer D

TMUA 2018 · Paper 1 · Q8

Let us write the terms of the GP (geometric progression) as a, ar, ar^2 and so on. Then we are told that

$$S_\infty = a + ar + ar^2 + \cdots = \frac{a}{1-r} = 6 \tag{1}$$

We can rearrange this final equation to $a = 6 - 6r$ if we wish to.

The squares of each term are a^2, a^2r^2, a^2r^4, a^2r^6 and so on. These themselves form a GP with first term a^2 and common ratio r^2. Therefore, given that their sum is 12, we have for this GP

$$S_\infty = a^2 + a^2r^2 + a^2r^4 + \cdots = \frac{a^2}{1-r^2} = 12$$

We can write this last equation as $\dfrac{a^2}{(1+r)(1-r)} = 12$

But we can now substitute in equation (1) to give

$$6\left(\frac{a}{1+r}\right) = 12$$

so that $\dfrac{a}{1+r} = 2$, or $a = 2 + 2r$

Thus we have simultaneous equations for a and r:

$$a = 6 - 6r$$
$$a = 2 + 2r$$

Subtracting these gives $4 - 8r = 0$, so $r = \frac{1}{2}$ and $a = 3$

We can now work out the sum to infinity of the cubes: the sequence of terms becomes a^3, a^3r^3, a^3r^6 and so on, which as before is a GP; this time the first term is a^3 and the common ratio is r^3. Hence the sum to infinity is

$$S_\infty = \frac{a^3}{1-r^3} = \frac{3^3}{1-(\frac{1}{2})^3} = \frac{27}{(\frac{7}{8})} = \frac{216}{7}$$

and the correct answer is D.

9.22 Answer B

TMUA 2022 · Paper 1 · Q8

We use the formula $S_n = \dfrac{a(r^n - 1)}{r - 1}$ and substitute into the given equation:

$$\frac{a(r^{30} - 1)}{r - 1} - \frac{a(r^{20} - 1)}{r - 1} = k\frac{a(r^{10} - 1)}{r - 1}.$$

We can multiply this by $r - 1$ and divide by a to get

$$(r^{30} - 1) - (r^{20} - 1) = k(r^{10} - 1),$$

which simplifies to

$$r^{30} - r^{20} = k(r^{10} - 1).$$

Factorising the left hand side then gives

$$r^{20}(r^{10} - 1) = k(r^{10} - 1);$$

as $r > 1$, we can divide by $r^{10} - 1 \neq 0$ to obtain

$$r^{20} = k.$$

Therefore k is a 20th power, and the smallest possible value is 2^{20}, which is option B.

9.23 Answer C

TMUA 2022 · Paper 2 · Q8

We first determine the number of terms in the whole sequence: there are $\frac{70-1}{3} + 1 = 23 + 1 = 24$ terms.

We next determine the number of pairs that sum to 74. The pairs are:

$$\begin{gathered} 4 + 70 \\ 7 + 67 \\ \vdots \\ 34 + 40 \\ [37 + 37] \end{gathered}$$

The final pair is excluded because the two terms have to be distinct to fit the rule in $(*)$. The number of valid pairs is therefore $\frac{34-4}{3} + 1 = 10 + 1 = 11$.

If a selection S does *not* satisfy the rule in $(*)$, then it can have at most one term from each of these pairs, so it has to leave out at least 11 terms. So it can have at most $24 - 11 = 13$ terms. Therefore if S has at least 14 terms, it must satisfy $(*)$.

We also need to show that if S only has 13 terms, it might not satisfy $(*)$. We can include 1, 37 and one term from each listed pair, giving $2 + 11 = 13$ terms. These 13 terms do not contain a distinct pair summing to 37.

Hence the smallest value of n which forces $(*)$ to be true is 14, and the correct answer is option C..

9.24 Answer D

TMUA 2023 · Paper 2 · Q9

We first note that the interior angles in a pentagon sum to $3 \times 180° = 540°$ (since a pentagon can be cut up into three triangles). Therefore the mean interior angle is $540°/5 = 108°$.

I The angles could be 108°, 108°, 108°, 107° and 109°: start with a regular pentagon and slightly rotate one of the sides. These do not form an arithmetic sequence, so statement $(*)$ is false.

II The contrapositive of a statement always has the same truth value as the statement itself, so this is also false.

III The converse of $(*)$ states: **if** all the interior angles in P form an arithmetic sequence, **then** at least one of the interior angles in P is 108°. If the interior angles in P form an arithmetic sequence with common difference d, then the angles can be written as $c-2d$, $c-d$, c, $c+d$, $c+2d$. (One could also write them as a, $a+d$, …, $a+4d$, but this approach is easier.) Then the sum of the interior angles is $5c = 540°$, so $c = 108°$. Since one of the interior angles is c, the converse of $(*)$ is true.

Thus the answer is option D, III only.

9.25 Answer A

TMUA 2017 · Paper 1 · Q11

We have $x_1 = 7$, $x_2 = 3$, $x_3 = 1$. It is not at all clear what the pattern here is, so we calculate the next few terms:

$$x_4 = \frac{23 \times 1 - 53}{5 \times 1 + 1} = \frac{-30}{6} = -5$$
$$x_5 = \frac{23 \times (-5) - 53}{5 \times (-5) + 1} = \frac{-168}{-24} = 7$$

so $x_6 = 3$, $x_7 = 1$, $x_8 = -5$ and so on. Thus the terms repeat in a cycle with length 4, and as $100 = 25 \times 4$, we see that $x_{100} = -5$, which is option A.

9.26 Answer A

TMUA 2022 · Paper 1 · Q11

We have $\log_{10}(3^{1-n}) = (1-n)\log_{10} 3$, so the sum becomes

$$\sum_{n=1}^{100} \log_{10}(3^{1-n}) = \sum_{n=1}^{100} (1-n)\log_{10} 3$$
$$= (\log_{10} 3) \sum_{n=1}^{100} (1-n).$$

Now the sum is actually an arithmetic series: expanding it gives

$$\sum_{n=1}^{100} (1-n) = 0 + (-1) + (-2) + \cdots + (-99)$$
$$= -(0 + 1 + 2 + \cdots + 99)$$
$$= -\tfrac{1}{2} \times 100(0 + 99 \times 1)$$
$$= -4950$$

using the formula $S_n = \frac{1}{2}n(2a + (n-1)d)$. (Alternatively, we could have used the formula for the nth triangular number.)

Therefore the original sum evaluates to $-4950 \log_{10} 3$, which is option A.

9.27 Answer A

TMUA 2019 · Paper 2 · Q11

Let the first term of the series be a and the common difference be d, as usual. Therefore we have

$$\tfrac{1}{2}n(2a + (n-1)d) = 20.$$

Multiplying by 2 gives

$$n(2a + (n-1)d) = 40,$$

so we need to find integers n, a and d which satisfy this. We list values of n which are factors of 40 and the corresponding value of $2a + (n-1)d$.

n	$2a + (n-1)d$
1	$40 = 2a$
2	$20 = 2a + d$
4	$10 = 2a + 3d$
5	$8 = 2a + 4d$
8	$5 = 2a + 7d$
10	$4 = 2a + 9d$
20	$2 = 2a + 19d$
40	$1 = 2a + 39d$

With this table to hand, we consider the possibilities for each of the statements.

I $n = 1$ gives $a = 20$, which is even.

$n = 2$ gives $2a + d = 20$. If we take $d = 2$, this gives $a = 9$, and indeed $9 + 11 = 20$, so this statement need not be true.

II The table shows that we could have $n = 1$ and $a = 20$, or $n = 5$, $a = 2$, $d = 1$, giving $2 + 3 + 4 + 5 + 6 = 20$. So this statement need not be true.

III The example in the previous part, $2 + 3 + 4 + 5 + 6 = 20$ shows that this is not necessarily true.

Therefore the correct answer is A.

9.28 Answer G

TMUA 2020 · Paper 2 · Q11

From the nature of the pattern, we see that it sort-of repeats every 2 units along each axis. So let us simplify this question by replacing 99 and 100 by 3 and 4 (an odd number and the even number one more than it).

Looking at the portion of the graph we are given, we see the points $(3, 4)$ and $(4, 3)$ are both on the spiral (top right), $(3, -4)$ and $(4, -3)$ are too (bottom right), $(-4, -3)$ and $(-3, -4)$ are both on the bottom left and if we continue the line at the top, $(-3, 4)$ is at the top left. However $(-4, 3)$ is missed out: that is the gap at the end of one 'loop' as the spiral heads out to form the next 'loop'.

Therefore in the case of 99 and 100, $(-100, 99)$ will be missed, which is option G.

9.29 Answer F

TMUA 2016 · Paper 2 · Q12

The formula for S_n in terms of a and d is $S_n = \frac{1}{2}n(2a+(n-1)d)$. Substituting $n = 8$ and $n = 6$ gives

$$S_8 = 4(2a + 7d)$$
$$S_6 = 3(2a + 5d)$$

so the condition $S_8 > 3S_6$ becomes

$$4(2a + 7d) > 3 \times 3(2a + 5d).$$

Expanding the brackets gives $8a + 28d > 18a + 45d$, which on rearranging gives $-8a > 17d$, or $d < -\frac{8}{17}a$.

If a is positive, then $-\frac{8}{17}a$ is negative, so d must be negative. If a is negative, then $-\frac{8}{17}a$ is positive, so d could be positive or negative. So a can be positive or negative, as can d.

We therefore cannot deduce the sign of either a or d, and so the correct option is F.

9.30 Answer C

TMUA 2017 · Paper 1 · Q13

The term in x^4 is $\binom{5}{4}a(bx)^4 = 5ab^4x^4$, so the coefficient of x^4 is $5ab^4$

The term in x^2 is $\binom{5}{2}a^3(bx)^2 = 10a^3b^2x^2$, so the coefficient of x^2 is $10a^3b^2$

It follows, since the coefficient of x^4 is 8 times that of x^2, that

$$5ab^4 = 80a^3b^2$$

As a and b are non-zero, we can divide by $5ab^2$ to get $b^2 = 16a^2$. Since both a and b are positive, we can take square roots to get $b = 4a$. (If we didn't know about their signs, we could only say that $b = \pm 4a$.)

So the smallest possible value of $a + b$ occurs when $a = 1$, $b = 4$, giving $a + b = 5$, which is option C.

9.31 Answer F

TMUA 2020 · Paper 1 · Q13

We have not learnt a "trinomial theorem" for expanding expressions like $(1+x+y^2)^7$, so instead we treat it as a binomial expression. If we write it as $(1+(x+y^2))^7$, we will then have a series of complicated expressions like $(x+y^2)^3$ that we have to handle. But if we write it as $(x+(1+y^2))^7$, we will be separating the x's and the y's, which is much simpler for our needs.

We have

$$\begin{aligned}(1 + x + y^2)^7 &= (x + (1 + y^2))^7 \\ &= x^7 + \binom{7}{1}x^6(1 + y^2) + \cdots + \binom{7}{5}x^2(1 + y^2)^5 + \cdots\end{aligned}$$

so we need only consider the x^2 term; note also that $\binom{7}{5} = \binom{7}{2} = \frac{7\times6}{2} = 21$. So the x^2 term is $21x^2(1 + y^2)^5$.

We next want the y^4 term of $(1 + y^2)^5$, so we expand this too using the binomial theorem:

$$(1 + y^2)^5 = 1 + \binom{5}{1}y^2 + \binom{5}{2}y^4 + \cdots$$

so the y^4 term is $\binom{5}{2}y^4 = 10y^4$.

Therefore the x^2y^4 term is $21x^2 \times 10y^4 = 210x^2y^4$, and the coefficient is 210; this is option F.

Let us first work out the value of the integral in the summation (the summand). We have

$$\begin{aligned}\int_0^r \mathrm{f}(x)\,\mathrm{d}x &= \int_0^1 \mathrm{f}(x)\,\mathrm{d}x + \int_1^2 \mathrm{f}(x)\,\mathrm{d}x + \cdots + \int_{r-1}^r \mathrm{f}(x)\,\mathrm{d}x \\ &= (0+1) + (1+1) + \cdots + ((r-1)+1) \\ &= 1 + 2 + \cdots + r \\ &= \tfrac{1}{2}r(r+1).\end{aligned}$$

We are thus being asked to evaluate

$$\sum_{r=1}^{8} \tfrac{1}{2}r(r+1).$$

There are algebraic methods to do this (as mentioned below), but they are not on the specification. The most straightforward approach here is to just perform the calculation manually:

$$\sum_{r=1}^{8} \tfrac{1}{2}r(r+1) = 1 + 3 + 6 + 10 + 15 + 21 + 28 + 36 = 120,$$

which is option C.

In general, it turns out that

$$\sum_{r=1}^{n} \tfrac{1}{2}r(r+1) = \tfrac{1}{6}n(n+1)(n+2).$$

The expressions $r(r+1)$ and $n(n+1)(n+2)$ are known as *rising powers*, and can be written as $r^{\overline{2}}$ and $n^{\overline{3}}$ respectively. On multiplying by 2, the formula then becomes

$$\sum_{r=1}^{n} r^{\overline{2}} = \tfrac{1}{3}n^{\overline{3}},$$

which looks very similar to the calculus formula $\int_0^x t^2\,\mathrm{d}t = \frac{1}{3}x^3$. There is a much longer discussion of sums such as this in Graham, Knuth and Patashnik, *Concrete Mathematics*.

Let us write out the first few terms of the three sequences explicitly. Let the common ratio of T be R and the common ratio of U be r.

term	1	2	3	4	5	...	n	...
T	4	$4R$	$4R^2$	$4R^3$	$4R^4$	...	$4R^{n-1}$	...
U	4	$4r$	$4r^2$	$4r^3$	$4r^4$	...	$4r^{n-1}$	...
S	8	$4R+4r$	$4R^2+4r^2$	$4R^3+4r^3$	$4R^4+4r^4$	...	$4R^{n-1}+4r^{n-1}$	...

We are given the first three terms of S, so

$$\begin{aligned} 8 &= 8 \\ 4R+4r &= 3 \\ 4R^2+4r^2 &= \tfrac{5}{4} \end{aligned}$$

Therefore $R+r=\frac{3}{4}$ and $R^2+r^2=\frac{5}{16}$. We can work out the values of R and r by guessing, or we can work them out by solving these equations. We show the second approach here.

The first equation gives $R=\frac{3}{4}-r$, which we can substitute into the second equation to give

$$(\tfrac{3}{4}-r)^2+r^2=\tfrac{5}{16}$$

so $2r^2-\frac{3}{2}r+\frac{1}{4}=0$, hence $8r^2-6r+1=0$, which factorises as $(4r-1)(2r-1)=0$, giving $r=\frac{1}{2}$ or $r=\frac{1}{4}$ and $R=\frac{1}{4}$ or $R=\frac{1}{2}$ respectively.

We can therefore work out the sums to infinity of T and U, taking $R=\frac{1}{4}$ and $r=\frac{1}{2}$:

$$\text{sum to infinity of T} = \frac{4}{1-\frac{1}{4}} = \tfrac{16}{3}$$

$$\text{sum to infinity of U} = \frac{4}{1-\frac{1}{2}} = 8$$

giving the sum to infinity of S as $\frac{16}{3}+8=\frac{40}{3}$, since S is the sum of T and U. If we took the values of R and r to be the other way round, we would have the series T and U swapped, but their sum would be unchanged.

So the answer is D.

Approach 1: direct reasoning

We can see from the example exactly what is needed for a sequence T to have property P: the sum of the terms from the $(m+1)$th term to the $(2m)$th term must be zero.

Consider now the two statements.

I We will have $ad < 0$ if $a > 0$ and $d < 0$. But if $|d|$ is large enough, then the second term will be negative and there is no hope of a sum of terms giving zero. For example, the sequence $1, -2, -5, \ldots$ has $ad = 1(-2) = -2 < 0$, but does not have property P. So this statement is false.

II In the given example, d is even and m is also even. But if m is odd, we would need the middle term of the second half to be zero, and then it does not matter whether d is even or odd, for example in the sequence (and this was constructed by working backwards): 4, 3, 2, 1, 0, $-1, \ldots$, which has property P taking $m = 3$. An even more extreme example is the sequence $1, 0, \ldots$, where $d = -1$ again and $m = 1$. Therefore this statement is false.

Hence the correct option is A.

Approach 2: algebraic approach

The sum of the first m and the first $2m$ terms of the sequence are given by

$$S_m = \tfrac{1}{2}m(2a + (m-1)d)$$
$$S_{2m} = \tfrac{1}{2}(2m)(2a + (2m-1)d)$$

and these are equal if and only if

$$\tfrac{1}{2}m(2a + (m-1)d) = \tfrac{1}{2}(2m)(2a + (2m-1)d).$$

Dividing both sides by $\frac{1}{2}m$ gives

$$2a + (m-1)d = 2(2a + (2m-1)d).$$

Expanding the brackets and rearranging gives

$$2a + (3m-1)d = 0. \qquad (*)$$

I The equation $(*)$ can be rearranged to make m the subject, giving

$$m = \frac{1}{3}\left(-\frac{2a}{d} + 1\right).$$

There is no reason why this should be an integer whenever $ad < 0$. For example, taking $a = 1$ and $d = -5$ gives $m = \frac{1}{3} \times \frac{7}{5}$ which is not an integer. Therefore this statement is false.

II We can rearrange $(*)$ to make d the subject:

but it is actually more helpful to rearrange it as

$$3m - 1 = -\frac{2a}{d}.$$

To show that the statement is true, we need to show that whenever we have integer values for a, d and m, d will be even; to show that the statement is false, we need to find an example where d is odd. If we take $d = 1$, we need a value of a for which $-2a$ is one less than a (positive) multiple of 3; $-2a = 2$ and $-2a = 8$ both work, giving $a = -1$ and $a = -4$ respectively. Similarly, if we take $d = 3$, we need a value of a for which $-\frac{2a}{3}$ is one less than a (positive) multiple of 3; here, $-2a = 2 \times 3 = 6$ would work, giving $a = -3$.

Any one of these counterexamples would show that this statement is false.

Again, we see that the correct option is A.

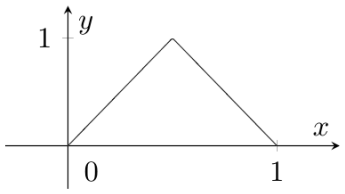
We can write the integral out in full by substituting the value of $g(x)$:

$$\begin{aligned}\int_0^1 g(x)\,dx &= \int_0^1 \sum_{r=1}^{10} f(2^{r-1}x)\,dx \\ &= \int_0^1 f(2^0x) + f(2^1x) + f(2^2x) + \cdots + f(2^9x)\,dx \\ &= \int_0^1 f(2^0x)\,dx + \int_0^1 f(2^1x)\,dx + \int_0^1 f(2^2x)\,dx + \cdots + \int_0^1 f(2^9x)\,dx\end{aligned}$$

So we can calculate the integral of $g(x)$ if we can calculate the integral of each $f(2^{r-1}x)$.

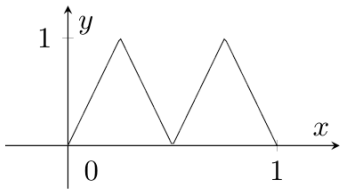
Now $\int_0^1 f(2^0x)\,dx = \int_0^1 f(x)\,dx$ is just the area under the graph of $y = f(x)$ between $x = 0$ and $x = 1$, which is $\frac{1}{2}$ (it is a right angled triangle with base and height both 1).

To work out $\int_0^1 f(2x)\,dx$, we consider the graph of $y = f(2x)$. This is the graph of $y = f(x)$ scaled by a factor of $1/2$ in the x-direction, so looks like this:

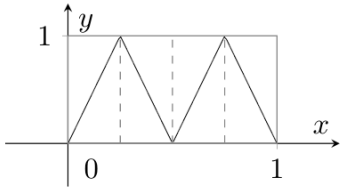


The area between $x = 0$ and $x = 1$ is again $\frac{1}{2}$.

The graph of $y = f(2^2x)$ between $x = 0$ and $x = 1$ consists of two triangles:



Helpfully, the area under this graph is still $\frac{1}{2}$: this is perhaps easiest to see by noticing that we can draw a rectangle (actually a square) around the graph:



In each rectangular slice, half of the area is below the graph and half is above, so the total area below the graph is $\frac{1}{2}$.

This same reasoning will apply for each of the functions $f(2^{r-1}x)$, and so the total area is

9.36 Answer D

TMUA 2017 · Paper 2 · Q15

From the question, it seems that we are expected to work out the first few values of the functions $\mathrm{f}(n)$ and $\mathrm{g}(n)$, and to look for a pattern.

n	$\mathrm{f}(n)$	$\mathrm{g}(n)$
1	5	3
2	16	8
3	8	4
4	4	2
5	2	1
6	1	6
7	4	3
8	2	8
9	1	4
10	4	2
11	2	1
12	1	6

So after the first three terms, $\mathrm{f}(n)$ repeats every 3 terms: for $k \geqslant 1$, $\mathrm{f}(3k+1) = 4$, $\mathrm{f}(3k+2) = 2$, $\mathrm{f}(3k+3) = 1$, while $\mathrm{g}(n)$ repeats every 6 terms, and we can write similar expressions for $\mathrm{g}(6k+r)$.

We could either calculate $\mathrm{f}(n) - \mathrm{g}(n)$ for all of these terms, and find the repeating pattern there, or find $\mathrm{f}(1000)$ and $\mathrm{g}(1000)$ separately. The latter approach seems like less effort, so we will go for that.

Note that $1000 = 3 \times 333 + 1 = 6 \times 166 + 4$. So $\mathrm{f}(1000) = 4$, $\mathrm{g}(1000) = 2$.

Thus $\mathrm{f}(1000) - \mathrm{g}(1000) = 4 - 2 = 2$ and the answer is D.

9.37 Answer D

TMUA 2020 · Paper 2 · Q15

It is very helpful to write out the values of the summands (the terms being added) in terms of $\sqrt{3}$ and similar, and then work out the first few values of the sum.

k	1	2	3	4	5	6	7	8	9	10	…
$\frac{k\pi}{3}$	$\frac{\pi}{3}$	$\frac{2\pi}{3}$	π	$\frac{4\pi}{3}$	$\frac{5\pi}{3}$	2π	$\frac{7\pi}{3}$	$\frac{8\pi}{3}$	3π	$\frac{10\pi}{3}$	…
$\sin(\frac{k\pi}{3})$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	0	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	0	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	0	$-\frac{\sqrt{3}}{2}$	…
$\sum_{k=1}^{n} \sin(\frac{k\pi}{3})$	$\frac{\sqrt{3}}{2}$	$2\frac{\sqrt{3}}{2}$	$2\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	0	0	$\frac{\sqrt{3}}{2}$	$2\frac{\sqrt{3}}{2}$	$2\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	…

We can see that the sum repeats every 6 steps, and that it equals $\frac{\sqrt{3}}{2}$ twice in every cycle of 6, specifically when $n = 1, 4, 7, 10, \ldots$. This sequence of n values are the numbers which are 1 more than a multiple of 3, and so the correct option is D.

Approach 1

Using the rule for working out the value of a base 2 number, we have

$$\begin{aligned}0.\dot{0}01\dot{1} &= 0.001100110011\ldots \\ &= 0\times 2^{-1}+0\times 2^{-2}+1\times 2^{-3}+1\times 2^{-4}+0\times 2^{-5}+0\times 2^{-6}+1\times 2^{-7}+1\times 2^{-8}+\cdots\end{aligned}$$

Grouping these into sets of four consecutive terms gives

$$\begin{aligned}0.\dot{0}01\dot{1} &= (0\times 2^{-1}+0\times 2^{-2}+1\times 2^{-3}+1\times 2^{-4})+ \\ &\qquad (0\times 2^{-5}+0\times 2^{-6}+1\times 2^{-7}+1\times 2^{-8})+\cdots \\ &= (0\times 2^{-1}+0\times 2^{-2}+1\times 2^{-3}+1\times 2^{-4})+ \\ &\qquad 2^{-4}(0\times 2^{-1}+0\times 2^{-2}+1\times 2^{-3}+1\times 2^{-4})+2^{-8}(\cdots)+\cdots \\ &= \left(\frac{1}{8}+\frac{1}{16}\right)+2^{-4}\left(\frac{1}{8}+\frac{1}{16}\right)+2^{-8}\left(\frac{1}{8}+\frac{1}{16}\right)+\cdots \\ &= \frac{3}{16}(1+2^{-4}+2^{-8}+\cdots)\end{aligned}$$

The term in brackets is an infinite geometric series with first term 1 and common ratio 2^{-4}, so its sum is

$$\frac{1}{1-2^{-4}}=\frac{1}{1-\frac{1}{16}}=\frac{16}{15}.$$

Therefore

$$0.\dot{0}01\dot{1}=\frac{3}{16}\times\frac{16}{15}=\frac{3}{15}=\frac{1}{5}$$

which is option B.

Approach 2

We have, on dropping zero terms and rearranging:

$$\begin{aligned}0.\dot{0}01\dot{1} &= 0.001100110011\ldots \\ &= 0\times 2^{-1}+0\times 2^{-2}+1\times 2^{-3}+1\times 2^{-4}+0\times 2^{-5}+0\times 2^{-6}+1\times 2^{-7}+1\times 2^{-8}+\cdots \\ &= 2^{-3}+2^{-4}+2^{-7}+2^{-8}+2^{-11}+2^{-12}+\cdots \\ &= (2^{-3}+2^{-7}+2^{-11}+\cdots)+(2^{-4}+2^{-8}+2^{-12}+\cdots)\end{aligned}$$

(Technical note: we are allowed to rearrange this infinite series as all of the terms are positive.)

We can now calculate the sums of these infinite geometric series to get

$$\begin{aligned}0.\dot{0}01\dot{1} &= \frac{2^{-3}}{1-2^{-4}} + \frac{2^{-4}}{1-2^{-4}} \\ &= \frac{2}{16-1} + \frac{1}{16-1} \\ &= \frac{3}{15} \\ &\quad 1\end{aligned}$$

Approach 3

We can adapt the method used to evaluate a base 10 recurring decimal.

Let $x = 0.\dot{0}01\dot{1} = 0.001100110011\ldots$. Then writing a subscript '2' to indicate base 2 and a subscript '10' to indicate the usual base 10, we have

$$\begin{aligned} & & x &= (0.001100110011\ldots)_2 \\ &\text{so} & 16_{10}x &= (11.001100110011\ldots)_2 \\ &\text{giving} & 15_{10}x &= 11_2\end{aligned}$$

Now $11_2 = 1 \times 2^1 + 1 \times 2^0 = 3_{10}$, so we have (in base 10) $15x = 3$, yielding $x = \frac{1}{5}$, which is option B.

9.39 Answer B

TMUA 2021 · Paper 1 · Q16

We can expand the binomial expression in ascending powers of x using the binomial theorem:

$$(a+bx)^n = a^n + \binom{n}{1}a^{n-1}bx + \binom{n}{2}a^{n-2}b^2x^2 + \binom{n}{3}a^{n-3}b^3x^3 + \cdots + \binom{n}{n-1}ab^{n-1}x^{n-1} + b^nx^n$$

The third term in ascending powers of x is $\binom{n}{2}a^{n-2}b^2x^2 = 105x^2$.

The fourth term in ascending powers of x is $\binom{n}{3}a^{n-3}b^3x^3 = 210x^3$.

The fourth term in descending powers of x is $\binom{n}{n-3}a^3b^{n-3}x^{n-3} = 210x^3$.

From the last statement, we see that $n = 6$. Then the first two statements give:

$$\binom{6}{2}a^4b^2 = 105$$
$$\binom{6}{3}a^3b^3 = 210$$

Now $\binom{6}{2} = 15$ and $\binom{6}{3} = 20$, so

$$a^4b^2 = 7$$
$$a^3b^3 = \frac{21}{2}$$

Dividing these two equations gives

$$\frac{a}{b} = \frac{2}{3}$$

and so $\left(\frac{a}{b}\right)^2 = \frac{4}{9}$, which is option B.

We start by calculating the first few terms of the sequence:

$$\begin{aligned} a_1 &= 2 \\ a_2 &= 2 + (-1)^1 = 1 \\ a_3 &= 1 + (-1)^2 = 2 \\ a_4 &= 2 + (-1)^3 = 1 \\ a_5 &= 1 + (-1)^4 = 2 \end{aligned}$$

The pattern is now clear: every pair of terms is 2, 1 and sums to 3. So $\sum_{n=1}^{100} a_n$ is the sum of 50 pairs of terms, each of which sums to 3. Hence

$$\sum_{n=1}^{100} a_n = 50 \times 3 = 150$$

and the answer is A.

> ***Commentary:*** *Note that this is not an arithmetic series, as there is not a common difference (that is, equal difference) between consecutive terms, and neither is it a geometric series. So we cannot use the formulae for the sum of an arithmetic or geometric series.*

Finding the median of the first n terms depends on whether n is even or odd.

If n is odd, then the median of the first n terms is the middle term, which is $x_{(n+1)/2}$, so it is certainly an integer, whatever the sequence.

We can consider the three given statements in this case:

I $n+2$ is also odd, so the median of the first $n+2$ terms is a term of the sequence, specifically $x_{(n+3)/2}$, so it is an integer.

II $2n$ is even, so the median of the first $2n$ terms is midway between the nth and $(n+1)$th terms. This may or may not be an integer. For example, if the sequence is 1, 2, 3, ... and $n = 3$, then the median of the first $2n = 6$ terms is midway between 3 and 4, so is 3.5. This is not an integer. (On the other hand, it might be: if the sequence were 2, 4, 6, ..., with $n = 3$ still, then the median of the first $2n = 6$ terms is midway between 6 and 8, which is 7; this is an integer.)

III There are n terms listed here, so the median of them is the middle one, which is certainly an integer.

Thus in the case that n is odd, I and III must be true, but II need not be.

Next we consider the case of n being even. In this case, the median of the first n terms is midway between $x_{\frac{1}{2}n}$ and $x_{\frac{1}{2}n+1}$, and we are told that this is an integer. If we call the common difference of this arithmetic progression d, then midway between these is $x_{\frac{1}{2}n} + \frac{1}{2}d$, so $\frac{1}{2}d$ is an integer. (Alternatively, note that both the middle two terms are even or both the middle two terms are odd, so either all the terms of the arithmetic progression are even or all of them are odd. This is equivalent to saying that the common difference is even.)

We again consider the three statements:

I Now $n+2$ is even, and the median is midway between two terms (namely $x_{\frac{1}{2}n+1}$ and $x_{\frac{1}{2}n+2}$), so it is an integer since the common difference is even. (Alternatively, it is an integer because either all of the terms are even or all of the terms are odd.)

II We do not need to consider this statement, as we know it is not always true from before. (Nevertheless, this statement is always true for this case, using the same reasoning as for I.)

III This time, we have n terms, and the common difference between consecutive pairs of them is $2d$. The median is midway between central two terms (as n is even), so is an integer as d is an integer. Alternatively, the median is midway between x_n and x_{n+2}, so is x_{n+1}, which is an integer.

Therefore all three statements are true in the case that n is even.

Putting this together, statements I and III must be true, whatever the value of n, but statement II is not necessarily true. Hence the answer is F.

I Since the highest common factor of a and b is 7, they are both multiples of 7. So u_1 and u_2 are multiples of 7, hence $u_3 = u_1 + u_2$ is a multiple of 7. Then $u_4 = u_2 + u_3$ is a multiple of 7, and so on. So every u_n is a multiple of 7 and this statement is true.

II We have $u_3 = u_1 + u_2 = u_1(1 + u_2/u_1)$. As u_1 is not a factor of u_2, the expression in brackets is not an integer, so u_1 is not a factor of u_3.

Next, we have $u_4 = u_2 + u_3 = u_1 + 2u_2 = u_1(1 + 2u_2/u_1)$. It is feasible that u_1 is a factor of u_4: it will be if $2u_2/u_1$ is an integer. We could take $u_1 = 14$ and $u_2 = 7$ to achieve this. (Explicitly, in this case we have $u_3 = 21$, $u_4 = 28$, and 14 is a factor of 28.)

So this statement may be false.

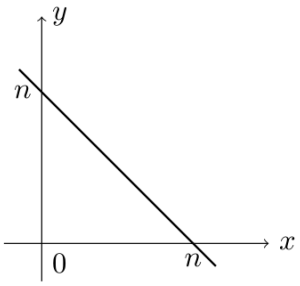
III Following on from our calculations in II, we have $u_5 = u_3 + u_4 = (u_1 + u_2) + (u_1 + 2u_2) = 2u_1 + 3u_2$. The highest common factor of u_1 and u_5 is therefore the same as the highest common factor of u_1 and $3u_2$, and it is possible that this is 21 if u_1 is a multiple of 21 but u_2 is not. Let us check this explicitly: take $u_1 = 21$ and $u_2 = 7$, so the highest common factor of u_1 and u_2 is 7, as required. Then $u_5 = 2 \times 21 + 3 \times 7 = 3 \times 21$, and so the highest common factor of $u_1 = 21$ and $u_5 = 3 \times 21$ is 21. Therefore this statement may be false.

The correct answer is therefore option B, I only.

It seems that we need to calculate $\mathrm{F}(n)$ before we can find $\mathrm{G}(n)$. We could do this either using integration or geometry. We start with an integration method:

$$\begin{aligned}\mathrm{F}(n) &= \frac{1}{n}\int_0^n (n-x)\,\mathrm{d}x \\ &= \tfrac{1}{n}\left[nx - \tfrac{1}{2}x^2\right]_0^n \\ &= \tfrac{1}{n}(n^2 - \tfrac{1}{2}n^2) \\ &= \tfrac{1}{2}n\end{aligned}$$

We could reach the same conclusion by sketching a graph of the function:



The area of this triangle is $\frac{1}{2}n^2$, so $\mathrm{F}(n) = \frac{1}{n} \times \frac{1}{2}n^2 = \frac{1}{2}n$

We now have:

$$\begin{aligned}\mathrm{G}(n) &= \sum_{r=1}^{n} \mathrm{F}(r) \\ &= \sum_{r=1}^{n} \tfrac{1}{2}r \\ &= \tfrac{1}{2}\sum_{r=1}^{n} r\end{aligned}$$

where for the last line we have used $(\frac{1}{2}\mathrm{F}(1) + \frac{1}{2}\mathrm{F}(2) + \cdots) = \frac{1}{2}(\mathrm{F}(1) + \mathrm{F}(2) + \cdots)$

Now $\sum_{r=1}^{n} r = 1 + 2 + \cdots + n = \frac{1}{2}n(n+1)$, either using the formula for triangular numbers or of the sum of an arithmetic series.

So $\mathrm{G}(n) = \frac{1}{4}n(n+1)$.

Therefore

$$\begin{aligned} & \mathrm{G}(n) > 150 \\ \iff \quad & \tfrac{1}{4}n(n+1) > 150 \\ \iff \quad & n(n+1) > 600\end{aligned}$$

So we can try values: 25 is a nice number to calculate with, so we start with options C and D:

$$24 \times 25 = 600$$
$$25 \times 26 = 650$$

It seems from the given sketch that each of the circles passes through the origin, and that all the circles are tangent to the line $y = -x$ at the origin. This means that the shaded area between C_n and C_{n+1} is just the area of C_{n+1} minus the area of C_n, as C_n lies entirely within C_{n+1} except for the single point of tangency at the origin.

We can prove this. Firstly, the circle C_n passes through the origin as $(x, y) = (0, 0)$ satisfies the equation of the circle. Next, we can solve the equation of the circle C_n simultaneously with $y = -x$. Substituting $y = -x$ into the equation of the circle gives

$$x^2 + x^2 = 2n(x + (-x)) = 0$$

so $2x^2 = 0$, which has a repeated root $x = 0$. Therefore this line is tangent to the circle at the origin.

Next, we need to find the area of C_n. We rearrange and complete the square to do this: we have

$$\begin{aligned} & x^2 - 2nx + y^2 - 2ny = 0 \\ \text{so} \quad & (x-n)^2 - n^2 + (y-n)^2 - n^2 = 0 \\ \text{so} \quad & (x-n)^2 + (y-n)^2 = 2n^2 \end{aligned}$$

This shows that the centre of C_n is at (n, n) and it has squared radius $2n^2$. Its area is therefore $2\pi n^2$.

We deduce that the shaded area between C_{n+1} and C_n is

$$2\pi(n+1)^2 - 2\pi n^2 = 2\pi(2n+1).$$

Summing this for $n = 1, 3, 5, \ldots, 99$ (as we want the difference between circles C_1 and C_2, and between C_3 and C_4, and so on), the total shaded area is

$$2\pi(3 + 7 + 11 + \cdots + 199)$$

The sum here is an arithmetic series; we have first term $a = 3$, last term $l = 199$, and $n = 50$ terms (as there are 50 pairs of circles), so the total area is

$$2\pi \times \tfrac{1}{2}n(a + l) = \pi \times 50 \times 202 = 10100\pi$$

which is option E.

9.45 Answer F

TMUA 2023 · Paper 2 · Q17

The value of $2^{\lceil x \rceil}$ changes at every integer. We have

$$2^{\lceil x \rceil} = \begin{cases} 2^0 & \text{when } x = 0 \\ 2^1 & \text{when } 0 < x \leq 1 \\ 2^2 & \text{when } 1 < x \leq 2 \\ 2^3 & \text{when } 2 < x \leq 3 \\ \dots & \\ 2^{99} & \text{when } 98 < x \leq 99 \end{cases}$$

We therefore break the integral up into unit intervals. The value of $2^{\lceil x \rceil}$ at the endpoints does not influence the value of the integral, so we have

$$\begin{aligned} \int_0^{99} 2^{\lceil x \rceil}\,\mathrm{d}x &= \int_0^1 2^{\lceil x \rceil}\,\mathrm{d}x + \int_1^2 2^{\lceil x \rceil}\,\mathrm{d}x + \int_2^3 2^{\lceil x \rceil}\,\mathrm{d}x + \cdots + \int_{98}^{99} 2^{\lceil x \rceil}\,\mathrm{d}x \\ &= \int_0^1 2^1\,\mathrm{d}x + \int_1^2 2^2\,\mathrm{d}x + \int_2^3 2^3\,\mathrm{d}x + \cdots + \int_{98}^{99} 2^{99}\,\mathrm{d}x \\ &= 2^1 + 2^2 + 2^3 + \cdots + 2^{99} \end{aligned}$$

as the integral of a constant is just the constant times the interval width. We now have a geometric series with first term $a = 2$, common ratio $r = 2$ and $n = 99$ terms, so the sum is

$$\frac{a(r^n - 1)}{r - 1} = \frac{2^1(2^{99} - 1)}{2 - 1} = 2^{100} - 2$$

which is option F.

9.46 Answer E

TMUA 2023 · Paper 1 · Q18

The sum is an infinite geometric series with first term $a = 4$ and common ratio $r = \frac{2k}{7}$. For this series to have a finite sum, we require $|r| < 1$, so $-7 < 2k < 7$, that is $-\frac{7}{2} < k < \frac{7}{2}$. Thus we can only consider $k = -3, -2, \dots, 2, 3$ out of the 11 possible values of k. (Note that there are 11 values: $k = 0$ is a possibility!)

In these cases, the sum is given by

$$S = \frac{a}{1 - r} = \frac{4}{1 - 2k/7} = \frac{28}{7 - 2k}.$$

We need to solve the inequality $S > 3$, so

$$\frac{28}{7 - 2k} > 3.$$

Since $7 - 2k > 0$ for all $k < \frac{7}{2}$, which are the only values of k of interest to us, we can multiply by $7 - 2k$ to get

$$28 > 3(7 - 2k).$$

Expanding and rearranging gives $6k > -7$, so $k > -\frac{7}{6}$. Therefore the values of k that satisfy the conditions are -1, 0, 1, 2, 3, a total of 5 values out of the 11 possible values. Therefore the required probability is $\frac{5}{11}$, which is option E.

The first four terms are 4, $4r$, $4r^2$ and $4r^3$. We are also given that 4, $4r$ and $4r^3$ are three successive terms of an arithmetic series. We thus have

$$4r = 4 + d$$
$$4r^3 = 4 + 2d$$

where d is the common difference of this arithmetic series.

Therefore $d = 4(r-1)$ from the first equation, and $d = 4r(r^2-1)$ on subtracting the two equations.

Equating these expressions for d gives

$$4(r-1) = 4r(r^2-1)$$

so $r - 1 = r(r-1)(r+1)$ on factorising and dividing by 4.

Since $0 < r < 1$, we can divide by $r-1$ to obtain $1 = r(r+1)$, so $r^2 + r - 1 = 0$. The solutions to this quadratic are

$$r = \frac{-1 \pm \sqrt{1^2+4}}{2} = \frac{-1 \pm \sqrt{5}}{2}.$$

Since $r > 0$, we have $r = \frac{-1+\sqrt{5}}{2}$, and so the sum to infinity (given by the standard formula $S_\infty = \frac{a}{1-r}$) is

$$\begin{aligned}\frac{4}{1-\frac{-1+\sqrt{5}}{2}} &= \frac{8}{2-(-1+\sqrt{5})} \\ &= \frac{8}{3-\sqrt{5}} \\ &= \frac{8(3+\sqrt{5})}{9-5} \\ &= 2(3+\sqrt{5})\end{aligned}$$

and the answer is D.

9.48 Answer B

TMUA 2016 · Paper 1 · Q19

We expand the first expression up to powers of x^3 to find its coefficient. We can think of the first expression as $(1+(2x+3x^2))^6$ and use the binomial theorem to expand it. We note that the first few binomial coefficients are

$$\binom{6}{0}=1; \qquad \binom{6}{1}=6; \qquad \binom{6}{2}=\frac{6\times 5}{2!}=15; \qquad \binom{6}{3}=\frac{6\times 5\times 4}{3!}=20.$$

We thus have

$$\begin{aligned}(1+(2x+3x^2))^6 &= 1+\tbinom{6}{1}(2x+3x^2)+\tbinom{6}{2}(2x+3x^2)^2+\tbinom{6}{3}(2x+3x^2)^3+\cdots\\ &= 1+6(2x+3x^2)+15((2x)^2+2(2x)(3x^2)+\cdots)+20((2x)^3+\cdots)+\cdots\end{aligned}$$

where we have stopped when the powers reach 3. We can read off the coefficient of x^3 from this; it is

$$15\times 2\times 2\times 3+20\times 8=340.$$

We can expand $(1-ax^2)^5$ similarly, and obtain

$$1+\tbinom{5}{1}(-ax^2)+\tbinom{5}{2}(-ax^2)^2+\cdots=1-5ax^2+10a^2x^4+\cdots$$

so the coefficient of x^4 is $10a^2$.

We are told how these two coefficients relate to each other: we have

$$340=2\times 10a^2$$

so $a^2=17$ and $a=\pm\sqrt{17}$, giving the answer as option B.

9.49 Answer C

TMUA 2019 · Paper 1 · Q19

We start by writing out the first few terms of the summation, and writing each of them in terms of $\sin 10^\circ$ or $\sin 100^\circ$:

$$\begin{aligned}k=0:&\quad \sin 10^\circ\\ k=1:&\quad \sin 100^\circ\\ k=2:&\quad \sin 190^\circ=-\sin 10^\circ\\ k=3:&\quad \sin 280^\circ=-\sin 100^\circ\\ k=4:&\quad \sin 370^\circ=\sin 10^\circ\\ k=5:&\quad \sin 460^\circ=\sin 100^\circ\\ k=6:&\quad \sin 550^\circ=-\sin 10^\circ\\ k=7:&\quad \sin 640^\circ=-\sin 100^\circ\end{aligned}$$

and we see that we repeat every four terms. The first four terms sum to zero, as do the next four, and so on.

In total, there are 91 terms, and $91=22\times 4+3$, so the only terms which contribute to the final sum are the last three. Therefore the whole sum is that from $k=88$, $k=89$ and $k=90$, giving

$$\sin 10^\circ+\sin 100^\circ+(-\sin 10^\circ)=\sin 100^\circ.$$

Hence the answer is C.

9.50 Answer G

TMUA Specimen · Paper 1 · Q20

We expand each of the terms up to the term in x^2 using the binomial theorem:

$$\begin{aligned}(1+2x+3x^2)^6 &= 1+6(2x+3x^2)+15(2x+3x^2)^2+\cdots\\ &= 1+12x+18x^2+15(4x^2+\cdots)+\cdots\\ &= 1+12x+78x^2+\cdots\\ (1+4x^3)^5 &= 1+5(4x^3)+\cdots = 1+\cdots\end{aligned}$$

so the whole expression is

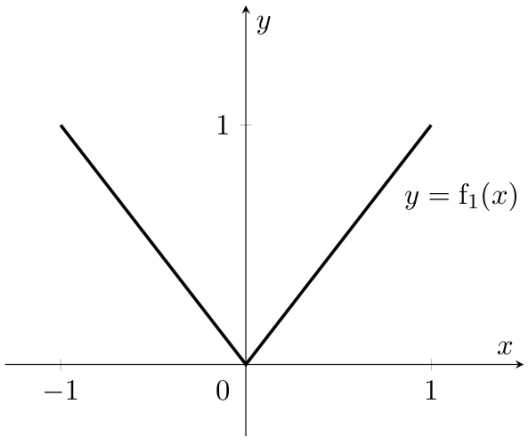
$$\begin{aligned}(4-x^2)[(1+2x+3x^2)^6-(1+4x^3)^5] &= (4-x^2)[(1+12x+78x^2+\cdots)-(1+\cdots)]\\ &= (4-x^2)(12x+78x^2+\cdots)\\ &= 4(12x+78x^2+\cdots)-\cdots\\ &= 48x+312x^2+\cdots\end{aligned}$$

so the coefficient of x^2 is 312, and the answer is G.

9.51 Answer E

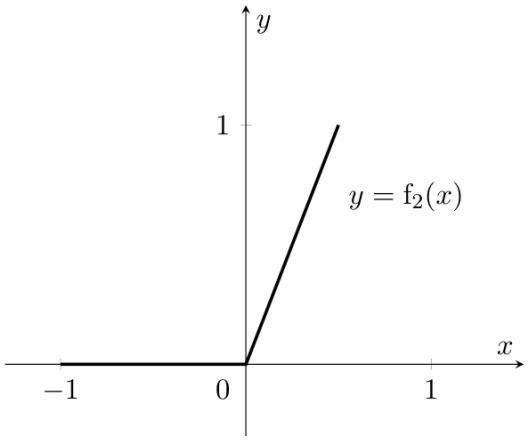
TMUA 2021 · Paper 2 · Q20

It is helpful to get a feeling for what this sequence of functions looks like, so we will sketch the first few. We are only interested in the range from -1 to 1 (because that is what the integral asks for). The first function, $\mathrm{f}_1(x) = |x|$ is standard:



Note that when $x < 0$, this is the graph of $y = -x$, and when $x \geq 0$, this is the graph of $y = x$.

We next have $\mathrm{f}_2(x) = |\mathrm{f}_1(x) + x|$. We can calculate this on the two regions $x < 0$ and $x \geq 0$ separately: when $x < 0$, we get $\mathrm{f}_2(x) = |-x+x| = 0$, and when $x > 0$, we get $\mathrm{f}_2(x) = |x+x| = 2x$, giving this graph:



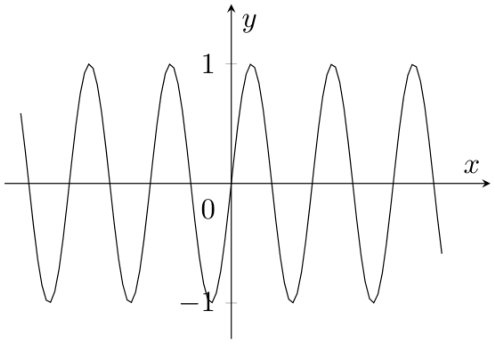
We can determine $\mathrm{f}_3(x)$ and $\mathrm{f}_4(x)$ in a similar way:

Calculus 2

10.1 Answer A

TMUA 2023 · Paper 2 · Q5

The graph of $y = \sin 2x$ is periodic with period π: it is the graph of $y = \sin x$ stretched by a factor of $\frac{1}{2}$ in the x-direction:



So if R is true, we see from the graph that $\int_0^k \sin 2x \, dx = 0$ as the areas above and below the x-axis cancel. Thus R is sufficient for S.

Conversely, if $\int_0^k \sin 2x \, dx = 0$, the areas above and below the x-axis must cancel. We can see from the graph that the only way to achieve this is if k is a multiple of π, so that we have an exact number of periods of $y = \sin 2x$. (The integral increases as k increases from 0 to $\frac{\pi}{2}$, and then decreases back to 0 as k increases from $\frac{\pi}{2}$ to π, and the same happens in every period.) Therefore S implies R, or R is necessary for S.

It follows that the correct answer is option A: R is necessary and sufficient for S.

We can check whether the calculation stands any chance of being correct by checking whether line (V) is correct. We can work out the integral directly:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_x^{2x} t^2\,\mathrm{d}t\right) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\left[\tfrac{1}{3}t^3\right]_x^{2x}\right)\\ &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\tfrac{1}{3}(2x)^3 - \tfrac{1}{3}x^3\right)\\ &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\tfrac{7}{3}x^3\right)\\ &= 7x^2\end{aligned}$$

So an error has certainly been introduced somewhere.

From this, it is clear that line (IV) is also incorrect.

Line (I) is correct, as $\int_a^b \mathrm{f}(t)\,\mathrm{d}t + \int_b^c \mathrm{f}(t)\,\mathrm{d}t = \int_a^c \mathrm{f}(t)\,\mathrm{d}t$ for all functions $\mathrm{f}(t)$ and real numbers a, b and c.

Therefore the error must lie in line (II) or (III). We can work out the left hand sides of the equations explicitly as we did for line (V):

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_0^{x} t^2\,\mathrm{d}t\right) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\left[\tfrac{1}{3}t^3\right]_0^{x}\right)\\ &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\tfrac{1}{3}x^3\right)\\ &= x^2\end{aligned}$$

so line (II) is correct. However,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_0^{2x} t^2\,\mathrm{d}t\right) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\left[\tfrac{1}{3}t^3\right]_0^{2x}\right)\\ &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\tfrac{8}{3}x^3\right)\\ &= 8x^2\end{aligned}$$

so line (III) is incorrect. Hence the correct option is D.

The reason for the error is that the fundamental theorem of calculus is stated for the case where the upper limit is x; if the upper limit is $2x$, then the integral grows twice as fast as x increases, so we actually have $\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_0^{2x} t^2\,\mathrm{d}t\right) = 2(2x)^2 = 8x^2$.

Proof and Logic

11.1 Answer C

TMUA Specimen · Paper 2 · Q3

We can determine whether $x = -4$ and/or $x = -1$ are solutions by substituting them back into the original equation.

Substituting $x = -4$ gives $\sqrt{x+5} = \sqrt{-4+5} = 1$ and $x + 3 = -4 + 3 = -1$, so this is not a solution.

Substituting $x = -1$ gives $\sqrt{x+5} = \sqrt{-1+5} = 2$ and $x + 3 = -1 + 3 = 2$, so this is a solution.

Where does $x = -4$ arise?

Working backwards, once we have reached the line $(x + 4)(x + 1) = 0$, it is clear that there are the two proposed solutions.

The previous line is just the expansion of this; the equations $x^2+5x+4 = 0$ and $(x+4)(x+1) = 0$ are algebraically equivalent, so step (3) is fine.

Likewise, $x + 5 = x^2 + 6x + 9$ is algebraically equivalent to these, obtained by a simple rearrangement, so step (2) is fine. And indeed, $x = -4$ is a solution to this equation, as is easy to check.

Therefore the extra solution was introduced at step (1), when we squared the equation: on the first line, $x = -4$ gives the incorrect $-1 = 1$, while squaring this gives the correct $1 = 1$. Hence the correct answer is C.

We can go further, though, and ask how we could correct the solution. If we introduce implication symbols between pairs of lines, it will be clearer what is happening. The symbol ' $\iff$ ' indicates that this line is equivalent to the previous line, while ' $\implies$ ' indicates that this line follows from the previous line, but not vice versa.

The argument now reads as follows:

$$\begin{aligned} & \sqrt{x+5} = x+3 \\ \implies \quad & x+5 = x^2+6x+9 \\ \iff \quad & x^2+5x+4 = 0 \\ \iff \quad & (x+4)(x+1) = 0 \\ \iff \quad & x = -4 \text{ or } x = -1 \end{aligned}$$

The first line is only a one-way implication, as $a^2 = b^2$ does not imply that $a = b$, for example if $a = 1$ and $b = -1$ (as in our case).

Since not every line is an 'if and only if' implication, all we can say is the following: *if* the original equation is true, *then* $x = -4$ or $x = -1$. However, this argument does not show that if $x = -4$ or $x = -1$, then the original equation is true. So to correctly complete the argument, we *must* check whether these possibilities satisfy the original equation.

11.2 Answer F

TMUA 2018 · Paper 2 · Q3

A counterexample to this statement is an example which shows that it is not true.

So we consider the journeys described, and determine whether or not the average speed for the whole journey is, indeed, $\frac{1}{2}(u+v)$.

Perhaps the simplest way to do this is to make a table. We know that average speed is distance travelled divided by time, so we will need these for each part of the journey.

Filling in the information given, we end up with the following. We write d_1, d_2 and D for the distances (in km) in the first part, second part and total, respectively. Likewise, t stands for time (in h), and we use V for the average speed for the whole journey (in km/h).

	First part			Second part			Overall		
	d_1	t_1	u	d_2	t_2	v	D	T	V
I	100		50	100		40			
II		1	50		1	40			
III	80		50	100		40			

We can now use $u = d_1/t_1$ and so on to fill in the rest of the first and second part of the journey, and then sum the distances and times for the overall journey. We can finally calculate the average speed. This gives:

	First part			Second part			Overall		
	d_1	t_1	u	d_2	t_2	v	D	T	V
I	100	2	50	100	2.5	40	200	4.5	$\frac{400}{9}$
II	50	1	50	40	1	40	90	2	45
III	80	1.6	50	100	2.5	40	180	4.1	$\frac{1800}{41}$

In every case, $\frac{1}{2}(u+v) = 45$ so both I and III are counterexamples (as the fractions cannot be simplified, so they cannot equal 45), and the answer is F.

11.3 Answer A

TMUA 2019 · Paper 2 · Q3

It might be tempting to divide by a to get $b = c$, but a might be zero, and we cannot divide by zero. Instead, we can rearrange and factorise to get $ab - ac = 0$, so $a(b-c) = 0$. From this, we can deduce that $a = 0$ or $b - c = 0$ (or both).

I This might not be true; we could have $b = c$ instead.

II This might not be true; we could have $b = c = 1$, for example.

III This might not be true; we could have $a = 0$ instead.

So the correct answer is A.

11.4 Answer C

TMUA 2020 · Paper 2 · Q3

Lines (I) to (V) are algebraic manipulations; we need to check whether each manipulation is correct.

Line (I) writes $4 = 2 \times 2$, which is correct.

Line (II) multiplies out the inner brackets, but does not do it correctly: the inner brackets should expand to $(9n + 1) - (3n - 1)$, which simplifies to $9n + 1 - 3n + 1$.

The remaining manipulations are correct.

Thus the first error occurs on line (II), and the correct option is C.

11.5 Answer C

TMUA 2022 · Paper 2 · Q3

I $n = 2$ is prime and $n^2 + 2 = 6$ is not prime, so this case satisfies the statement and is not a counterexample.

II $n = 3$ is prime but $n^2 + 2 = 11$ is prime, so this is a counterexample as it does not satisfy the statement.

III $n = 4$ is not prime, so it satisfies the statement and is not a counterexample.

Therefore only II provides a counterexample, which is option C.

11.6 Answer C

TMUA 2023 · Paper 2 · Q3

We substitute the values into the two sides of the equation:

I $\sqrt{1^{16}} = \sqrt{1} = 1$ and $1^{\sqrt{16}} = 1^4 = 1$, so this is not a counterexample.

II $\sqrt{2^8} = 2^4 = 16$ and $2^{\sqrt{8}} \neq 16$, so this is a counterexample.

III $\sqrt{3^4} = 3^2 = 9$ and $3^{\sqrt{4}} = 3^2 = 9$, so this is not a counterexample.

Therefore the answer is option C, II only.

11.7 Answer A

TMUA Specimen · Paper 2 · Q4

The statement is a 'for all' statement, and can be expressed as a conditional statement:

For every card, **if** it has a vowel on its front, **then** it has an even number on its back.

Thus a counterexample to this statement is a card for which '**if** it has a vowel on its front, **then** it has an even number on its back' is false. This is false if the card has a vowel on its front and it does not have an even number on its back.

The only cards with vowels on their front are A and E; of these, E has an even number on its back but A does not.

So the only counterexample is A.

11.8 Answer C

TMUA 2016 · Paper 2 · Q4

We work through the possibilities systematically.

- Urn P's statement is true, so the urns each contain one or four balls. As Urn Q has a false statement, there cannot be two or four balls in the urns, and so there is one ball in each urn. But then Urn S has a true statement, which is impossible.
- Urn Q's statement is true, so the urns each contain two or four balls. As Urn P has a false statement, there cannot be one or four balls in each urn, so there are two balls in each urn. But then Urn S has a true statement, which is impossible.
- Urn R's statement is true, so there are three or four balls in each urn. As Urn P's statement is false, there cannot be four balls in the urn, so there are three balls in each urn. This means that Urn Q, Urn S and Urn T each have a false statement, and therefore this is possible.
- Urn S's statement is true, so the urns each contain one or two balls. As Urn P's statement is false, there cannot be one ball in each urn, so there are two balls in each urn. But then Urn Q has a true statement, which is impossible.
- Urn T's statement is true, so the urns each contain one or two balls. But then Urn S's statement is true, which is impossible.

Therefore the only urn which has a true statement is Urn R, so option C is correct.

11.9 Answer D

TMUA 2019 · Paper 2 · Q4

A counterexample with have the antecedent ("N is a positive integer that consists of the digit 1 followed by an odd number of 0 digits and then a final digit 1") true, and the consequent ("N is a prime number") false. We can now check each of the three offered values of N:

I The form of N is correct: there is 1 zero, and 1 is odd, so the antecedent is true. However, the consequent is also true, so this is not a counterexample.

II The form of N is not correct, as there are 2 zeros, so the antecedent is false. Hence this is not a counterexample.

III The form of N is correct, as there are 3 zeros, and the consequent is false, as N is not prime. Hence this is a counterexample.

The correct solution is D.

11.10 Answer G

TMUA 2020 · Paper 2 · Q4

A counterexample is a positive integer N that is greater than 6 but it is not the case that N can be written as the sum of two non-prime integers that are greater than 1.

- $N = 5$ is not greater than 6, so is not a counterexample.
- $N = 7 = 2 + 5 = 3 + 4$ cannot be written in this way, so this provides a counterexample.
- $N = 9 = 2 + 7 = 3 + 6 = 4 + 5$ also cannot be written in this way.

Therefore both II and III provide counterexamples; the correct option is G.

11.11 Answer C

TMUA 2021 · Paper 2 · Q4

I $bc = 200$, a is a factor of 200, a is a factor of b, so not a counterexample.

II $bc = 16$, a is a factor of 16, a is not a factor of b and a is not a factor of c, so this is a counterexample.

III $bc = 84$, a is a factor of 84, a is not a factor of b but a is a factor of c, so not a counterexample.

The correct option is therefore C.

11.12 Answer G

TMUA 2023 · Paper 2 · Q4

The argument looks very convincing and there is no obvious error. However, writing down the first few odd integers and underlining those which are prime shows that there is a problem:

$$1, \underline{3}, \underline{5}, \underline{7}, 9, \underline{11}, \underline{13}, 15, \underline{17}, \underline{19}, \ldots$$

There are three consecutive odd numbers that are all prime right at the start (3, 5, 7), so the student's final answer in line VII is incorrect. We must therefore find the first error in the argument.

I This line is correct: 17 and 19 are consecutive odd integers that are prime.

II This is correct: consecutive odd integers differ by 2, and we can call the middle one n.

III If $n = 3k+1$, then $n+2 = 3k+3 = 3(k+1)$, which is a multiple of 3, so this line is correct.

IV If $n = 3k+2$, then $n-2 = 3k$, which is a multiple of 3, so this line is correct.

V All integers are either multiples of 3, one more than a multiple of 3 or two more than a multiple of 3; a number which is three more than a multiple of 3 is itself a multiple of 3. Similarly, a number which is four more than a multiple of 3 is one more than the next multiple of 3 (in algebraic notation: $3k+4 = 3(k+1)+1$), and so on. So this statement is correct.

VI In each case, one of $n+2$, $n-2$ and n is a multiple of 3. But our example of 3, 5, 7 shows that the last part of this statement is false: 3 is a multiple of 3, but *is* prime.

So the first error is on line VI, which is option G.

We can rewrite the statement $(*)$ in the form

> **If** a whole number n is 1 less or 5 less than a multiple of 6, **then** n is prime.

A counterexample to this statement is a whole number n which makes this **if** statement false, that is, it is a whole number n that is 1 less or 5 less than a multiple of 6, but which is **not** prime. So we list these numbers and check whether they are prime:

n	Prime?
1	no
5	yes
7	yes
11	yes
13	yes
17	yes
19	yes
23	yes
25	no
29	yes
31	yes
35	no
37	yes
41	yes
43	yes
47	yes
49	no

Thus there are 4 numbers which are counterexamples in this range, and so the answer is C.

We note from the context of the statements that $91 = 7 \times 13$ shows that 91 is not prime. It also shows that 91 is a multiple of 7 and a multiple of 13.

We use a counterexample when we are trying to disprove a statement of the form '**if** A **then** B'; the counterexample will be a case where A is true but B is false. We can also use counterexamples to disprove statements of the form '**for all** x, A is true': a single example of an x for which A is false shows that the statement is false.

We first rewrite each of the statements given into an explicit '**if** ... **then**' statement or an explicit '**for all**' statement.

(With experience, this is not actually necessary; once you have seen enough statements, it becomes clear what a counterexample 'looks like'. But for the purposes of reaching this point, it can be useful to go through these formal steps.)

1 The word 'when' in this context has the same logical meaning as 'if', so the statement can be rewritten as '**if** p is an odd prime, **then** $10p^2 + 1$ and $10p^2 - 1$ are both prime'.

So if $91 = 7 \times 13$ is to provide a counterexample, we need an odd prime p with $10p^2 + 1 = 91$ or $10p^2 - 1 = 91$, since in that case, '$10p^2 + 1$ and $10p^2 - 1$ are both prime' will be false. It is clear that $p = 3$ achieves this, for then $10p^2 + 1 = 91$.

2 We can write this statement as an explicit '**for all**' statement, as 'Every prime...' suggests this meaning. The statement becomes: '**For all** primes p where $p > 5$, $p = 6n + 1$ for some integer n'.

So a counterexample would be a prime p with $p > 5$ but for which $p = 6n+1$ is false. However, 91 is not prime, so $91 = 7 \times 13$ does not provide a counterexample. The fact that $91 = 6n + 1$ for $n = 15$ is immaterial, and the fact that the statement is false (as $11 = 6 \times 2 - 1$) does not help either.

3 We can again write this as an explicit '**for all**' statement; the English is better with the equivalent '**for each**', though: '**For each** n which is a multiple of 7 greater than 7, n is not prime.'

If we now look at $91 = 7 \times 13$, we see that 91 is a multiple of 7 greater than 7. However, 91 is not prime, so 91 is an example of where the statement does hold, rather than being a counterexample. In fact, the given statement is true, so there cannot be any counterexamples, and we need not have looked at $91 = 7 \times 13$ at all.

So the answer is B.

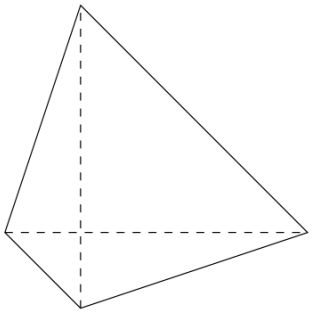
> ***Commentary:*** *With a question such as this, it is very wise to try drawing a sketch. We are told that the two diagonals of Q are perpendicular, so it makes sense to start by drawing the two diagonals.*

If statement I is **necessarily** true for Q, it means that it is always true, no matter how we choose to draw Q (subject to the condition that it has perpendicular diagonals), and likewise for statement II.

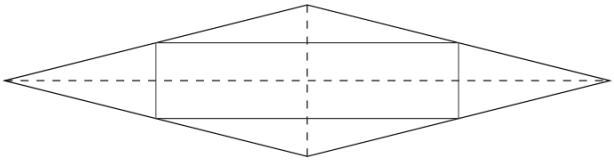
We could therefore either try proving these statements or start by looking for counterexamples.

Following the above comments, we start by drawing two perpendicular lines, and mark vertices on them with the aim of finding a counterexample to each statement. This turns out not to be that difficult after some playing around.

To make a diagonal *not* be a line of symmetry, we simply need the off-diagonal vertices to be different distances from the diagonal. So this will do it:



For II, to make the midpoints of the sides not be the vertices of a square, we can make the quadrilateral be "long and skinny":



So neither I nor II is **necessarily** true, so the correct answer is A.

> ***Commentary:*** *It turns out that the midpoints of the sides of Q are always the vertices of a rectangle, and for a general quadrilateral, they are always the vertices of a parallelogram, with sides parallel to the diagonals of the original quadrilateral.*

Saying that 'either a or b is divisible by n' is **necessary** for $(*)$ is the same as saying that **if** ab is divisibly by n, **then** either a or b is divisible by n. We can look for a counterexample or proof for this assertion.

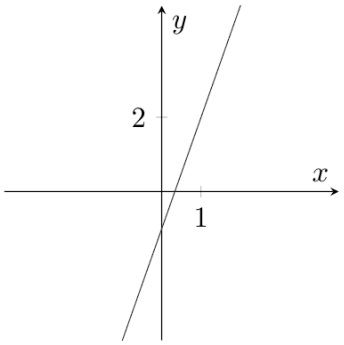
On the surface of it, it seems surprising: when a product is divisible by a number, why should one of the two factors also be? We know something similar to this for a product which equals zero, and we may know that if a product is divisible by a prime, then one of the factors must be. But there is no mention of zero or primes here. So let us try a simple example where n is not prime, such as $n = 4$ (the smallest non-prime other than 1). Then the statement becomes: **if** ab is divisible by 4, **then** either a or b is divisible by 4. A small amount of exploration finds counterexamples, such as $a = 2$, $b = 2$ or $a = 6$, $b = 2$. So the condition is **not necessary** for $(*)$.

What about sufficiency? Saying that 'either a or b is divisible by n' is **sufficient** for $(*)$ is the same as saying that **if** either a or b is divisible by n, **then** ab is divisibly by n. We can again look for a counterexample or proof for this assertion.

This time, though, a proof is clear: if a is divisible by n, then a is a multiple of n, so ab is also a multiple of n (as b is an integer). This means that ab is divisible by n. The same is the case if b is divisible by n, so the condition is **sufficient**.

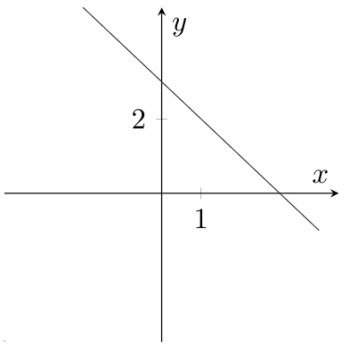
The correct answer is therefore B.

I Let us sketch a line through $(1, 2)$ with a negative y-intercept:



It is clear from this sketch that if the line has a negative y-intercept, it must have a positive x-intercept. (A more formal proof goes as follows: since $y < 0$ when $x = 0$ and $y > 0$ when $x = 1$, there must be some value of x with $0 < x < 1$ where $y = 0$. Therefore the x-intercept must lie between 0 and 1.)

II The converse of P reads: If the x-intercept of L is positive, then the y-intercept of L is negative. This is the case if the x-intercept is between 0 and 1, as in the previous sketch, but if the x-intercept is greater than 1, this is no longer true:



So the converse of P is false.

III The contrapositive of P has the same truth value as P itself, so the contrapositive of P is true.

Therefore only I and III are true, and the correct answer is option F.

11.18 Answer B

TMUA 2019 · Paper 2 · Q7

We work through these in turn:

A A counterexample here is a case for which the antecedent (the 'if' part) is true but the consequent (the 'then' part) is false. In this case, taking $a = 12$, $b = 16$, $c = 20$ to match $12^2 + 16^2 = 20^2$ does not satisfy the antecedent, as they do have a common divisor of 4. Even if we were to divide all three by 4 to get $3^2 + 4^2 = 5^2$, we would still not have a counterexample, as two of them are odd and the other is even. (The statement is actually true, so there are no counterexamples.)

B A counterexample in this case is an example to show that there is a solution. Since $16 = 4^2$, we can rewrite $12^2 + 16^2 = 20^2$ as $4^4 + 12^2 = 20^2$, so this does provide a counterexample.

C Since neither 12 nor 20 is a perfect square, we cannot use this fact to obtain a counterexample. (And that is quite a relief, since Fermat proved that this statement is true! It is the simplest case of Fermat's Last Theorem.)

D We note that $12^2 + 16^2 = 20^2$ satisfies the antecedent, but as 16 is the arithmetic mean of 12 and 20, this does not provide a counterexample. (The statement is false, and a counterexample is $5^2 + 12^2 = 13^2$.)

The correct answer is therefore B.

11.19 Answer H

TMUA 2020 · Paper 2 · Q7

To show that a condition is sufficient, we must explain why having the condition means that the parallelogram must be a square. To show that a condition is not sufficient, we must find an example of a parallelogram that satisfies the condition but is not a square.

For condition I, if $PQRS$ is a rhombus that is not a square, then it satisfies the condition but is not a square. So condition I is not sufficient.

For condition II, the same example works: the diagonals of a rhombus intersect at right angles even when the rhombus is not a square.

For condition III, we know that the adjacent angles $\angle PQR$ and $\angle QRS$ sum to 180°, so if condition III is satisfied, they must each be 90°, a right angle. But a rectangle that is not a square satisfies this condition and is a parallelogram.

Therefore none of these conditions is sufficient, and the correct option is H.

11.20 Answer F

TMUA 2022 · Paper 2 · Q7

Searching for small counterexamples to the claim does not find any; the first few differences are all prime. So we do have to check the proof in detail, as it might be correct.

The algebraic expansion on line I is correct.

The expression in line II is also correct, using the expansion from line I.

For line III, if we could write $3x^2 + 3x + 1 = (ax + b)(cx + d)$ with real coefficients a, b, c and d, then we would have real roots of the equation $3x^3 + 3x + 1 = 0$, namely $x = -\frac{b}{a}$ and $x = -\frac{d}{c}$. But the discriminant is $3^2 - 4 \times 3 \times 1 = -3 < 0$, so this is impossible. So this statement is true.

Line IV is more tricky: just because the algebraic expression is not factorisable into algebraic factors does *not* mean that the integer it represents for any particular value of x is not factorisable in integers. As a simpler example, the algebraic expression $x^2 + 1$ is not factorisable into real algebraic factors, yet substituting integers for x shows that the integer value can sometimes be factorised: taking $x = 3$ gives $3^2 + 1 = 10 = 2 \times 5$, for example. So this line of the argument is not valid.

Line V does follow from lines II and IV.

Therefore the proof is wrong, and the first (and only) error occurs on line IV (option F).

It is worth noting that the 'opposite' argument does work: if an algebraic expression does factorise as $(ax + b)(cx + d)$ with a, b, c and d integers, then the value of original expression always factorises (as an integer) when x is an integer, just by substituting the value of x into the factorised form. (It is possible, though, that one of the factors is 1, so the integer may be prime for some value(s) of x.)

It also turns out that in this case, if we search far enough, we do find a counterexample to the claim: $6^3 - 5^3 = 216 - 125 = 91 = 7 \times 13$. (It might have been the case that the claim was true even though the attempted proof had an error.)

11.21 Answer C

TMUA Specimen · Paper 2 · Q8

We could start by trying a few examples to get a feeling for the statement.

- $n = 0$ gives $0 + 1 + 2 + 3 = 6$, so $(*)$ is true in this case.
- $n = 1$ gives $1 + 2 + 3 + 4 = 10$, so $(*)$ is false in this case.
- $n = 2$ gives $2 + 3 + 4 + 5 = 14$, so $(*)$ is false in this case.
- $n = 3$ gives $3 + 4 + 5 + 6 = 18$, so $(*)$ is true in this case.

Of the given options, this exploration shows: A is false; B is false (as $(*)$ is false for $n = 1$); C may be true; D is false (as $(*)$ is true for $n = 3$), and E is false.

So by elimination, the correct answer must be C.

We can also prove this. The sum of the four consecutive integers is

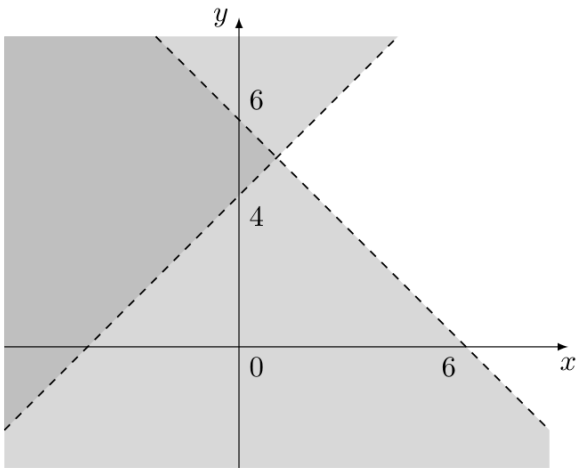
$$n + (n + 1) + (n + 2) + (n + 3) = 4n + 6.$$

This is a multiple of 6 whenever $4n$ is a multiple of 6. An integer is a multiple of 6 if and only if it is a multiple of both 2 and 3. $4n$ is always a multiple of 2, and is a multiple of 3 when n is a multiple of 3 and for no other value of n. Hence option C is correct.

11.22 Answer B

TMUA 2016 · Paper 2 · Q8

A useful way to approach this question is to sketch the graph of these inequalities; we shade the *invalid* regions. (You might find it helpful to rewrite the inequalities as $y > 6 - x$ and $y < x + 4$.)



The boundaries of the regions intersect when $x + y = 6$ and $x - y = -4$; adding these equations gives $2x = 2$, so $x = 1$ and $y = 5$, that is, they intersect at $(1, 5)$.

Therefore for every point in the white region, condition 1, $x > 1$, is satisfied.

However, there are clearly points for which condition 2, $y > 5$, is not satisfied, for example $(7, 0)$.

Finally, $(x + y)(x - y) > -24$ is not immediately clear. This is obtained by multiplying the two inequalities together, but is this permissible? We could try finding a counterexample to this condition. $x + y > 6$ means that $x + y$ is always positive, so a counterexample must have $x - y < 0$ (that is, $y > x$) to give a product less than -24. We could take $y = x + 3$, giving $x - y = -3$, and then if we had $x + y > 8$, the product would be less than -24. (Recall that the first inequality is equivalent to $x + y > 6$, so we may do this.) So we have a counterexample: take $x = 5$, $y = 8$, giving $x + y = 13$, and $(x + y)(x - y) = -39 < -24$. Hence this third condition is not true for every point in the region.

Therefore the answer is option B.

11.23 Answer D

TMUA 2018 · Paper 2 · Q8

I Every valid code begins with a 'U' and ends with a 'D', so the reverse code begins with a 'D', meaning it immediately dips below sea level. So it is never a valid code, and this statement is false.

II As with I, the first letter of this substituted code will always be 'D', which will never give a valid code, so this statement is false.

III The new 'U' at the beginning of the code and 'D' at the end mean that the whole middle part of the profile is lifted one unit higher than the original profile. So this will never dip below sea level, and will always be a valid code. So this statement is true.

Therefore the correct answer is D.

When thinking about inequalities involving negative numbers, we must take care to reverse the sign of any inequality when multiplying or dividing by a negative number. In this case, all three of a, b and c are negative.

I We start with $a < b < c$ and multiply by $a < 0$ to obtain

$$a^2 > ab > ac$$

so this statement must be true.

II Since $a < 0$ and $c < 0$, $a + c < 0$. Since $b < 0$, it follows that $b(a + c) > 0$. So this statement must be true.

III We start with $c > a$ (since $a < c$). Dividing by $b < 0$ then gives

$$\frac{c}{b} < \frac{a}{b}$$

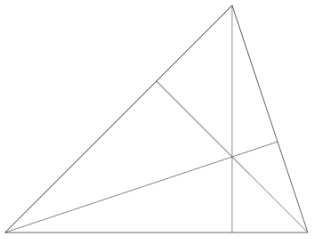
and so this statement must be false.

The correct answer is E.

11.25 Answer D

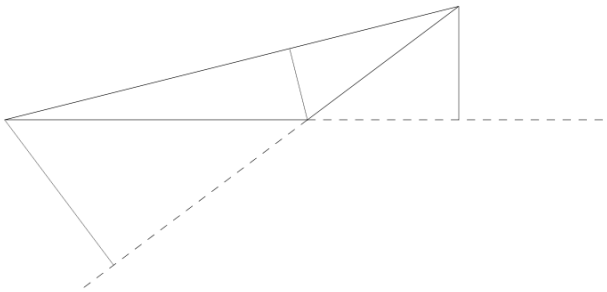
TMUA 2023 · Paper 2 · Q8

Let us start with a triangle and see how many such straight lines it has. If we take an acute-angled triangle, then it has three such lines (shown in red), which are the altitudes of the triangle, going from each vertex perpendicular to the opposite side:

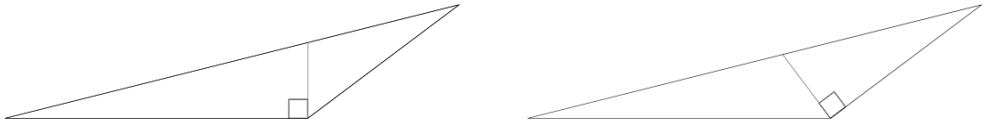


Therefore statement III is true.

But can the student draw a triangle for which there are fewer than 3 such straight lines? There is always such a straight line (altitude) from every vertex in an acute-angled triangle, so the only hope for fewer than 3 such straight lines is to consider an obtuse-angled triangle. There is only one usable altitude in this case, and that is from the obtuse angle; the other two altitudes land outside the triangle:



So it may appear that there is only one such line in this case. However, we are not restricted to altitudes! Instead, we can split the obtuse angle into a right angle and an acute angle in two different ways, giving another two lines:



Neither of these lines can meet the side opposite the obtuse angle perpendicularly, as if they did, we would have a triangle with two right angles, so they are always distinct from the altitude.

So in this case, we also get three such staight lines.

Hence the correct answer is option D, III only.

11.26 Answer E

TMUA Specimen · Paper 2 · Q9

If a statement is false, then its negation is true.

The negation of 'for all x, P' is 'there exists x such that not P'. So in this case, a negation is

> There exists a day next week such that it is not the case that Fred will do at least one maths problem.

'It is not the case that Fred will do at least one maths problem' is equivalent to 'Fred will do less than one maths problem'; assuming that one cannot do a fraction of a maths problem, this is equivalent to 'Fred will do no maths problems'. So the whole negation can be written as 'Some day next week, Fred will do no maths problems', which is option E.

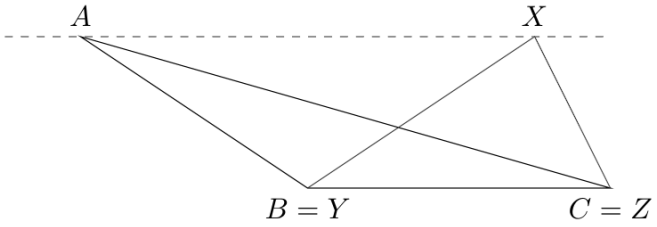
11.27 Answer D

TMUA 2016 · Paper 2 · Q9

We work through the conditions in the order given. We recall three relevant conditions for these two triangles to be congruent:

- side-side-side (SSS): all three corresponding pairs of sides are equal in length
- side-angle-side (SAS): two corresponding pairs of sides are equal in length, and the corresponding included angles are equal
- angle-side-angle (ASA): two corresponding pairs of angles are equal, and the corresponding sides between these angles are equal in length

(1) If we knew that either $AC = XZ$ or the angle B equals the angle Y, then they would certainly be congruent by SSS or SAS. We know that the areas are equal, so we have $\frac{1}{2}AB.BC\sin B = \frac{1}{2}XY.YZ\sin Y$, and so $\sin B = \sin Y$. This does not imply that $B = Y$ though: we could have $Y = 180° - B$, for example:



(2) We now have an angle at the end of the given side, so we can use the equal-area property to work out a second side: the angle $\angle ABC$ is included between AB and BC, and likewise for XYZ. Therefore, as the areas are $\frac{1}{2}AB.BC\sin(\angle ABC)$ and $\frac{1}{2}XY.YZ\sin(\angle XYZ)$, it follows that $BC = YZ$. The triangles are therefore congruent by SAS.

(3) We now have two pairs of corresponding equal angles, and so the third pair of corresponding angles is also equal and the triangles are similar. Since the areas are equal, they must be congruent.

Hence conditions (2) and (3) each imply that the triangles are congruent, and the answer is option D.

11.28 Answer D

TMUA 2017 · Paper 2 · Q9

> ***Commentary:*** *This is Fermat's Last Theorem for the case $n = 3$; it was first proven by Euler in the 18th century. The attempted proof here, though, is deeply flawed, as we will see.*

Line I is fine: this is a simple rearrangement.

Line II is fine: this is a standard algebraic identity which is easy to check.

Line III is problematic. It is certainly true that $a \leqslant a^2$ and $c - b < c + b \leqslant c^2 + b^2 < c^2 + cb + b^2$. It is therefore plausible that $a = c - b$ and $a^2 = c^2 + cb + b^2$, but it is far from necessary. Of course we cannot give an explicit example to show this, as there are no sets of positive integers a, b and c with $a^3 + b^3 = c^3$. But we can observe that if $c - b = 1$, we must have $a^3 = c^2 + bc + b^3$, and this possibility has not been considered; alternatively, we could consider a case where a is not prime such as $a = 6$; then we can write $a^3 = 6^3 = 3 \times 72$, and we could have $c - b = 3$ and $c^2 + cb + b^2 = 72$. Either way, it does not necessarily follow that $a = c - b$ and $a^2 = c^2 + cb + b^2$, so this step is wrong.

The correct answer is D.

11.29 Answer F

TMUA 2018 · Paper 2 · Q9

Given the choices, we first check to see what the proposed solutions $x = \frac{5}{8}$ and $x = \frac{5}{2}$ work in the original equation.

When $x = \frac{5}{8}$ the left side is $4 \times \frac{5}{8} \times \sqrt{2 \times \frac{5}{8} - 1} = \frac{5}{4}$ and the right side is $10 \times \frac{5}{8} - 5 = \frac{5}{4}$ so this is a valid solution.

When $x = \frac{5}{2}$ the left side is $4 \times \frac{5}{2} \times \sqrt{2 \times \frac{5}{2} - 1} = 20$ and the right side is $10 \times \frac{5}{2} - 5 = 20$ so this is also a valid solution.

This therefore eliminates options B, C and D.

There may, though, be additional solutions. Let us work through the argument step-by-step, looking for steps which are not reversible; these are often the cause of errors when attempting to solve equations.

Step (I) is just factorising the right hand side, so is fine.

Step (II) involves squaring. In general, squaring an equation is not reversible, as we do not know whether to take the positive or negative square root when going backwards. However, if we do have a solution to the original equation, it will still be a solution when we square it; the worst that can happen is that we introduce new solutions. Since the two solutions found do work, this step is fine, as is the conclusion (though the author of the attempt should have checked that this is the case).

Step (III) involves dividing by $2x - 1$. But what if this is zero? In that case, we would have $x = \frac{1}{2}$. Trying this in the original equation makes the left hand side $4 \times \frac{1}{2} \times \sqrt{2 \times \frac{1}{2} - 1} = 0$, while the right side is $10 \times \frac{1}{2} - 5 = 0$. So $x = \frac{1}{2}$ is another solution.

The correct option is therefore F.

(The remaining steps are rearrangements, and are all valid.)

11.30 Answer D

TMUA 2019 · Paper 2 · Q9

If the next person **must** sit next to someone, there cannot be three spaces between one person and the next. But if there are only two spaces between them, someone sitting down in one of those spaces must sit next to someone. So the smallest number of people will be achieved when all of the gaps are exactly two spaces, except perhaps for the final person to the first person.

So counting the seats around the table as 1, 2, 3, ..., we place people in seats 1, 4, 7, 10, ..., 37, 40. (The final person is sitting immediately next to the first person, but that does not matter.) There are 14 seats in this list, so the answer is D.

11.31 Answer A

TMUA 2022 · Paper 2 · Q9

The statement $(*)$ does not restrict us to *positive* real numbers x, so whatever the value of k, we can always find a large negative number x with $x < k$ and $x^2 \geq k$.

We can even be explicit about this (though that is not required to answer this question): if $k \leq 0$, we can take $x = k - 1$ in which case $x^2 > 0 \geq k$, and if $k > 0$, we can take $x = -\sqrt{k}$ so $x < 0 < k$ and $x^2 = k \geq k$.

Therefore $(*)$ is true for no real values of k, and the correct option is A.

A key point in this question is that x and y are permitted to be positive or negative.

To show that a condition is not sufficient, we need to find values of x and y where the condition holds but $x \geqslant y$. To show that a condition is sufficient, we must prove it to be the case.

We can argue algebraically or graphically; we start with an algebraic approach.

A If $y = -2$ and $x = 1$, the condition holds but $x > y$.

B We do not even need to look at negative cases here: if x and y are both positive, we can take fourth roots and deduce that $y < x$, for example $y = 1$ and $x = 2$.

C If $x = 2$ and $y = 1$, then $x^{-1} < y^{-1}$ but $x > y$.

D Positive values of x and y do not work, but if we take $x = 1$ and $y = -1$, then $y^{-1} < x^{-1}$ but $x > y$.

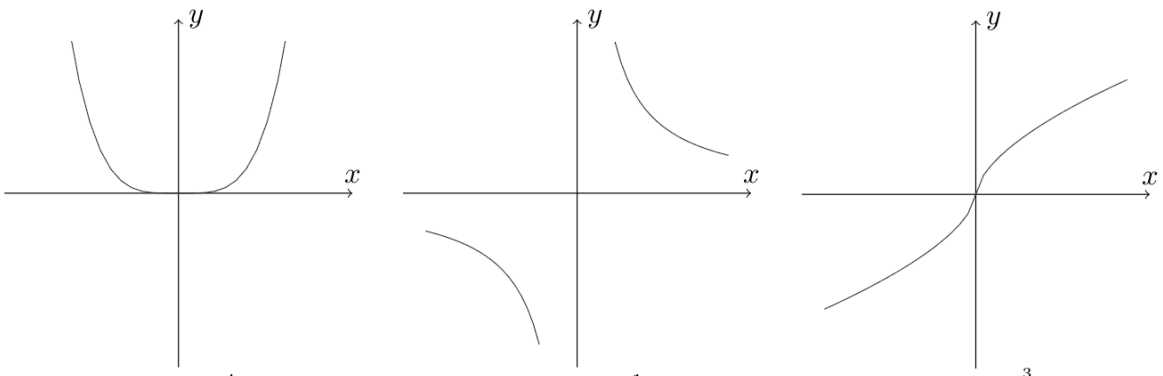
E This looks quite plausible: if we raise this inequality to the power of 5 (which we are allowed to do, as $\mathrm{f}(x) = x^5$ is an increasing function), we get $x^3 < y^3$. We can now take cube roots (again, this is permissible as $\mathrm{f}(x) = x^{\frac{1}{3}} = \sqrt[3]{x}$ in an increasing function) to give $x < y$. So this condition is sufficient.

F Arguing as in part E, $y^{\frac{3}{5}} < x^{\frac{3}{5}}$ implies $y < x$, so this is not sufficient. Alternatively, we can just substitute in the values $x = 1$, $y = -1$.

Thus the answer is option E.

We can also see this by looking at the graphs of the functions $\mathrm{f}(x) = x^4$, $\mathrm{f}(x) = x^{-1}$ and $\mathrm{f}(x) = x^{\frac{3}{5}}$: if these are strictly increasing functions, then $x < y$ if and only if $\mathrm{f}(x) < \mathrm{f}(y)$, while if they are strictly decreasing functions, then $x < y$ if and only if $\mathrm{f}(x) > \mathrm{f}(y)$.

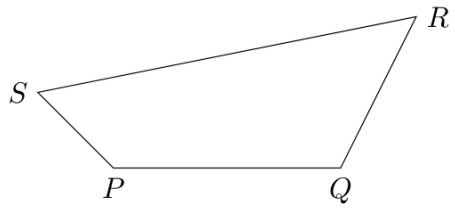
These are sketches of the graphs of these functions:



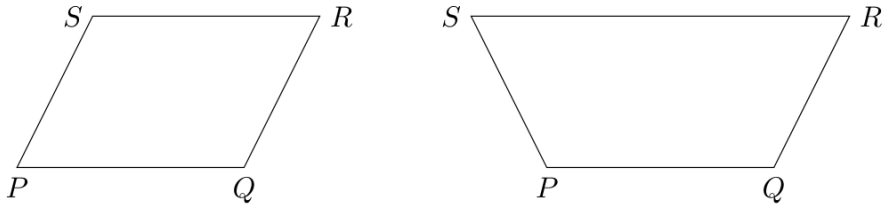
$\mathrm{f}(x) = x^4$ $\mathrm{f}(x) = x^{-1}$ $\mathrm{f}(x) = x^{\frac{3}{5}}$

Neither of the first two functions is strictly increasing or strictly decreasing (for $\mathrm{f}(x) = x^{-1}$, note that the function decreases when $x < 0$, but $\mathrm{f}(-1) < \mathrm{f}(1)$, so it is not a decreasing function). The third function is strictly increasing, so the answer is option E.

We sketch a quadrilateral for ease of reference.



A The sides PQ and SR are equal in length, which is certainly necessary if $PQRS$ is a parallelogram, and PS being parallel to QR is also necessary. If these are both the case, then on first glance it may seem that $PQRS$ must be a parallelogram. However, if we draw two parallel lines and ask how we can join them with two equal-length sides. We discover there are two ways:



So this condition is necessary but not sufficient.

B As before, $PQ = SR$ is necessary, and PQ being parallel to SR is obviously necessary, so the whole condition is necessary.

Unlike in the previous part, though, this is also sufficient: if two sides of the quadrilateral are both equal and parallel, the remaining two sides must also be equal and parallel. (This may seem obvious, and it can be proven using geometric or vector methods; see below for details.)

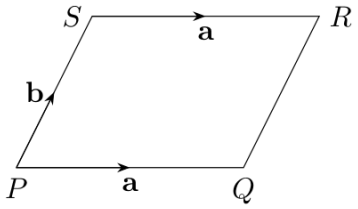
C This condition is not necessary, as not all four sides of a parallelogram need to be the same length. It does mean that the parallelogram is a rhombus, so it is a sufficient condition.

D The diagonals of a parallelogram need not be equal in length, so this is not a necessary condition. It is also not sufficient.

The correct answer is therefore A.

We now give two proofs that condition B is sufficient, that is, if $PQ = SR$ and PQ is parallel to SR, then $PQRS$ is a parallelogram.

We first give a vector proof. Since $PQ = SR$ and they are parallel, $\overrightarrow{PQ} = \overrightarrow{SR}$; we call this vector $\mathbf{a}$. Let $\overrightarrow{PS} = \mathbf{b}$, as shown in this diagram:

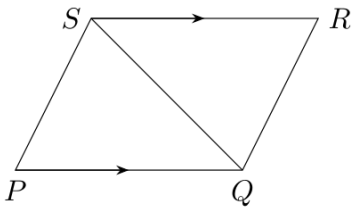


Then

$$\overrightarrow{QR} = \overrightarrow{QP} + \overrightarrow{PS} + \overrightarrow{SR} = (-\mathbf{a}) + \mathbf{b} + \mathbf{a} = \mathbf{b}$$

so $\overrightarrow{QR} = \overrightarrow{PS}$, showing that PS and QR are equal and parallel, and thus $PQRS$ is a parallelogram.

And here is a geometric approach. We are given that $PQ = RS$ and that they are parallel. Draw in the diagonal QS, as shown.



Then $\angle PQS = \angle RSQ$ as they are alternating angles.

Therefore triangles PQS and RSQ are congruent by side-angle-side, as $PQ = RS$ (given), $QS = SQ$ (common side) and $\angle PQS = \angle RSQ$.

Thus $\angle QSP = \angle SQR$, so SP is parallel to QR (by alternating angles) and $PS = RQ$.

Hence $PQRS$ is a parallelogram.

11.34 Answer F

TMUA 2020 · Paper 2 · Q10

One result about inequalities is that if $p < q$ and $r < s$, then $p + r < q + s$. This is because

$$p + r < q + r < q + s$$

where the first inequality follows by adding r to $p < q$ and the second follows by adding q to $r < s$.

Since we have $a + b < c + d$ and $a + c < b + d$, we can therefore add these to get

$$2a + b + c < b + c + 2d.$$

Subtracting $b + c$ from both sides gives $2a < 2d$, so $a < d$. Hence I must be true.

A similar approach does not seem to work to show $b < c$, which suggests that perhaps we should look for a counterexample. If d is much larger than a, then the values of b and c should not make much of a difference to the inequality if they are both small. So let us try $a = 1$, $d = 100$, $b = 2$, $c = 1$. Then we do indeed have $0 < 1 + 2 < 1 + 100$ and $0 < 1 + 1 < 2 + 100$, so the first two inequalities hold but $b > c$. Therefore II might not be true.

For III, we note that $a + b > 0$ and $c + d > a + b > 0$, so adding $a + b > 0$ and $c + d > 0$ gives $a + b + c + d > 0$, and thus III must be true.

The correct option is F.

11.35 Answer E

TMUA 2021 · Paper 2 · Q10

To be a counterexample, we need the antecedent (the 'if' part) to be true but the consequent (the 'then' part) to be false.

n	n is prime	a_n multiple of 3 or 5
1	false	–
2	true	true
3	true	true
4	false	–
5	true	false
6	false	–
7	true	false

Therefore $n = 5$ and $n = 7$ are both counterexamples, and the smallest counterexample is $n = 5$, which is option E.

11.36 Answer G

TMUA 2022 · Paper 2 · Q10

I We can find a counterexample to this statement: taking $x = 2$ and $n = 1$ shows that this statement is false.

II This is true; if $x \leq 0$, we can take $n = 1$, and if $x > 0$, we can take n to be the smallest integer greater than x.

III This is true; we can take $x = 0$. (If, though, we replaced 'positive integers' with 'integers', the resulting statement would be false.)

Therefore II and III are true, and the correct option is G.

11.37 Answer A

TMUA 2023 · Paper 2 · Q10

Substituting $x = 0$ gives $x^4 - 2x^2 - 3 = -3 < 0$, so the final conclusion on line VI is at least plausible. We will check the argument line-by-line.

I This is obtained by adding 4 to both sides, and this is reversible. This line is correct.

II This is an algebraic rearrangement: $(x^2 - 1)^2 = x^4 - 2x^2 + 1$ is an identity, so this line is correct.

III We have $u^2 < 4$ if and only if $-2 < u < 2$. Taking $u = x^2 - 1$, we see that this line is equivalent to line II, so this step is correct.

IV This looks suspicious, as one of the inequalities has been dropped. The statement $x^2 - 1 < 2$ certainly follows from $-2 < x^2 - 1 < 2$, but the claim is that they follow from each other. Can we deduce $-2 < x^2 - 1 < 2$ from $x^2 - 1 < 2$?

Perhaps surprisingly, it turns out that we can. The function $x^2 - 1$ has a minimum value of -1, so we always have $x^2 - 1 \geq -1$, and therefore $-2 < x^2 - 1$ is true for all x. Hence the statement in line III does follow from $x^2 - 1 < 2$, and this line is correct.

V This is obtained by adding 1 to both sides, which is reversible, so this line is correct.

VI Similarly to our observation in line III, $x^2 < 3$ is true if and only if $-\sqrt{3} < x < \sqrt{3}$, so this line is correct.

Thus the entire argument is correct, and the correct option is A.

11.38 Answer C

TMUA 2021 · Paper 2 · Q11

We start by checking that each step works.

Line (II) follows from line (I) by expanding the brackets.

The algebraic expression in line (III) expands as $x^2+2xy+y^2-4xy$, so it follows from line (II).

Line (IV) follows from line (III) by multiplying by x^2y^2. Since $x^2y^2 \geq 0$, the inequality keeps the same direction, so this is a valid deduction.

However, how does line (V) follow from line (IV)? It assumes that $a = xy(x+y)$ and $b = xy$ in order to make the deduction.

This argument has therefore shown that *if* $a = xy(x+y)$ and $b = xy$, *then* $a^2 - 4b^3 \geq 0$, or even more explicitly: *if* there exist real numbers x and y such that $a = xy(x+y)$ and $b = xy$, *then* $a^2 - 4b^3 \geq 0$. However, the problem posed is the converse: 'show that *if* $a^2 - 4b^3 \geq 0$, *then* there exist real numbers x and y such that $a = xy(x+y)$ and $b = xy$.

So this attempt has proved the converse, though the argument would have been slightly better presented if the student had explicitly stated 'assume that x and y are real numbers with $a = xy(x+y)$ and $b = xy$'.

The correct option is therefore C.

11.39 Answer C

TMUA 2022 · Paper 2 · Q11

The angle SPQ is a right angle if and only if the triangle SPQ satisfies Pythagoras's theorem, that is, if and only if

$$SP^2 + PQ^2 = SQ^2.$$

Since $PQRS$ is a kite, its diagonals intersect at right angles, so both SOP and POQ are right-angled triangles. Therefore, again by Pythagoras,

$$SP^2 = x^2 + z^2$$
$$PQ^2 = x^2 + y^2.$$

Substituting this, along with $SQ = y + z$, into the above equation gives the following necessary and sufficient condition for angle SPQ to be a right angle:

$$x^2 + z^2 + x^2 + y^2 = (y+z)^2.$$

Expanding and simplifying, this is equivalent to

$$2x^2 = 2yz$$

or $x^2 = yz$.

Therefore option C gives a necessary and sufficient condition.

(One should also check that none of the others are necessary and sufficient; we could take $x = 2$, $y = 4$, $z = 1$: this satisfies option C but none of the others.)

11.40 Answer G

TMUA 2023 · Paper 2 · Q11

I This is the converse of $(*)$, so is not equivalent to $(*)$.

II We can rewrite this as an 'if… then' statement, noting that '**A** only if **B**' means the same as 'if **A**, then **B**'. So this becomes '**if** 2^k+1 is **not** prime, **then** k is **not** a power of 2'. This is the contrapositive of '**if** k is a power of 2, **then** 2^k+1 is prime', which is the statement in I, namely the converse of $(*)$. So this statement is equivalent to the converse of $(*)$ but not equivalent to $(*)$ itself. (The contrapositive of the converse, which is the same as the converse of the contrapositive, is called the *inverse* of the original statement.)

III '**A** is a sufficient condition for **B**' is the same as 'if **A**, then **B**', so this statement can be rewritten as '**if** 2^k+1 is prime, **then** k is a power of 2'. (Note the subtlety of language here: the condition **A** is '2^k+1 is prime' and **B** is 'k is a power of 2', but they are written in the order 'a sufficient condition for **B** is **A**'.) This is equivalent to $(*)$ (and is in fact just a different way of saying the same thing).

The correct option is G.

> ***Commentary:*** *It turns out that statement I is false. It was conjectured by Pierre de Fermat in the 17th century, who noted that* $2^{2^n}+1$ *is prime for* $n=1$, *2, 3 and 4. Such primes are called Fermat primes. The smallest counterexample to statement I is* $2^{2^5}+1=641\times 6\,700\,417$, *which was discovered by Leonhard Euler in the 18th century. It is not known whether* $2^{2^n}+1$ *is prime for any* $n>4$.

11.41 Answer F

TMUA 2018 · Paper 2 · Q12

The negation of '**for any** N, P' is '**there exists** N such that **not** P'. So none of options A-D can be a negation of the original statement, but all of E-H still have the potential to be.

In this case, the statement P is 'there is a postiive integer K such that $N(Km+1)-1$ is not prime for any positive integer m'; this is a statement of the form '**there exists**', and we wish to negate it. The negation of '**there exists** K such that Q' is '**for all** K, **not** Q', so we are looking for a statement which continues with 'for all positive integers K', and this eliminates option H. ('For any' and 'for all' are equivalent ways of expressing this.) So we are now looking at options E, F and G.

The statement Q that we wish to negate is '$N(Km+1)-1$ is not prime for any positive integer m', which we can rewrite with the quantifier ('for any') at the start of the statement to give 'for any positive integer m, $N(Km+1)-1$ is not prime'. And the negation of '**for all** m, R' is '**there exists** m such that **not** R', which in this case is 'there exists a positive integer m such that it is **not** the case that $N(Km+1)-1$ is not prime', so option G is not correct.

Now '**not not** S' is equivalent to S, so this becomes 'there exists a positive integer m such that $N(Km+1)-1$ is prime'. The only option which matches this is option F, and this is therefore the correct answer.

11.42 Answer C

TMUA Specimen · Paper 2 · Q13

Let us indicate the runners by their initials, and write $A > B$ to mean that A beat B.

We are given $F > G$, $H > L$, $L > G$ and $R > G$, which combine to give $F > G$, $H > L > G$, $R > G$. So G came last, and the other four runners could have come in any order, except that $H > L$. There are $4! = 24$ ways of ordering these four runners. In half of them, $H > L$, while in the other half, $L > H$. (More precisely, we can pair every finishing order with $H > L$ together with the finishing order in which H and L are swapped, so there are the same number of finishing orders with $H > L$ as there are with $L > H$.)

So the number of possible finishing orders is $\frac{24}{2} = 12$, and the answer is C.

11.43 Answer E

TMUA 2016 · Paper 2 · Q13

Working through the lines of the argument one at a time:

1. This looks like it might be the contrapositive of the statement we are aiming to prove; let us check this. The original statement is:

   If a divides bc, **then** a divides b **or** a divides c

   so the contrapositive is

   If not (a divides b **or** a divides c), **then not** (a divides bc)

   which simplifies to

   If not (a divides b) **and not** (a divides c), **then** a does not divide bc

   or

   If a does not divide b **and** a does not divide c, **then** a does not divide bc

   So line 1 is correct.
2. "Suppose …" starts the proof: we suppose the antecedent of the "**if**" statement. The part about remainders is correct; this is what it means for a to not divide b.
3. This is an expression of division with remainder: $b = ax + r$ could be read as "b divided by a is x remainder r." So this line is correct.
4. A small amount of algebra shows that this line is correct.
5. This is not clearly correct: what happens if $rs > a$, for example? We will return to this line in a moment.
6. This is clearly a correct assertion.
7. If line 5 were correct, then this would be correct.

Since we are told that the statement is false, the error must lie in line 5. We have already identified one possible problem, but we can do even better by finding a case where the remainder when dividing bc by a is 0, showing that a *does* divide bc. Let us take, for example, $a = 6$. Then if $r = 2$ and $s = 3$, we have $rs = 6 = a$, so that the remainder is 0. An explicit example would then be $a = 6$, $b = 2$ and $c = 3$.

The correct answer is therefore E.

One way to think about this is to write it as a column addition. If we write $a \times 10^{-3}$ as $0.00a_1a_2\ldots$, where $a = a_1.a_2a_3\ldots$ (as a decimal expansion), and similarly for the other numbers, the sum becomes:

$$\begin{array}{rcccccc} & 0 & . & 0 & 0 & a_1 & a_2 \ \ldots \\ + & 0 & . & 0 & b_1 & b_2 & b_3 \ \ldots \\ \hline & 0 & . & c_1 & c_2 & c_3 & c_4 \ \ldots \end{array}$$

We see that, as $c_1 \geqslant 1$, there must be a carry from the hundredths column to the tenths column, and hence $b_1 = 9$, so $b \geqslant 9$. If $b = 9$ exactly, then there cannot be a carry; we would just get a sum of $0.09a_1a_2\ldots$ in that case. So we must have $b > 9$, and **II must be true**.

There is no need to have $a_1 = 9$; we could, for example, have $a_1 = 8$ and $b_2 = 3$ and still have a carry. So **I is not necessary true**.

Comparing a and b to c, we know that $1 \leqslant c < 2$, as $c_1 = 1$. We saw that $a_1 = 8$ is possible, so $a > 8$ is possible, and $a < c$ **(statement III) is not necessarily true**.

We have already shown that $b > 9$, so since $c < 2$, $b > c$ **(statement IV) is never true**.

So the answer is B.

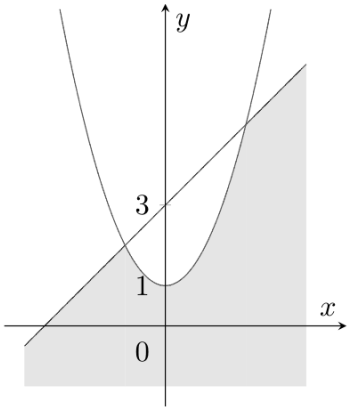
11.45 Answer A

TMUA 2021 · Paper 2 · Q13

Manipulating inequalities algebraically requires care. For example, we cannot (in general) multiply two inequalities together and obtain a valid inequality (which is what statement II has done). Neither can we subtract two inequalities (which is where statement I comes from).

In this case, one approach is to sketch the region R described by the pair of inequalities and then determine whether the inequalities in statements I–III are satisfied in the whole of R. Another approach is to search for counterexamples for each statement, that is, values of x and y that do not satisfy the statement but do satisfy the original inequalities. This, however, might be harder to do, and if the inequalities are true, it may take a while to discover this. So let us sketch the given inequalities.

The inequality $y - x < 3$ can be rearranged as $y < x + 3$, so it is the region below $y = x + 3$. Similarly, $y - x^2 < 1$ can be rearranged as $y < x^2 + 1$, so it is the region below the quadratic $y = x^2 + 1$. The region R is therefore the shaded area shown here (excluding the boundaries):



We can now consider the three statements.

I It is clear that this is not true in general, as the region R extends in all directions. For example, the point $(10, 0)$ lies in R (it is easy to check that this satisfies the original inequalities: $0 - 10 < 3$ and $0 - 10^2 < 1$) but $x = 10$ does not satisfy $-1 < x < 2$.

II If we take $x = 0$, the inequality becomes $y^2 < 3$, but $(0, -2)$ is in R and $(-2)^2 = 4$. So this statement is not true for all points in R.

III We could take, for example, $(6, 6)$ as a point in R with $y \geq 5$, so this statement is not true in general.

The correct option is therefore A.

11.46 Answer E

TMUA 2022 · Paper 2 · Q13

For each fixed value of x, we can think about this expression as being a function of y. We can write the expression as $(1 - x)y + x$, which is just $my + c$ where $m = 1 - x$ and $c = x$. This is the equation of a straight line graph, so it takes every real value, both positive and negative, as long as $m \neq 0$.

The case $m = 0$ occurs exactly when $x = 1$, and in this case, the expression becomes $(1-1)y+1 = 1$, so it is always 1, for every value of y.

Therefore $(*)$ is true for all except exactly one value of x, namely $x = 1$, and the correct answer is option E.

11.47 Answer C

TMUA 2023 · Paper 2 · Q13

We work through them sequentially:

A If $x \geq 0$, then B may be false, C may be false, D may be false and E may be false

B If $x = 1$, then A is true, C is true, D is true and E is true

C If $x = 0$ or $x = 1$, then A is true, B may be false, D is true and E is true

D If $x \geq 0$ or $x \leq 1$, then A may be false (for example if $x = -1$), B may be false, C may be false and E may be false

E If $x \geq 0$ and $x \leq 1$, then A is true, B may be false, C may be false and D is true

The correct option is therefore C, which is sufficient for exactly three of the other four statements.

11.48 Answer E

TMUA Specimen · Paper 2 · Q15

1 If we multiply the original inequality $a \geqslant b$ by -1, we obtain $-a \leqslant -b$, since multiplying by a negative number reverses the inequality. This is equivalent to $-b \geqslant -a$, so this statement must be true.

2 We can rearrange this inequality as $a^2 - 2ab + b^2 \geqslant 0$, and then factorise the left hand side to get $(a-b)^2 \geqslant 0$. As $x^2 \geqslant 0$ for every real number x, this statement must be true. (Note that the condition $a \geqslant b$ is irrelevant here.)

3 This is related to the original inequality by multiplying it by c. If $c = -1$, then this inequality states $-a \geqslant -b$, which will not be the case unless $a = b$. For example, $a = 2$, $b = 1$, $c = -1$ provides an example where $a \geqslant b$ but $ac \geqslant bc$ is false. So this statement might not be true.

Hence the correct option is E.

11.49 Answer C

TMUA 2016 · Paper 2 · Q16

There are several pairs of similar triangles in this diagram.

We note that SUX and SPQ are similar, RXT and RPQ are similar, and XSR and XQP are similar.

The last pair is the most helpful to start with, as we know the lengths of two corresponding sides: $SR = 3$ and $QP = 12$, so the scale factor is $\frac{12}{3} = 4$. Thus the height of XQP (from X) is 4 times the height of XSR (again from X).

It follows that $UP = 4SU$, and so $SP = 5SU$. Since SUX is similar to SPQ, and $SP = 5SU$, we must have $PQ = 5UX$, so $UX = \frac{1}{5}PQ = \frac{12}{5}$. Likewise, $XT = \frac{12}{5}$, so $UT = \frac{24}{5} = 4.8$ and the correct answer is C.

We could alternatively have used the similarity of triangles PUX and PSR, and of triangles QXT and QSR; we would reach the same result.

For each of these, we can try to construct a counterexample, and if that doesn't work, we may understand why the statement must be true.

For simplicity in the explanations below, let's write

$$A_{\min} = \text{minimum of } a_1, \ldots, a_{100}$$
$$B_{\min} = \text{minimum of } b_1, \ldots, b_{100}$$
$$C_{\min} = \text{minimum of } c_1, \ldots, c_{100}$$

and similarly for $A_{\max}$ etc.

I Let's suppose that $B_{\min} = 0$ and $C_{\min} = 0$. Can we then make the minimum of the a_ns greater than zero?

Yes, we can: if we take $b_1 = 0$ and the rest of the b_ns to be 100, and we take $c_{100} = 0$ and the rest of the c_ns to be 100, then we can have $a_n = 100$ for every n, so the minimum of the a_ns is 100.

Therefore this statement is not necessarily true.

II If we try the same example as in I, we find that $A_{\min} = 100$, which is greater than $B_{\min} + C_{\min}$, so the inequality holds in this case.

But to get a counterexample, we want to make $A_{\min}$ *small*. Helpfully, the condition $a_n \leq b_n + c_n$ allows us to make each a_n as small as we like. So if we take $a_n = 0$, $b_n = 1$ and $c_n = 1$ for all n, then the condition will be satisfied for each n, but we will have $A_{\min} = 0$ and $B_{\min} = C_{\min} = 1$, so $A_{\min} < B_{\min} + C_{\min}$ in this case.

Therefore this statement is not necessarily true either.

III Both of the counterexamples we used for I and II satisfy this statement. It is not obvious how to construct a counterexample: every simple example satisfies this statement. So let us instead try to prove it.

Each a_n satisfies the condition $a_n \leq b_n + c_n$. Now $b_n \leq B_{\max}$ and $c_n \leq C_{\max}$ for each n, so $a_n \leq B_{\max} + C_{\max}$ for each n. But this means that the maximum of all of the a_ns also satisfies this condition, that is $A_{\max} \leq B_{\max} + C_{\max}$, so the given statement must be true.

Therefore only statement III must be true, and the correct answer is option D.

11.51 Answer E

TMUA Specimen · Paper 2 · Q17

We are being asked to negate the statement 'n is an S-number', that is, to negate the statement

> for every factor m of n with $m > 1$, the number m is a multiple of some number in S.

The negation of 'for every m, P' is 'there exists m such that not P', a negation is

> there exists a factor m of n with $m > 1$ such that it is not the case that the number m is a multiple of some number in S.

(Note that 'a factor m of n with $m > 1$' is the condition on the numbers m which are being considered by the 'for every', so it is not changed by the negation.) 'There exists' is the same as 'for some', so the correct answer is either D, E or F.

We next need to simplify the second half of our new statement, namely 'it is not the case that the number m is a multiple of some number in S'. If it is not the case that 'the number m is a multiple of some number in S', then it must be that m is not a multiple of any number in S, or equivalently, that no number in S is a factor of m.

Therefore a full negation is

> for some factor m of n with $m > 1$, there is no number in S which is a factor of m,

so the correct answer is E.

11.52 Answer F

TMUA 2017 · Paper 2 · Q17

We begin with the original statement with S replaced by T:

> T is stapled **if and only if for every** whole number a which is in T, **there exists** a prime factor of a which divides **at least one** other number in T.

Therefore, negating this:

> T is **not** stapled **if and only if** it is **not** the case that **for every** whole number a which is in T, **there exists** a prime factor of a which divides **at least one** other number in T.

We can negate the outer '**for every**' to get '**there exists** ... such that it is **not** the case that ...':

> T is **not** stapled **if and only if there exists** a whole number a which is in T such that it is **not** the case that **there exists** a prime factor of a which divides **at least one** other number in T.

So the answer is one of E, F, G, H.

Our current phrasing is 'it is **not** the case that **there exists**'; we could convert this into '**for every** ..., it is **not** the case that ...', but none of these options use this wording. Instead, E, F and G use the wording 'there is no prime factor', which is the same as 'there does not exist a prime factor', which in turn is the same as 'it is not the case that there exists a prime factor'. So we can rewrite the previous statement as:

> T is **not** stapled **if and only if there exists** a whole number a which is in T such that there is no prime factor of a which divides **at least one** other number in T.

Hence the correct answer is F.

11.53 Answer C

TMUA 2018 · Paper 2 · Q17

We work through the given statements in turn.

A This is a statement about numbers of the form N^2. If we write N as a product of prime factors, then N^2 will have the same prime factors, but each to double the power. For example, if $N = 3 \times 17^3 \times 41^4$ then $N^2 = 3^2 \times 17^6 \times 41^8$. So each prime factor occurs to an even power in its prime factorisation, so it is a squaresum.

Alternatively, we can ignore the theorem completely, and note that $N^2 = N^2 + 0^2$, so N^2 is a squaresum.

So this statement is true.

B If N is a squaresum, then each of its awkward prime factors occurs to an even power in its prime factorisation, by the theorem. Likewise, each of the awkward prime factors of M occurs to an even power in its prime factorisation. The prime factorisation of NM is the product of the prime factorisations of M and N, so each awkward prime occurs to an even power in the prime factorisation of NM, and therefore NM is a squaresum.

Thus this statement is also true.

C If NM is a squaresum, then each of its awkward prime factors occurs to an even power in its prime factorisation. But it is not clear why that would mean that each of the awkward prime factors of M and N each occur to an even power in the factorisations of M and N. And we can find a counterexample to this statement: 23 is awkward, so if $N = M = 23$ then $NM = 23^2$ is a squaresum, but neither N nor M is.

So this statement is false.

D As N is not a squaresum, some awkward prime factor occurs to an odd power in its prime factorisation. If we multiply N by this awkward prime factor, then the result will have this awkward prime factor to an even power. But N might have more than one awkward prime factor occurring to an odd power, so we multiply by all such awkward prime factors. The result will be a number with every awkward prime factor occurring to an even power. Thus letting k be the product of all the awkward prime factors of N which occur to an odd power, kN is a squaresum.

Hence this statement is true.

The only false statement is C.

11.54 Answer D

TMUA 2019 · Paper 2 · Q17

As exactly one of the options was correct, if option X is implied by option Y, then option Y cannot be correct (as then option X would also be correct).

Option A implies option B: if A is correct, then the statement is true **if and only if** $x > 1$, which means the statement is true **if** $x > 1$ and the statement is true **only if** $x > 1$, so option B is also correct. Therefore option A cannot be the correct option.

For the same reason, option C cannot be correct, as it implies option D.

We must therefore consider options B and D. Let us assume that option B is correct and ask whether option D follows from it. Option B tells us that whenever $x > 1$, the statement is true, so certainly whenever $x > 2$, the statement is true. Therefore option D follows from option B, and option B cannot be the correct option. The opposite direction does not follow, though: the statement could be true for all $x > 2$, but be false for $x = \frac{3}{2}$, for example.

Therefore the correct option must have been option D.

Let us write $f(x) - g(x)$ explicitly:

$$f(x)-g(x) = (ax^3+bx^2+cx+d)-(px^3+qx^2+rx+s) = (a-p)x^3+(b-q)x^2+(c-r)x+(d-s).$$

We are given that this cubic is positive for every real x. But a cubic always crosses the x-axis! The only way to reconcile this is if the cubic $f(x)-g(x)$ is not actually a cubic, because $a-p=0$.

So we have $a - p = 0$ and statement I is false. We can also now write

$$f(x) - g(x) = (b - q)x^2 + (c - r)x + (d - s).$$

If this function is always positive, then there are two possibilities: either $b - q$ is nonzero and the quadratic has a negative discriminant, or $b-q=0$ and this is not actually a quadratic. The second option looks quite like statement II: if $b - q = 0$, that is, $b = q$, then we have

$$f(x) - g(x) = (c - r)x + (d - s).$$

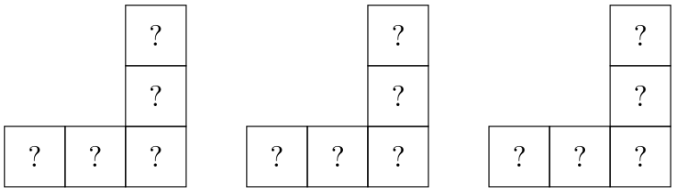
Since a linear function always crosses the x-axis, this cannot be always positive unless $c-r=0$, in which case the function is actually just the constant $f(x) - g(x) = (d - s)$. (To be more precise, this is still a linear function, just with zero x coefficient.) Therefore, if $f(x) - g(x) > 0$ for all real x, it follows that if $b = q$, then $c = r$, and so statement II is true.

We have reached the end of this line of reasoning, and we have not stumbled upon anything that looks like statement III. But d and s are the constant coefficients in the polynomials f and g, so if we take $x = 0$, we have $f(0) = d$ and $g(0) = s$. Since $f(0)-g(0) > 0$, this shows that $d-s > 0$, or $d > s$, hence statement III is true.

The correct option is G.

We first note that the object fits within a 3×3 cube. We begin with the plan view of the object: all of the cubes must be within three layers fitting within this L-shape.

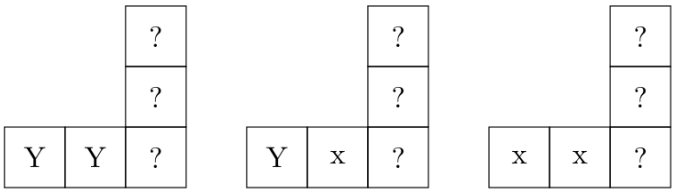
Let us show the three layers:



bottom layer middle layer top layer

We have used '?' to indicate that we do not yet know where there are cubes. We do, however, know that there must be a cube in at least one of the three layers for each of the five positions.

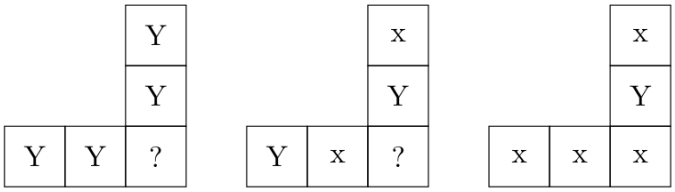
Next, let us look at the front elevation. This tells us that in the left hand column, we can see two cubes. Since there is only one possible location for the cubes in this column, the first two layers must have a cube and the third not. Likewise, for the middle column, there is only a cube in the bottom layer. In the right hand column, there must be at least one cube in each layer, but it is not yet clear where these would be, as there are three possible positions in each layer. We now know the following:



bottom layer middle layer top layer

Next, let us consider the side elevation. The left hand column, which corresponds to the bottom row of the plan view layers, has height 2, which means that the bottom right corner of the top layer cannot have a cube. The middle column, which corresponds to the middle row of the plan view layers, has height three, so all three of these cubes must be present (as there is only one possible position in the middle row). Finally, the right column, corresponding to the top row of the plan view layers, has height one, so only the bottom layer has a cube in this position.

We thus reach this position:



bottom layer middle layer top layer

One of the '?' cubes must be present, to give the required plan view, and the other one may or may not be – it makes no difference to any of the views. So there are either 8 or 9 cubes, and the answer is option F. (Note that we would need the corner of the bottom layer to be 'Y' if it is a single object.)

The person in the centre gives us a lot of information. If they are a truth-teller, then all four neighbours are liars, but if they are a liar, then at least one neighbour is a truth-teller. Since we are only interested in the number of truth-tellers and not their positions, we can assume the person directly 'above' them is a truth-teller. So we have these two possibilities (where T means truth-teller and L means liar):

	L	
L	T	L
	L	

	T	
	L	

In the first possibility, each corner position has two neighbours, both of whom are liars, so they must be occupied by truth-tellers. In the second possibility, the top two corners must be liars, giving these possibilities:

T	L	T
L	T	L
T	L	T

L	T	L
	L	

Case 1: 5 T, 4 L

In the right-hand possibility, the corner liars already have one neighbour who is a truth-teller, so the other neighbour could be either a truth-teller or a liar. This gives four possibilities for the left side and right side people: both liars, both truth-tellers or one of each in either order. By reflecting the grid, we only need to consider one case of one liar, one truth-teller, so we now have these three possible ways of continuing the grid:

L	T	L
L	L	L

L	T	L
T	L	T

L	T	L
L	L	T

Every truth-teller is surrounded by liars, and every liar must have at least one truth-teller as a neighbour, so we can fill in some of the bottom row in each of these three possibilities:

L	T	L
L	L	L
T		T

L	T	L
T	L	T
L		L

L	T	L
L	L	T
T		L

We can now fill in the final square in each of these, and check that the three people in the bottom row are consistent with the rules:

L	T	L
L	L	L
T	L	T

Case 2: 3 T, 6 L

L	T	L
T	L	T
L	T	L

Case 3: 4 T, 5 L

L	T	L
L	L	T
T	L	L

Case 4: 3 T, 6 L

We therefore have four cases. The smallest number of truth-tellers is in cases 2 and 4, with 3 truth-tellers, and the largest number is in case 1 with 5 truth-tellers. Therefore the correct option is E.

For ease of explanation, we say 'Mr X is true/false' as a shorthand for 'Mr X's statement is true/false'.

Of these statements, Ms Q and Mr T give the most explicit information, so we start with them. They cannot both be true, as they contradict each other, though they could both be false. So we look at these three possibilities in turn.

- Ms Q true, Mr T false.

  Since Ms Q is true, Ms S is also true, which means that either Mr P or Mr R is true (as Mr T is false).

  If Mr R is true, then Mr P is also true, contradicting Ms S. So Mr R must be false.

  Since Ms S is true, Mr P must be true; this does not pose a problem with Mr R being false, as it could be that Mr R's first name is not Robert. We must, though, check whether Mr P's statement being true is consistent with the remaining statements.

  We have Mr P, Ms Q and Ms S true and the other two false, giving three true statements. Thus Mr P's statement is, indeed, true, and this is a possible set of truth values for the statements.
- Ms Q false, Mr T true.

  Since Mr T is true, Ms S is false, and so either Mr P, Mr R or both must be true (as Mr T is already true).

  If Mr P is true, there are two true statements from Mr P, Ms Q, Ms S and Mr T, which is an even number, so Mr R must also be true. This is consistent, as Mr R says that Mr P's statement is true, which it is.

  If Mr R is true, then Mr P must also be true, and this is feasible as we have just seen.

  So in this case, we have Mr T, Mr P and Mr R all true, with the other two false, giving three true statements.
- Both Ms Q and Mr T false.

  Then exactly one statement made by a woman must be true, hence Ms S must be true.

  This requires exactly one of Mr P and Mr R to be true (as Mr T is false). If Mr R is true, then Mr P would also be true, which is not possible.

  If, on the other hand, Mr P is true and Mr R is false (say his first name is not Robert, as before), then we would have only Mr P and Ms S being true, and that is an even number of true statements. So this is not possible (as Mr P says there are an odd number of true statements).

Hence we must have one of the first two possibilities, and there are exactly three true statements; the answer is option D.

Commentary: *You might find this question easier to work out using a little table with ticks and crosses, for example something like:*

Mr P	*Ms Q*	*Mr R*	*Ms S*	*Mr T*
✓	×			✓

We can find the interior angle of a regular polygon with n sides as follows. The exterior angle is $\frac{360°}{n}$, so the interior angle is $180° - \frac{360°}{n}$. Therefore the given condition can be written as

$$180° - \frac{360°}{n} = \frac{3}{4}\left(180° - \frac{360°}{m}\right).$$

We start by dividing by $180°$ for simplicity, giving

$$1 - \frac{2}{n} = \frac{3}{4}\left(1 - \frac{2}{m}\right).$$

Now multiplying both sides by 4 and expanding the brackets gives

$$4 - \frac{8}{n} = 3 - \frac{6}{m},$$

which rearranges to

$$\frac{8}{n} - \frac{6}{m} = 1. \qquad (*)$$

We will look at two methods of solving this equation, starting with a case-based approach.

We know that n and m must each be at least 3. If $n \geqslant 8$, then $\frac{8}{n} \leqslant 1$, and so this cannot give a solution to $(*)$. So we need only consider $n = 3$, 4, 5, 6 and 7. For each of these, we can solve the equation to find m; if m is an integer greater than 2, we have a solution.

n	m
3	$\frac{18}{5}$
4	6
5	10
6	18
7	42

There are four pairs of n and m, and the answer is E.

A second approach to solving $(*)$ is to multiply both sides by mn to eliminate the fractions; this gives

$$8m - 6n = mn.$$

We can rearrange this to give $mn + 6n - 8m = 0$, and we can factorise the left hand side by adding or subtracting a constant to get

$$(m + 6)(n - 8) = -48.$$

So we need to find two integers which multiply to -48. Since m and n are both positive, we must have $n - 8 < 0$ and $m + 6 \geqslant 9$ (as $m \geqslant 3$), so the possibilities are:

$m+6$	$n-8$	m	n
12	-4	6	4
16	-3	10	5
24	-2	18	6
48	-1	42	7

We can write this out in a grid to see what the current state of knowledge is.

	Position				
	1	2	3	4	5
Tried	a	b	c	d	e
	c	d	b	e	a
	e	a	d	b	c
Possible	b	c	a	a	b
	d	e	e	c	d

So positions 1 and 5 contain `b` and `d` in some order, while positions 2, 3, 4 contain either `cea` or `eac` in that order.

We certainly do not know the correct password yet.

We could sensibly try `bcead` as our next attempt. What might we be told in response?

If we have the `b` and `d` in the correct order, these 2 letters will be correct, otherwise neither is correct.
If we have the middle three letters `cea` in the correct order, then these 3 letters will be correct, otherwise none of them is.
So:

- if both are correct, the computer will tell us that 5 letters are correct, and we are done;
- if the outer pair are correct but the middle ones are not, the computer will tell us that 2 letters are correct, and we can deduce that the correct password is `beacd`;
- if the outer pair are wrong but the middle ones are correct, the computer will tell us that 3 letters are correct, and we can deduce that the correct password is `dceab`;
- if both are wrong, the computer will tell us that 0 letters are correct, and we can deduce that the correct password is `deacb`.

Since the four cases give four distinct responses by the computer (0, 2, 3, 5 correct letters), we can distinguish between them, and therefore deduce the correct password with just this one further attempt.

The correct answer is B.

It is helpful to write all six statements as 'if ... then' statements:

A **if** $x \geq 0$ **then** $f(x) < 0$

B **if** $f(x) \geq 0$ **then** $x < 0$

C **if** $x \geq 0$ **then** $f(x) \geq 0$

D **if** $x < 0$ **then** $f(x) < 0$

E **if** $f(x) \geq 0$ **then** $x \geq 0$

F **if** $x < 0$ **then** $f(x) \geq 0$ **and if** $f(x) \geq 0$ **then** $x < 0$

If F is true, then B is also true, so F is false.

Let us suppose that A is true. Then its contrapositive is also true. The contrapositive of A is: **if** $f(x) \geq 0$ **then** $x < 0$, which is statement B. So A and B are either both true or both false. Since exactly one statement is true, A and B must both be false.

Similarly, D and E are contrapositives of each other, so they must both be false as well.

The only statement left is C, so that must be true.

Furthermore, C contradicts A, so A (and B) must be false; C is independent of D (knowing that C is true does not give any information about whether D is true or false), and C contradicts F, so F must be false.

Therefore having C true and the other five statements false is consistent; the function $f(x) = |x|$ is an example of a function for which C is true and the other five statements are false.

Statistical Calculations

12.1 Answer C

TMUA 2022 · Paper 2 · Q6

Suppose P is true. Since there are an odd number of integers in the list, there is a unique middle one (when they are written in increasing order), so this is the median. Therefore Q is true, and P is sufficient for Q.

Now suppose that Q is true. It does not follow that n is odd; here is a counterexample: if the list is 2, 2, then $n = 2$ and the median is 2, so Q is true, but P is false. (In general, any list with n even and the middle pair of numbers equal is a counterexample.) Therefore P is not necessary for Q.

The correct answer is option C.

12.2 Answer A

TMUA 2018 · Paper 1 · Q17

It is not obvious how to begin this question – there are lots of wordy descriptions. We also don't know how many pieces of data there are in each of the two sets.

So let's try naming things.

Let the first set of data have n pieces of data, and the second set have m pieces of data.

Since the mean of the first set of data is 15, the sum of the data is $15n$.

Likewise, the sum of the second set of data is $20m$.

When we exchange one piece of data from the first set with one piece of data from the second set, the overall total stays the same, but the totals of the two sets change. We are told what happens to the means, so the total of the first set is now $16n$ and of the second set is $17m$.

Since the overall total is the same, we have

$$15n + 20m = 16n + 17m$$

We can simplify this to give $n = 3m$

Therefore the overall total is $15n + 20m = 45m + 20m = 65m$ and there are $n + m = 4m$ pieces of data.

Thus the mean of the combined set of data is $\dfrac{65m}{4m} = \dfrac{65}{4} = 16\frac{1}{4}$ and the answer is A.

> ***Commentary:*** *At the end of this, we are still no wiser about how many pieces of data there were to begin with. All we know is that the first set contains three times as many as the second set.*

Let us write the six integers as two sets of three, each set in increasing order:

$$a,\ 8,\ b \qquad c,\ 9,\ d$$

We know the middle integers are 8 and 9 because these are the medians.

The first set of three has a mean of 10 and so they must sum to $3 \times 10 = 30$, therefore $a + b = 22$. Likewise, the second set sums to 36 and so $c + d = 27$.

We also know that $a \leq 8 \leq b$ and $c \leq 9 \leq d$.

Here are tables of some of the possibilities for a and b and for c and d; the two tables are independent of each other:

a	b
8	14
7	15
6	16
5	17
4	18
3	19

c	d
9	18
8	19
7	20
6	21
5	22
4	23

The range of the set of all six integers is the larger of b and d minus the smaller of a and c.

As all six integers are distinct, we cannot have $a = 8$ or $c = 9$ or $c = 8$. We also want b and d to be as small as possible while a and c are as large as possible to get a small range.

Let us try $a = 7$, $b = 15$. Then we cannot have $c = 7$, so we try $c = 6$, $d = 21$. This gives a range of $21 - 6 = 15$.

If we now try $a = 6$, $b = 16$, we can take $c = 7$, $d = 20$, and this gives a range of $20 - 6 = 14$, which is smaller.

If we next try $a = 5$, $b = 17$, we can again take $c = 7$, $d = 20$, but now the range has grown to $20 - 5 = 15$ again. And if we go smaller with the choice of a, say $a = 4$, $b = 18$, then the smallest range we can obtain becomes even larger: $20 - 4 = 16$. The range will keep growing as we take smaller values for a.

So the smallest possible range is 14, which is option E.

We can approach this algebraically or just by reasoning about numbers.

We begin with an algebraic approach. Let the five numbers be, in increasing order, a, b, c, d, e, so that the median is c. The range is 20, so $e - a = 20$ or $a = e - 20$. Also, the mean is zero, so $a + b + c + d + e = 0$.

Since $c \leqslant d \leqslant e$, $a = e - 20$ and $a \leqslant b \leqslant c$, we see that the sum of the five numbers is at least

$$(e - 20) + (e - 20) + c + c + c = 3c + 2e - 40$$

Since the sum is zero, this means that $0 \geqslant 3c + 2e - 40$, or that $3c \leqslant 40 - 2e$, so $3c + 2e \leqslant 40$. But $c \leqslant e$, so $5c \leqslant 3c + 2e \leqslant 40$, so $c \leqslant 8$. We can have $c = d = e = 8$ and then $a = b = -12$, so the largest possible median is 8, option E.

Alternatively, we can just reason as follows. Let's say that we have five numbers

$$a \quad b \quad c \quad d \quad e$$

arranged in increasing order, with mean 0 and range 20, and let's suppose that the median is as large as possible.

We ask if there is any way of increasing c while keeping the mean equal to 0 and the median equal to 20.

If $c < d$, then we can increase the median without changing the mean or range by replacing each of c and d by their mean. So we must have $c = d$.

Similarly, if $d < e$, we could increase both c and d by a small amount and decrease e and a by the same small amount. As long as d and e don't swap order in this process, the mean and range will not change, but the median will increase. (So we must take this small amount to be at most $\frac{1}{2}(e - d)$.) Therefore we must have $c = d = e$.

Now consider a and b. If $a < b$, we can do a similar trick: we increase a, c, d and e by a small amount k and decrease b by $4k$; this preserves the mean and range while increasing the median. (So that a and b don't swap places, we require k to be at most $\frac{1}{4}(b - a)$.) It follows that $a = b$.

We therefore have $a = b$ and $c = d = e$. Since the range is 20, the numbers must be a, a, $a + 20$, $a + 20$, $a + 20$. Their sum is $5a + 60 = 0$, so $a = -12$ and $a + 20 = 8$, which is the largest possible median.

Numerical Methods

13.1 Answer E

TMUA 2019 · Paper 1 · Q8

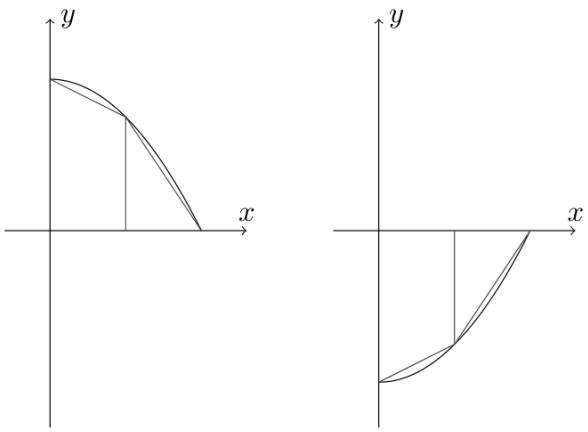
A This translates the graph upwards (in the positive y-direction) by 1 unit, so the trapezia will just be 1 unit taller. The approximation will therefore still be an underestimate.

B This stretches the graph by a factor of 2 in the y-direction, and the trapezia with it, so the approximation will still be an underestimate.

C This translates the graph by 1 unit leftwards (in the negative x-direction), and correspondingly calculates the area between -1 and 0 rather than between 0 and 1; the trapezia are correspondingly translated leftwards by 1 unit, and so we will still get an underestimate.

D This reflects the graph in the y-axis, and correspondingly calculates the area between -1 and 0. The trapezia are correspondingly reflected, so we will still get an underestimate.

E The graph is reflected in the x-axis and then translated upwards by 1 unit. Whether the approximation using the trapezium rule is an over- or underestimate is not affected by the vertical translation, as we noted in **A**, so we may as well not translate the graph at all, giving just $\int_0^1 (-\mathrm{f}(x))\,\mathrm{d}x$. Let us sketch an example using just two intervals:



The original (blue) trapezia give an underestimate for the integral. The red trapezia give an underestimate for the *area* of the region, but since the integral is negative (the graph lies below the x-axis), the trapezium rule gives an overestimate for the *integral* (for example, it may estimate -0.6 instead of the correct -0.65, and $-0.6 > -0.65$).

Hence the correct answer is E.

The interval has width $2 - \frac{1}{2} = \frac{3}{2}$, so each strip will have width $\frac{1}{2}$. The boundaries of the three strips will be at $\frac{1}{2}$, 1, $\frac{3}{2}$ and 2. The trapezium rule then gives the estimate (writing $\mathrm{f}(x) = 2\log_{10} x$):

$$\int_{\frac{1}{2}}^{2} \mathrm{f}(x)\,\mathrm{d}x \approx \tfrac{1}{2} \times \tfrac{1}{2}\big(\mathrm{f}(\tfrac{1}{2}) + 2\mathrm{f}(1) + 2\mathrm{f}(\tfrac{3}{2}) + \mathrm{f}(2)\big)$$

We can now work out the individual function values:

$$\begin{aligned}
\mathrm{f}(\tfrac{1}{2}) &= 2\log_{10}\tfrac{1}{2} = -2\log_{10} 2 \\
\mathrm{f}(1) &= 2\log_{10} 1 = 0 \\
\mathrm{f}(\tfrac{3}{2}) &= 2\log_{10}\tfrac{3}{2} = 2\log_{10} 3 - 2\log_{10} 2 \\
\mathrm{f}(2) &= 2\log_{10} 2
\end{aligned}$$

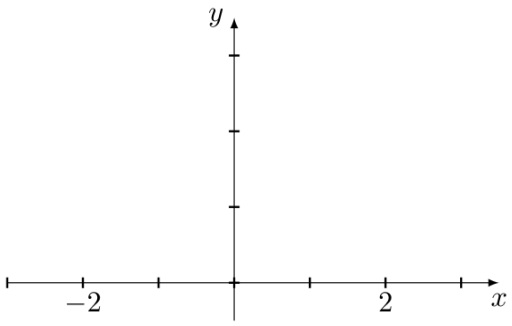
Therefore we get the approximation

$$\begin{aligned}
\int_{\frac{1}{2}}^{2} \mathrm{f}(x)\,\mathrm{d}x &\approx \tfrac{1}{2} \times \tfrac{1}{2}\big(-2\log_{10} 2 + 2 \times 0 + 2(2\log_{10} 3 - 2\log_{10} 2) + 2\log_{10} 2\big) \\
&= \tfrac{1}{4}(4\log_{10} 3 - 4\log_{10} 2) \\
&= \log_{10} 3 - \log_{10} 2 \\
&= \log_{10}\tfrac{3}{2}
\end{aligned}$$

which is option B.

Though integrating functions like this is not on the syllabus, the question assumes that we can calculate the value of this integral. There is also a hint from the options offered that it may have something to do with circles, as they all involve π.

Let us therefore consider the graph $y = \sqrt{4 - x^2}$. Squaring this gives $y^2 = 4 - x^2$ or $x^2 + y^2 = 4$. This is the equation of a circle with centre at the origin and radius 2. However, as the graph involves taking the square root of this, the resulting graph is just the upper semicircle (on and above the x-axis).



The integral is from $x = -2$ to $x = 2$, which is the whole diameter, so the integral is the area of the semicircle, that is, $\frac{1}{2} \times \pi \times 2^2 = 2\pi$.

We now need to calculate the area as determined by the trapezium rule with 4 strips. We can draw a table of values:

x	-2	-1	0	1	2
$\sqrt{4 - x^2}$	0	$\sqrt{3}$	2	$\sqrt{3}$	0

The trapezium rule then approximates the area as

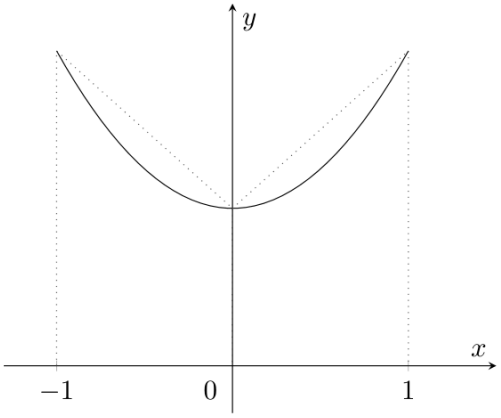
$$\tfrac{1}{2} \times 1 \times \left(0 + 2 \times \sqrt{3} + 2 \times 2 + 2 \times \sqrt{3} + 0\right) = \tfrac{1}{2} \times \left(4 + 4\sqrt{3}\right) = 2 + 2\sqrt{3}.$$

The positive difference (as the trapezium rule underestimates the area) is therefore

$$2\pi - \left(2 + 2\sqrt{3}\right) = 2\left(\pi - 1 - \sqrt{3}\right)$$

which is option B.

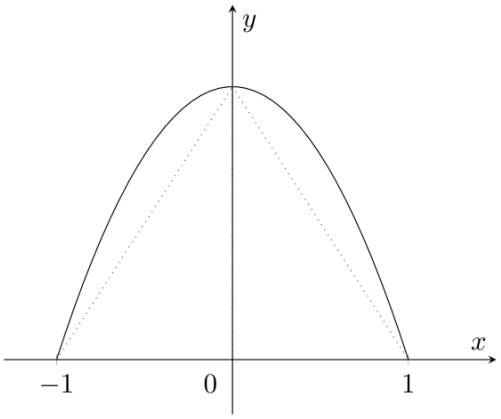
The trapezium rule can produce an overestimate even if the function's gradient goes from negative to positive, for example with the function $f(x) = x^2 + 1$ between $x = -1$ and $x = 1$:



Therefore neither the condition $f'(x) > 0$ for all x with $a < x < b$ nor the condition $f'(x) < 0$ for all x with $a < x < b$ is necessary for the trapezium rule to produce an overestimate. This rejects the 'only if' options (B, C, E, F), leaving us with only A and D.

The above quadratic can also be used to help distinguish between these two options. We have $f'(x) = 2x$ and $f''(x) = 2$. Thus for $-1 < x < 0$, $f'(x) < 0$ and $f''(x) > 0$ and the trapezium rule (with one strip) gives an overestimate. So D appears plausible.

On the other hand, we could reflect the quadratic to change the sign of $f''(x)$: consider the function $f(x) = 1 - x^2$.

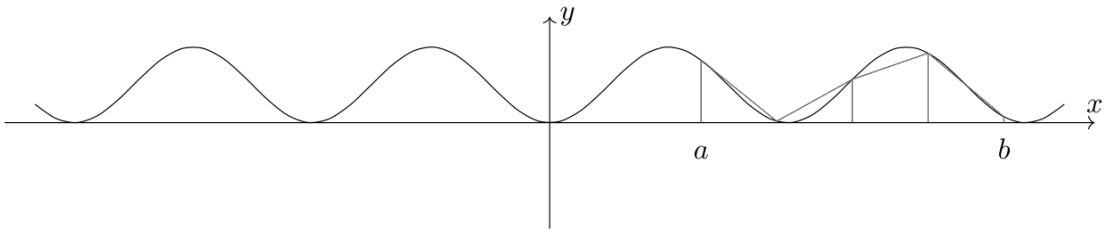


We now have $f'(x) = -2x$ and $f''(x) = -2$. So when $-1 < x < 0$, we have $f'(x) > 0$ and $f''(x) < 0$, yet the trapezium rule (with one strip) produces an underestimate. This shows that A cannot be correct.

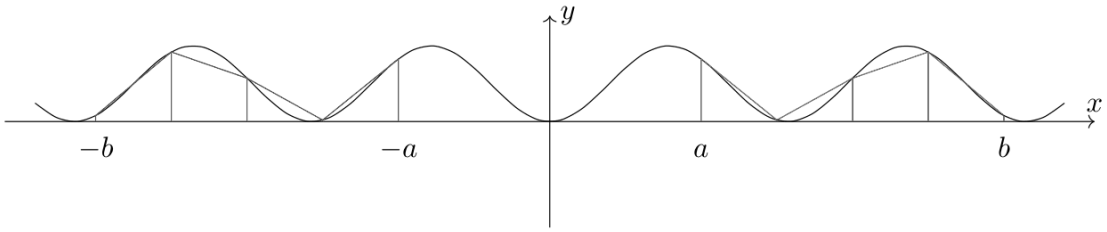
By elimination, the correct option must be D.

It is also possible to justify this: a function with $f''(x) > 0$ is 'bending upwards' (the technical term is *convex*). Any time we draw a chord between two points on the graph, the chord will lie above the graph (except at the endpoints), so the trapezium rule estimate will be an overestimate of the integral. In fact, this argument makes no assumptions about the sign of either $f(x)$ or of $f'(x)$; having $f''(x) > 0$ for $a < x < b$ is sufficient for the trapezium rule to give an overestimate (but it is not necessary).

It will help us to draw a sketch of this situation, remembering that we do not know that values of a and b. The exact shape of the graph does not matter; the only thing that is important for this question is that it is symmetrical about the y-axis, since $\sin^2(-x) = (-\sin x)^2 = \sin^2 x$.

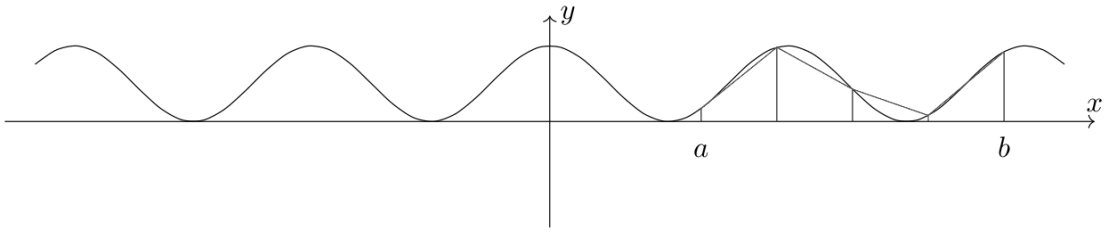


I We can now add to our sketch the trapezium rule calculation for the integral $\int_{-b}^{-a} \sin^2 x \, dx$:

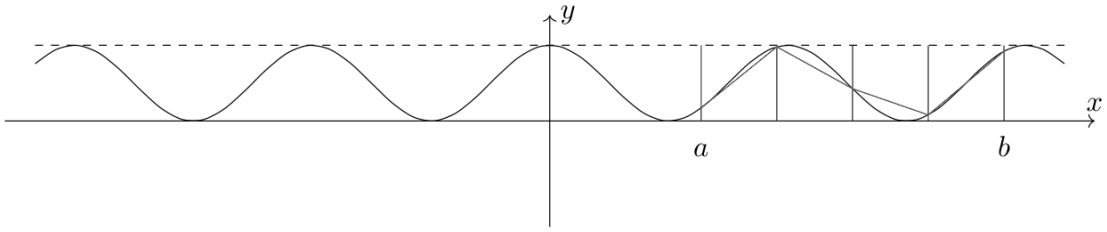


We see that the strips are just a reflection of the original 4 strips. Since the original approximation was an overestimate, this approximation is also an overestimate.

II We note that $\cos^2 x$ is related to $\sin^2 x$ by the identity $\sin^2 x + \cos^2 x = 1$, so $\cos^2 x = 1 - \sin^2 x$. Thus the graph of $y = \cos^2 x$ is a reflection of the graph of $y = \sin^2 x$ in the line $y = \frac{1}{2}$ (why?). Therefore the sketch of the graph and trapezia for $\int_a^b \cos^2 x \, dx$ is:



Since $\cos^2 x = 1 - \sin^2 x$, we can also draw on the original trapezia, coming down from the line $y = 1$:

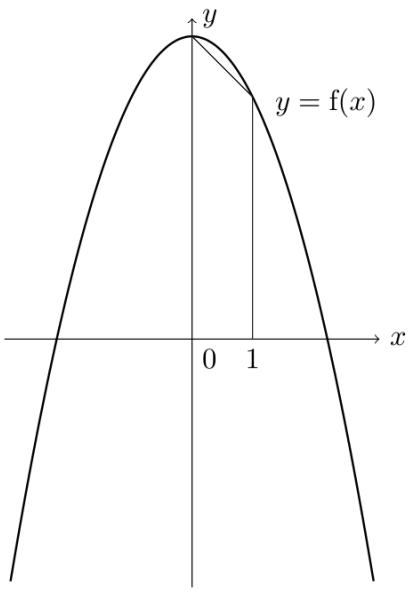


We can see from this sketch that the blue and red trapezia exactly sum to $\int_a^b \sin^2 x \, dx + \int_a^b \cos^2 x \, dx$; since the blue trapezia give an overestimate, the red ones must give an underestimate.

Hence this statement is also necessarily true.

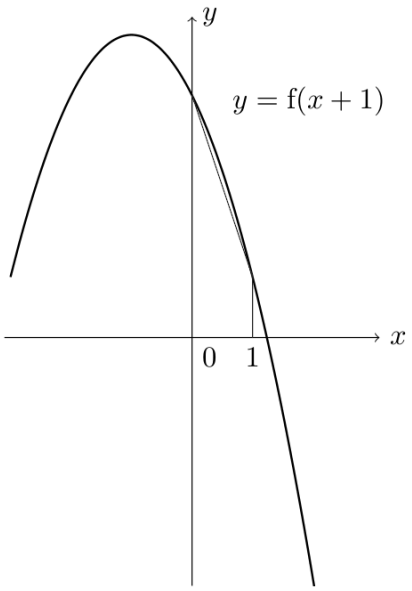
Therefore the correct answer is D.

We can either proceed by sketching the graph or by using a general theorem about the trapezium rule. We start with sketching. The simplest case of the trapezium rule is where we only have one trapezium, so that is what we will draw.



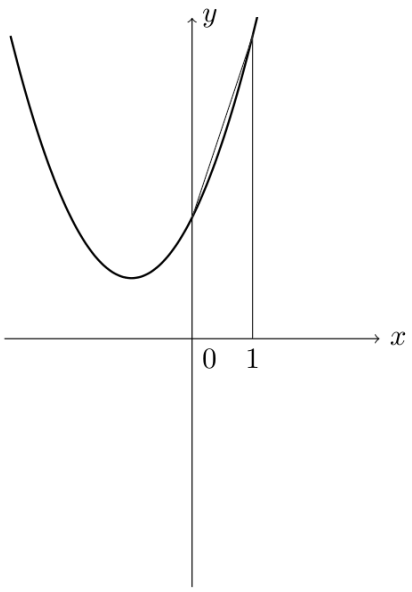
We see here that the trapezium rule underestimates the area of $y = \mathrm{f}(x)$

For the case $y = \mathrm{f}(x + 1)$ the graph of $y = \mathrm{f}(x)$ is translated one unit to the left:



Again, the trapezium rule gives an underestimate.

For the third case, we need to reflect this graph in the line $y = 6$, which gives the following graph:



We see that this time, the trapezium rule gives an overestimate.

So the correct answer is B.

An alternative approach is given by noting that the trapezium rule gives an overestimate of the area under the graph of $y = \mathrm{f}(x)$ if $\mathrm{f}''(x) > 0$ throughout the range and an underestimate if $\mathrm{f}''(x) < 0$ throughout. In our case, $\mathrm{f}''(x) = -4$, so the trapezium rule gives an underestimate for (1). For case (2), $\mathrm{f}''(x + 1) = -4$ again, so we still have an underestimate. For the third case, the equation of the graph is $y = 12 - \mathrm{f}(x + 1)$, so the second derivative is $-\mathrm{f}''(x + 1) = 4$, and therefore we have an overestimate.