



KING EDWARD VI  
HANDSWORTH GRAMMAR  
SCHOOL FOR BOYS



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# **Year 11**

## **2026      Mathematics      2027**

### **Unit 24 Tasks – Part 1**

**DO NOT WRITE INSIDE**



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# **Year 11**

## **2026 Mathematics 2027**

### **Unit 24 Tasks – Part 2**

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# **Year 11**

## **2026      Mathematics      2027**

### **Unit 24 Tasks – Part 3**

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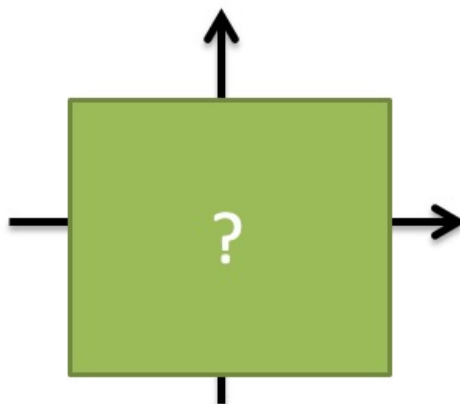
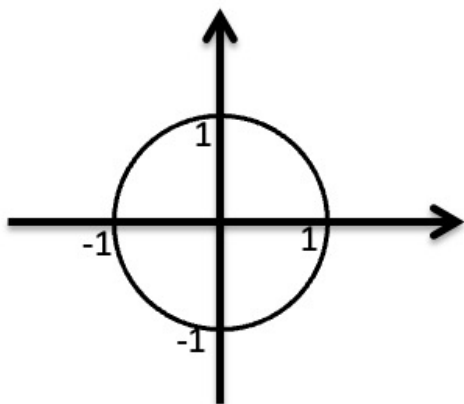
## Contents

- 1 [Equations of Circles and Tangents](#)
- 2 [Advanced Sequences](#)
- 3 [Iterations](#)
- 4 [Algebraic Proof](#)
- 5 [Advanced Vectors](#)

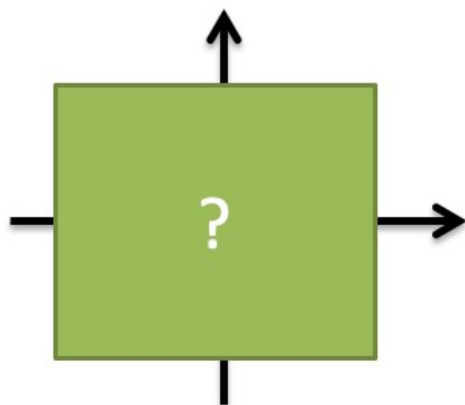
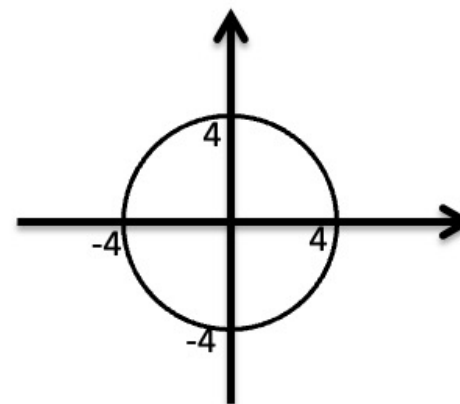


## 1 Equations of Circles and Tangents

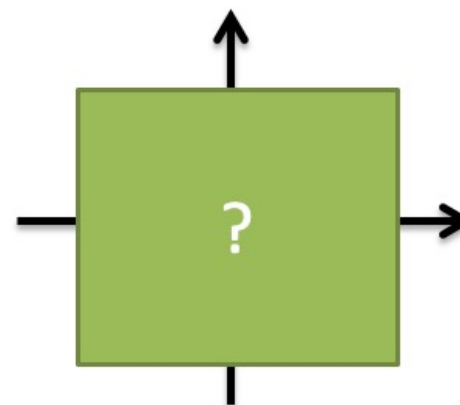
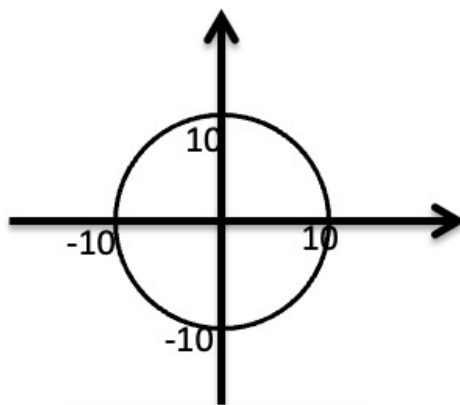
## Fluency Practice



$$x^2 + y^2 = 9$$



$$x^2 + y^2 = 64$$



$$x^2 + y^2 = 36$$

## Fluency Practice

|                                                                                               |                                                                                                  |
|-----------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------|
| Find the radius of the circle:<br>$x^2 + y^2 = 36$                                            | Find the radius of the circle:<br>$x^2 + y^2 = 45$                                               |
| Find the diameter of the circle:<br>$x^2 + y^2 = 100$                                         | Find the area of the circle:<br>$x^2 + y^2 = 144$                                                |
| Find the circumference of this circle:<br>$x^2 + y^2 = 9$                                     | Find the area of the circle:<br>$x^2 + y^2 = 48$                                                 |
| Write down the equation of the circle with centre (0, 0) and diameter 800.                    | Write down the equation of the circle with centre (0, 0) and area 201.                           |
| Write the equation of a circle with centre (0, 0) and radius $\sqrt{11}$ .                    | Write the equation of a circle with centre (0, 0) and radius $2\sqrt{3}$ .                       |
| The point (3, $a$ ) is on the circle<br>$x^2 + y^2 = 25$<br>Find the possible values of $a$ . | The point ( $b$ , 12) is on the circle<br>$x^2 + y^2 = 169$<br>Find the possible values of $b$ . |



①

Plotting Different Graphs

On the grid, plot the graph

$$x^2 + y^2 = 100$$

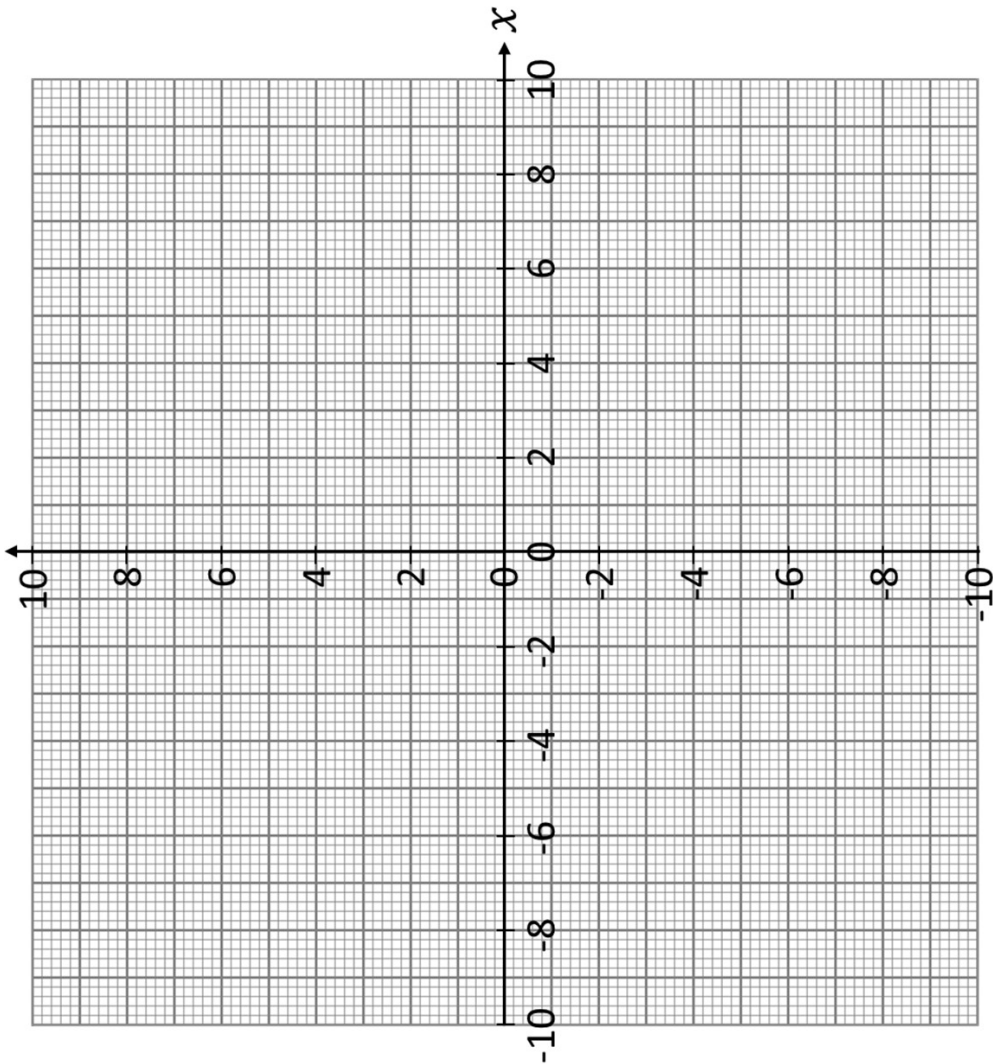
If  $x = 0$ ,  
what values will  $y$  be?

Form equations to help  
find coordinates.

| $x$ | $y$ |
|-----|-----|
| 10  |     |
| 8   |     |
| 6   |     |
| 4   |     |
| 2   |     |

| $x$ | $y$ |
|-----|-----|
| -10 |     |
| -8  |     |
| -6  |     |
| -4  |     |
| -2  |     |
| 0   |     |

$y$





②

Plotting Different Graphs

On the grid, plot the graph

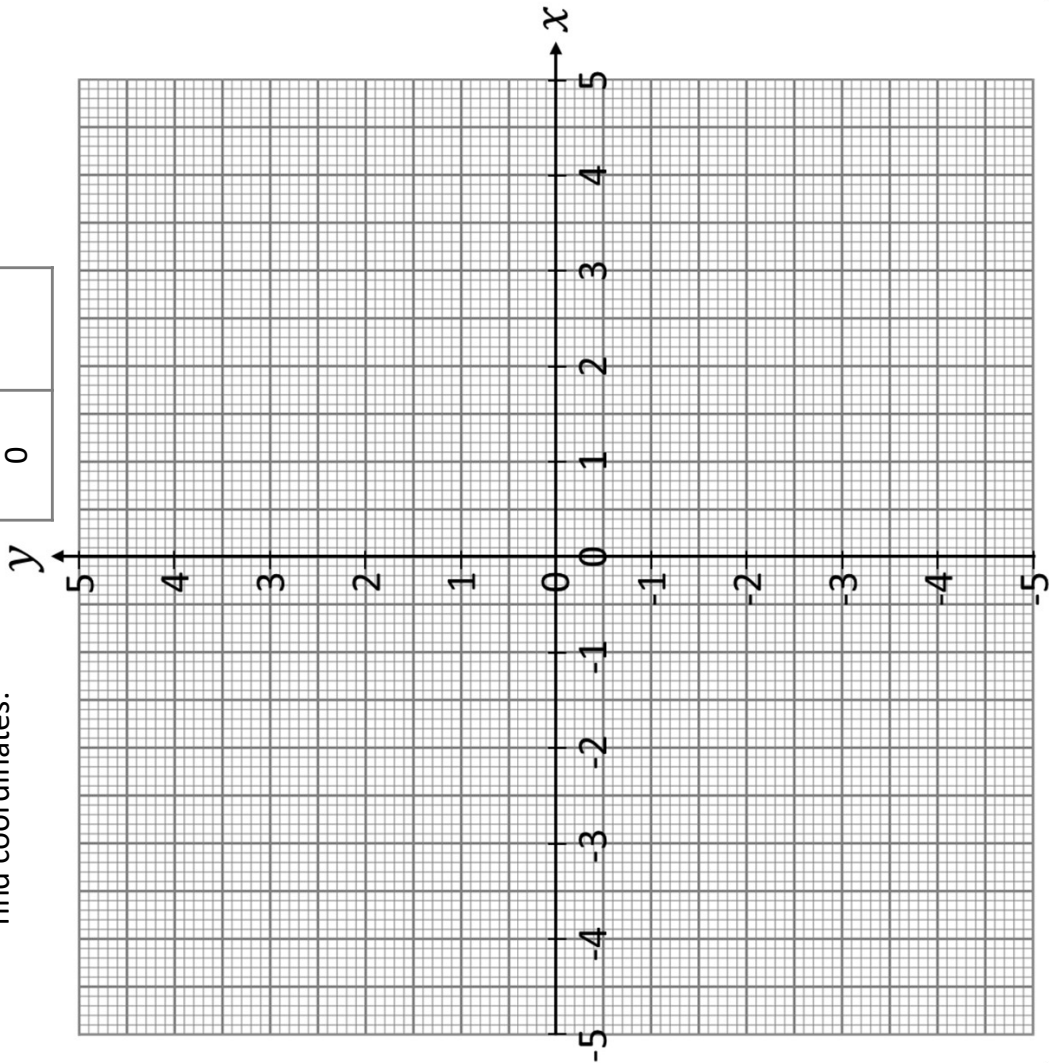
$$x^2 + y^2 = 16$$

If  $x = 0$ ,  
what values will  $y$  be?

Form equations to help  
find coordinates.

| $x$ | $y$ |
|-----|-----|
| 4   |     |
| 3   |     |
| 2   |     |
| 1   |     |

| $x$ | $y$ |
|-----|-----|
| -4  |     |
| -3  |     |
| -2  |     |
| -1  |     |
| 0   |     |





Plotting Different Graphs

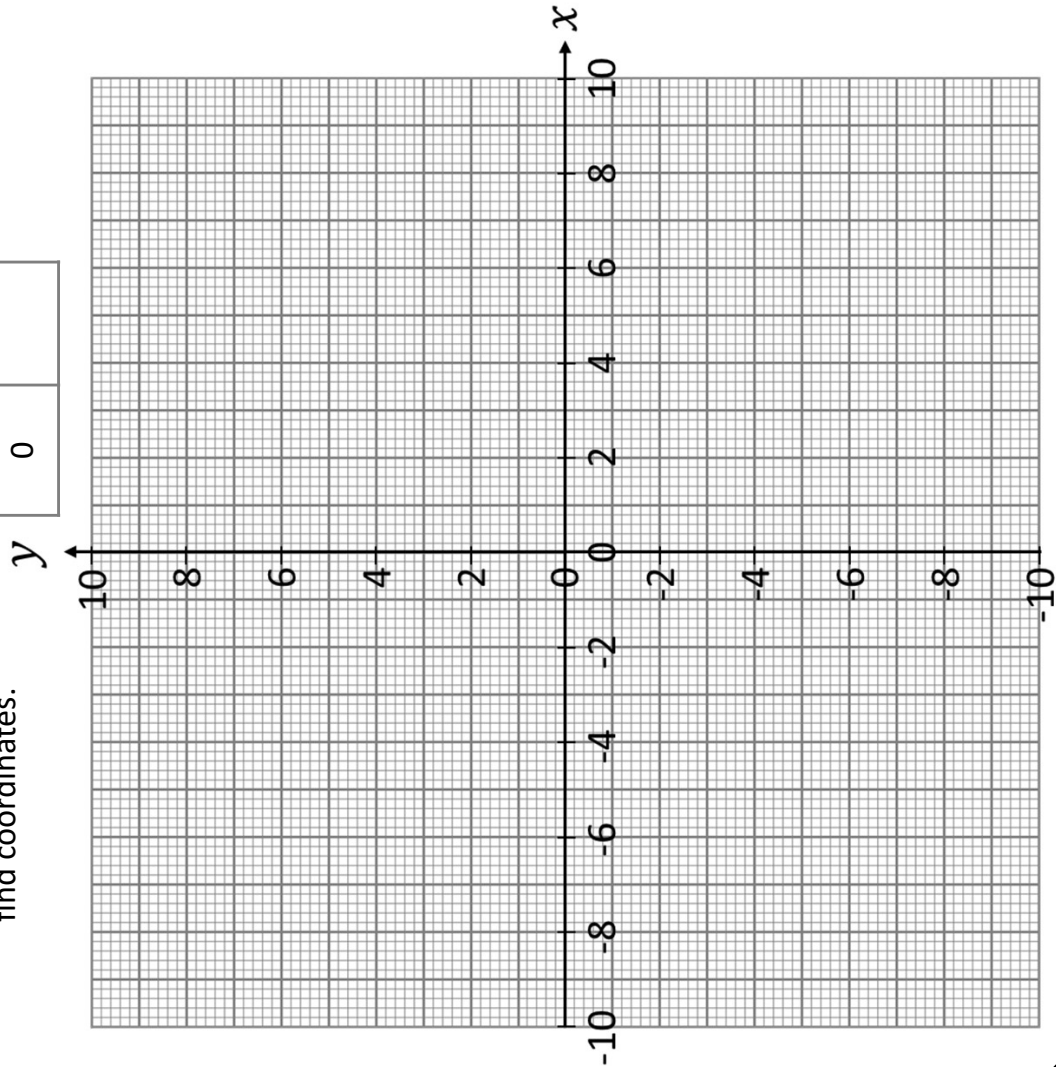
On the grid, plot the graph

$$x^2 + y^2 = 64$$

If  $x = 0$ ,  
what values will  $y$  be?

Form equations to help  
find coordinates.

| $x$ | $y$ | $x$ | $y$ |
|-----|-----|-----|-----|
| -8  |     | 8   |     |
| -6  |     | 6   |     |
| -4  |     | 4   |     |
| -2  |     | 2   |     |
| 0   |     |     |     |



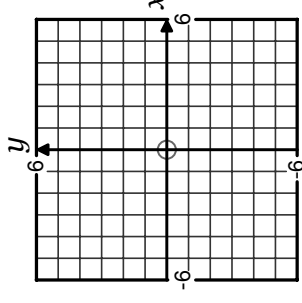
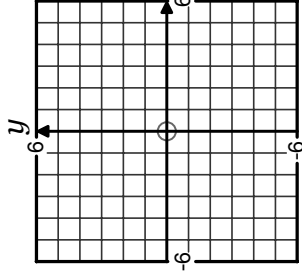
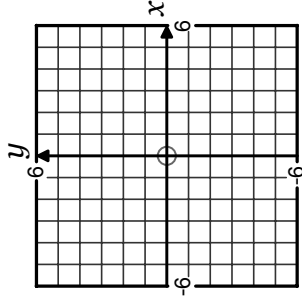
## Purposeful Practice

1. Draw the graph of each function:

a)  $x^2 + y^2 = 4$

b)  $x^2 + y^2 = 25$

c)  $x^2 + y^2 = 4^2$



2. A circle has equation  $x^2 + y^2 = 64$ . Work out the diameter of this circle.

3. Write down the equation of a circle centred on the origin with:

a) a radius of 7

b) a diameter of 20

c) an area of  $36\pi$

4. Which of the following points lie on the circle with equation  $x^2 + y^2 = 100$ ?  
Select all that apply.

a) (10, 1)

b) (-8, 6)

c) (7, -7)

d) (5,  $5\sqrt{3}$ )

5. Which of the following points lie on the circle with equation  $x^2 + y^2 = 80$ ?  
Select all that apply.

a) (0,  $4\sqrt{5}$ )

b) ( $3\sqrt{10}$ ,  $-\sqrt{10}$ )

c) (-7,  $-\sqrt{31}$ )

6. Ravi draws the graphs with equations  $x^2 + y^2 = 169$  and  $y = -5$ .  
Work out the coordinates of the points where Ravi's two graphs intersect.

7. Circle **C** has equation  $x^2 + y^2 = 81$ .

Write down the equations of the two horizontal lines that are tangent to **C**.

8. Which of the following are equations of circles? Select all that apply.

a)  $x^2 + y = 121$

b)  $(x + y)^2 = 9$

c)  $x^2 + y^2 - 24 = 0$

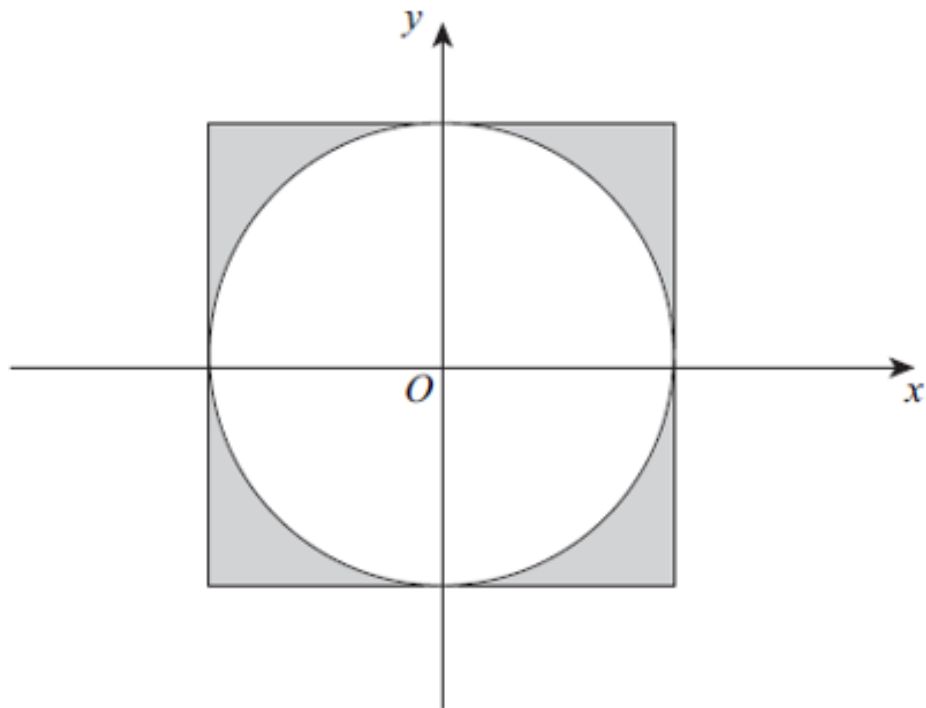
d)  $x = 16 - y^2$

e)  $20 = y^2 + x^2$

f)  $x^2 + y^2 = \sqrt{5}$

## Fluency Practice

- 1) Write down the centre and radius of the circle with equation  $x^2 + y^2 = 36$
- 2) A circle has the equation  $x^2 + y^2 = 36$ . Work out its circumference.
- 3) The circle  $x^2 + y^2 = 25$  touches each side of the square as shown. Work out the total shaded area.

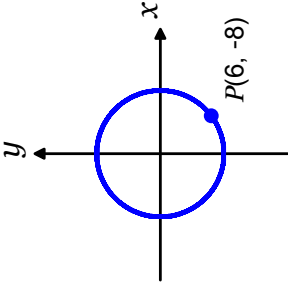
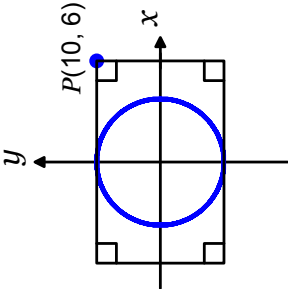
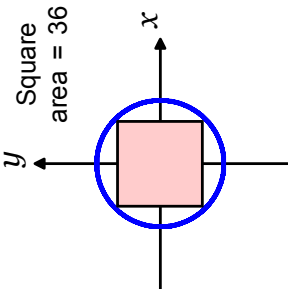
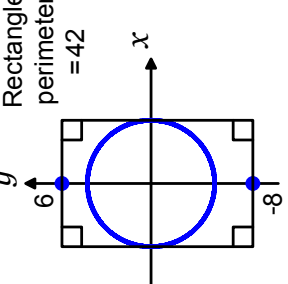
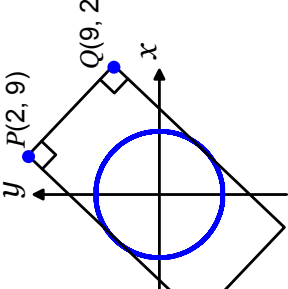
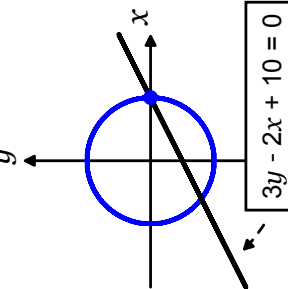
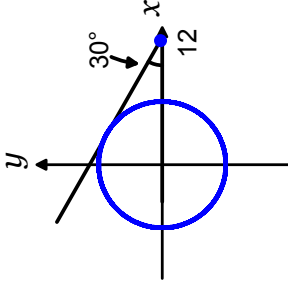
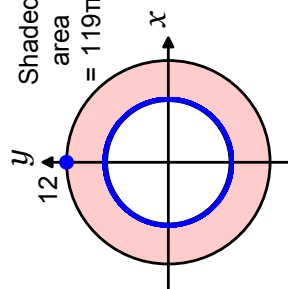
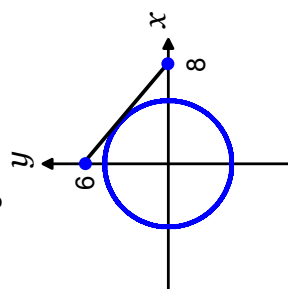




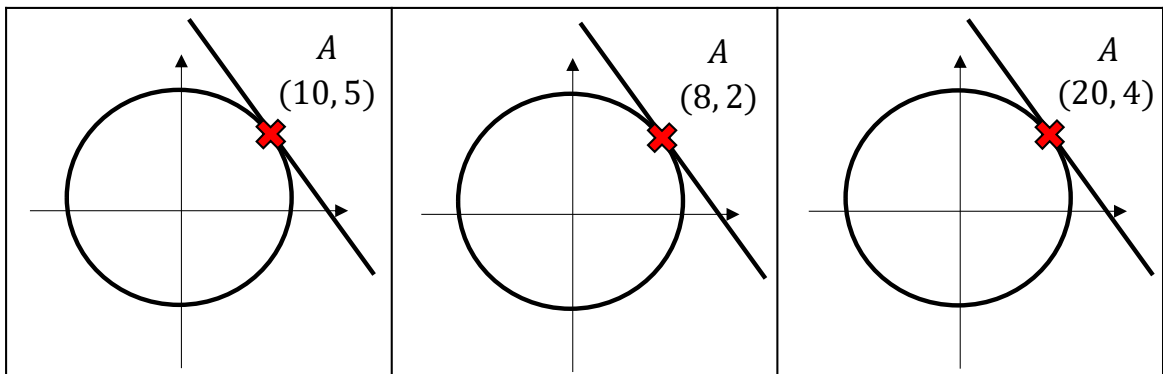
# Purposeful Practice

Circle graphs: find the radius

Work out the radius of each circle graph centred on the origin. Lines that look like tangents are tangents.

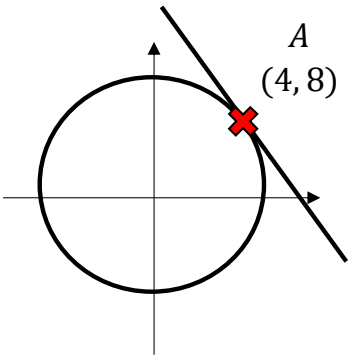
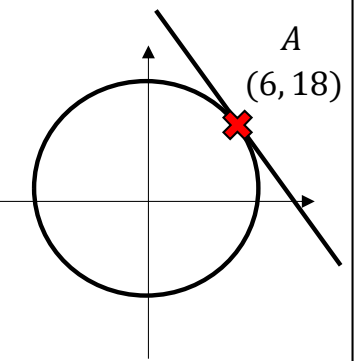
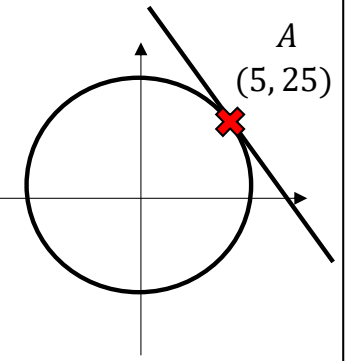
|                                                                                                                                             |                                                                                                                                                                   |                                                                                                                                         |
|---------------------------------------------------------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------|-----------------------------------------------------------------------------------------------------------------------------------------|
| <p><b>A</b></p>                                           | <p><b>B</b></p>                                                                  | <p><b>C</b></p>  <p>Square<br/>area = 36</p>           |
| <p><b>D</b></p>  <p>Rectangle<br/>perimeter<br/>= 42</p> | <p><b>E</b></p>                                                                 | <p><b>F</b></p>  <p><math>3y - 2x + 10 = 0</math></p> |
| <p><b>G</b></p>                                         | <p><b>H</b> Radius of inner circle?<br/>Shaded area = <math>119\pi</math></p>  | <p><b>I</b> Challenge</p>                            |

## Fluency Practice

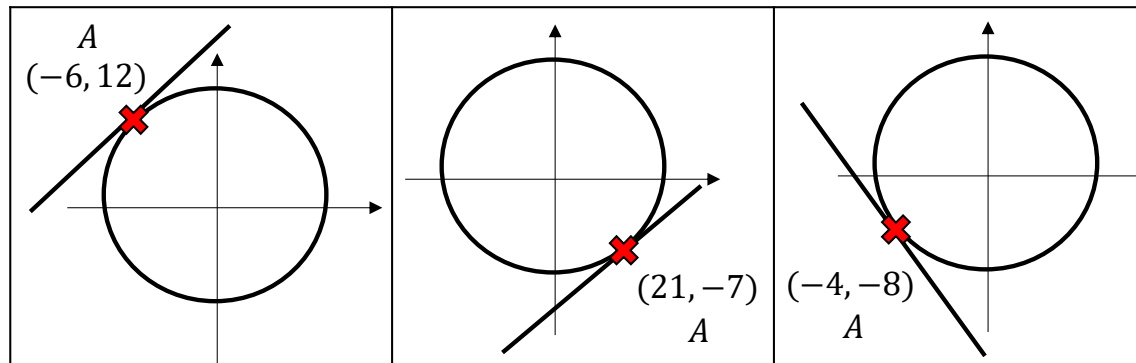


|                                                           |  |  |  |
|-----------------------------------------------------------|--|--|--|
|                                                           |  |  |  |
| What is the gradient between $A$ and the origin?          |  |  |  |
| What is the gradient of the tangent?                      |  |  |  |
| What is the equation of the tangent (so far)?             |  |  |  |
| Substitute the co-ordinate to find the value of $c$       |  |  |  |
| <b>Final Answer:</b><br>The equation of the tangent is... |  |  |  |

## Fluency Practice

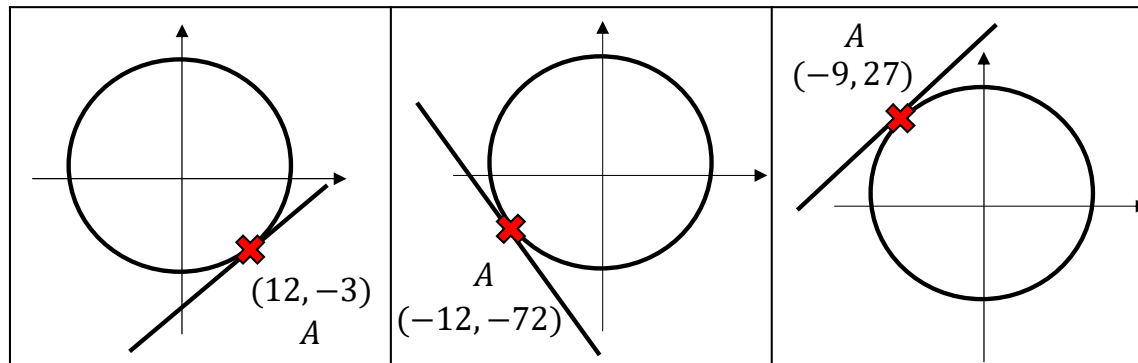
|                                                           |                                                                                    |                                                                                     |                                                                                     |
|-----------------------------------------------------------|------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------|
|                                                           |  |  |  |
| What is the gradient between $A$ and the origin?          |                                                                                    |                                                                                     |                                                                                     |
| What is the gradient of the tangent?                      |                                                                                    |                                                                                     |                                                                                     |
| What is the equation of the tangent (so far)?             |                                                                                    |                                                                                     |                                                                                     |
| Substitute the co-ordinate to find the value of $c$       |                                                                                    |                                                                                     |                                                                                     |
| <b>Final Answer:</b><br>The equation of the tangent is... |                                                                                    |                                                                                     |                                                                                     |

## Fluency Practice



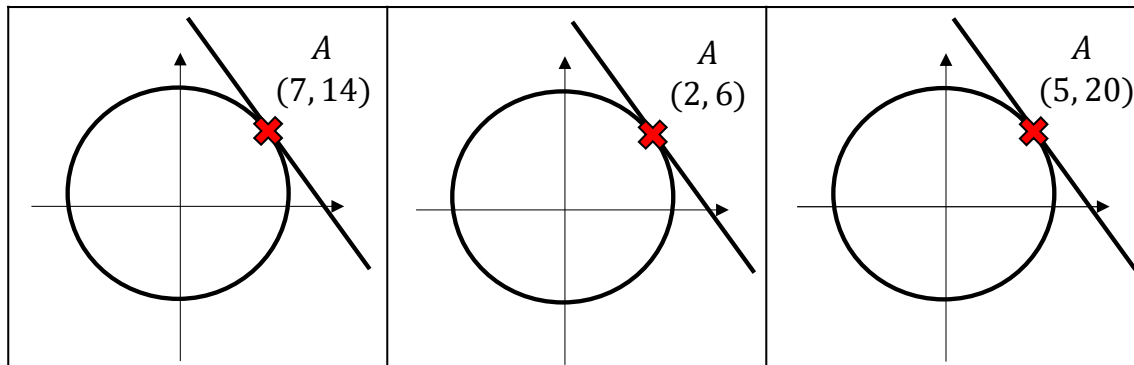
|                                                           |  |  |  |
|-----------------------------------------------------------|--|--|--|
| What is the gradient between A and the origin?            |  |  |  |
| What is the gradient of the tangent?                      |  |  |  |
| What is the equation of the tangent (so far)?             |  |  |  |
| Substitute the co-ordinate to find the value of $c$       |  |  |  |
| <b>Final Answer:</b><br>The equation of the tangent is... |  |  |  |

## Fluency Practice



|                                                           |  |  |  |
|-----------------------------------------------------------|--|--|--|
| What is the gradient between A and the origin?            |  |  |  |
| What is the gradient of the tangent?                      |  |  |  |
| What is the equation of the tangent (so far)?             |  |  |  |
| Substitute the co-ordinate to find the value of $c$       |  |  |  |
| <b>Final Answer:</b><br>The equation of the tangent is... |  |  |  |

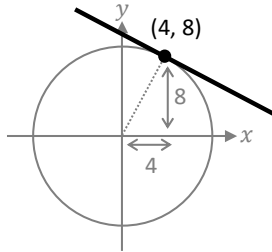
## Fluency Practice



|                                                           |  |  |  |
|-----------------------------------------------------------|--|--|--|
| What is the gradient between $A$ and the origin?          |  |  |  |
| What is the gradient of the tangent?                      |  |  |  |
| What is the equation of the tangent (so far)?             |  |  |  |
| Substitute the co-ordinate to find the value of $c$       |  |  |  |
| <b>Final Answer:</b><br>The equation of the tangent is... |  |  |  |

# Fluency Practice

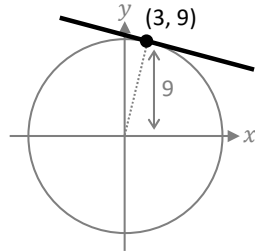
Rad. Grad. = 2    Tan. Grad. =  $-\frac{1}{2}$



$$y = -\frac{1}{2}x + 8$$

$$y = -\frac{1}{2}x + 10$$

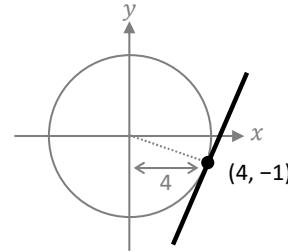
Rad. Grad. =    Tan. Grad. =



$$y = -\frac{1}{3}x + 10$$

$$y = -\frac{1}{3}x + 11$$

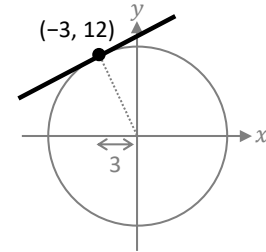
Rad. Grad. =  $-\frac{1}{4}$     Tan. Grad. =



$$y = 4x - 22$$

$$y = 4x - 17$$

Rad. Grad. = -4    Tan. Grad. =

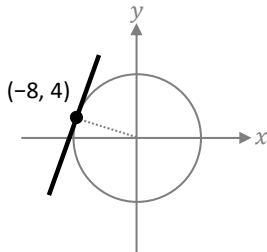


$$y = \frac{1}{4}x + 13$$

$$y = \frac{1}{4}x + 12\frac{3}{4}$$

**Tangent Match**

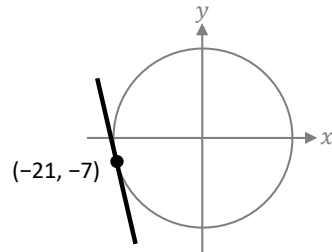
Rad. Grad. =    Tan. Grad. =



$$y = x + 20$$

$$y = x + 24$$

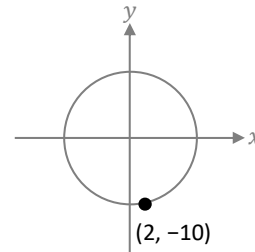
Rad. Grad. =    Tan. Grad. =



$$y = x - 56$$

$$y = x - 54$$

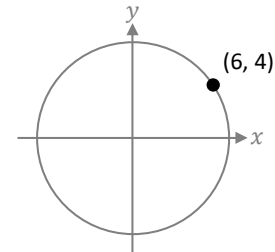
Rad. Grad. =    Tan. Grad. =



$$y = x - 10.2$$

$$y = x - 10.4$$

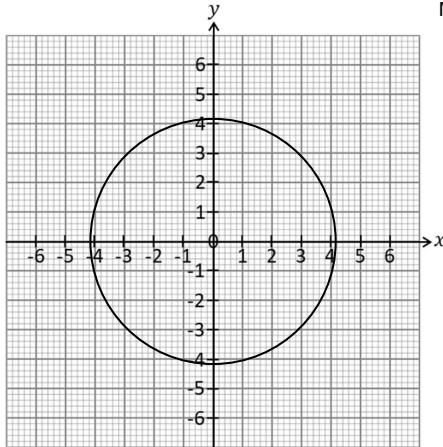
Rad. Grad. =    Tan. Grad. =



# Purposeful Practice

①

The graph  $x^2 + y^2 = 18$  is on the grid.



Mark point **A** on the circle at (3, 3), then draw line **OA** from the origin (**O**) to point **A**.

What is the equation of line **OA**?

\_\_\_\_\_

Plot the line  $y = 6 - x$

How are the gradients of these graphs related?

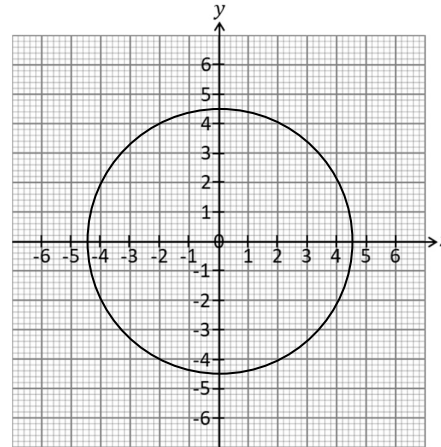
\_\_\_\_\_

$y = 6 - x$  is a \_\_\_\_\_ to the circle.

Draw & label the line that is parallel to  $y = 6 - x$  & touches the circle at one point.

## Lines around Circles

The graph  $x^2 + y^2 = 20$  is on the grid.



Mark point **B** on the circle at (2, 4), & draw line **OB** from the origin.

What is the gradient of the line **OB**?

\_\_\_\_\_

What is the gradient of a line perpendicular to **OB**?

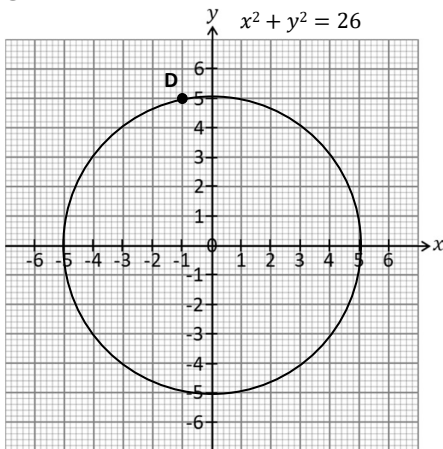
\_\_\_\_\_

Plot & label line **CD** - the line that is perpendicular to **OB** & passes through **B**.

Draw & label the line that is parallel to **CD** & touches the circle at one point.

②

③



Sketch the line **OD**.

Find the gradient of this radius, is it positive or negative?

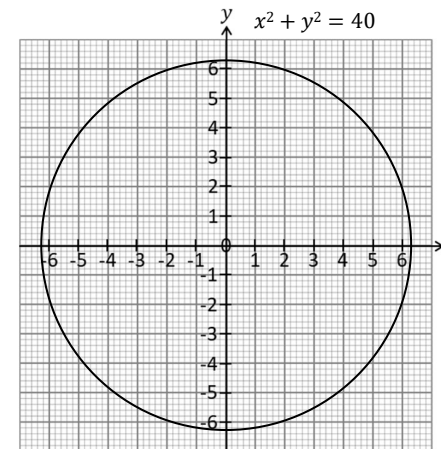
The tangent to the circle at **D** has a gradient of  $\frac{1}{5}$  and passes through (-1, 5).

We can express this as:

$$y = \frac{1}{5}x + c$$

Plot the tangent & calculate its y-intercept.

④



Find the equation of the tangent at (6, -2)

\_\_\_\_\_

Find the equation of the tangent at (-2, 6)

\_\_\_\_\_



## Purposeful Practice

(a) The point P  $(6, -2)$  lies on the circle with equation  $x^2 + y^2 = 40$ . Find the gradient of the tangent to the circle at point P.

(b) The point Q  $(2, 1)$  lies on the circle with equation  $x^2 + y^2 = 5$ . Find the gradient of the tangent to the circle at point Q.

(a) The point A  $(2, -5)$  lies on the circle with equation  $x^2 + y^2 = 29$ . Find the equation of the tangent to the circle at point A.

(b) The point B  $(-3, -2)$  lies on the circle with equation  $x^2 + y^2 = 13$ . Find the equation of the tangent to the circle at point A.

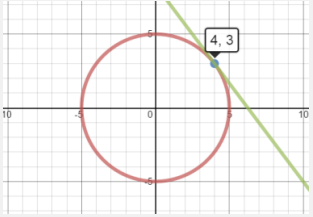
(a) The point R  $(-2, -3)$  lies on the circle with centre  $(1, 2)$  and radius  $\sqrt{34}$ . Find the equation of the tangent to the circle at point R.

(b) The point S  $(7, 1)$  lies on the circle with centre  $(x - 3)^2 + (y + 2)^2 = 25$ . Find the equation of the tangent to the circle at point S.

(a) Point P  $(-2, 8)$  lies on a circle with centre  $(-1, 6)$ . Point Q with coordinates  $(a, 5)$  lies on the tangent to the circle at P. Find the value of  $a$ .

(b) Point A  $(-1, -3)$  lies on the circle with equation  $(x - 3)^2 + (y + 2)^2 = 17$ . The line L is the tangent to the circle at point A. Find the coordinates of the point where line L crosses the  $x$ -axis.

## Fluency Practice

| Question                                                                                                                                         | Radius | Sketch                                                                             | Gradient of radius at point | Gradient of tangent at point | Equation of tangent at point       |
|--------------------------------------------------------------------------------------------------------------------------------------------------|--------|------------------------------------------------------------------------------------|-----------------------------|------------------------------|------------------------------------|
| <b>Example:</b> Find the equation of the tangent to $x^2 + y^2 = 25$ at $(4, 3)$                                                                 | 5      |  | $\frac{3}{4}$               | $-\frac{4}{3}$               | $y = -\frac{4}{3}x + \frac{25}{3}$ |
| 1. Find the equation of the tangent to $x^2 + y^2 = 5$ at $(2, 1)$                                                                               |        |                                                                                    |                             |                              |                                    |
| 2. Find the equation of the tangent to $x^2 + y^2 = 100$ at the point on the circumference with $x$ -coordinate 6 and a positive $y$ -coordinate |        |                                                                                    |                             |                              |                                    |
| 3. Find the equation of the tangent to $x^2 + y^2 = 45$ at $(-6, 3)$                                                                             |        |                                                                                    |                             |                              |                                    |

## Fluency Practice

4. Find the equation of the tangent to  
 $x^2 + y^2 - 20 = 0$  at  $(-4, -2)$

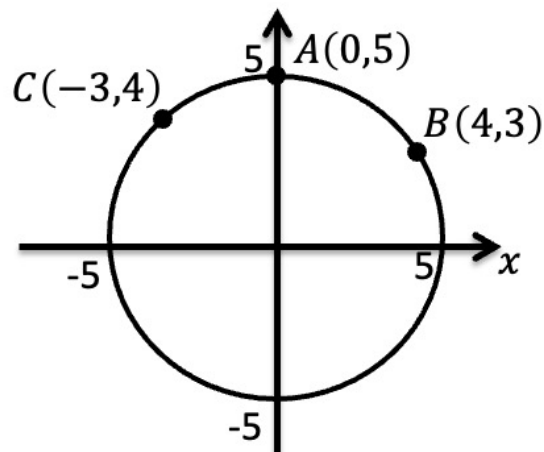
5. Find the equation of the tangent to  
 $x^2 + y^2 = 13$  at the point on the  
 circumference with  $x$ -coordinate 3  
 and a negative  $y$ -coordinate

6. Find the equation of the tangent to  
 the circle with centre  $(0, 0)$  and  
 diameter  $\sqrt{32}$  at the point  $(2, 2)$

7. Find the equation of the tangent to  
 $x^2 + y^2 = 25$  at the point  $(5, 0)$

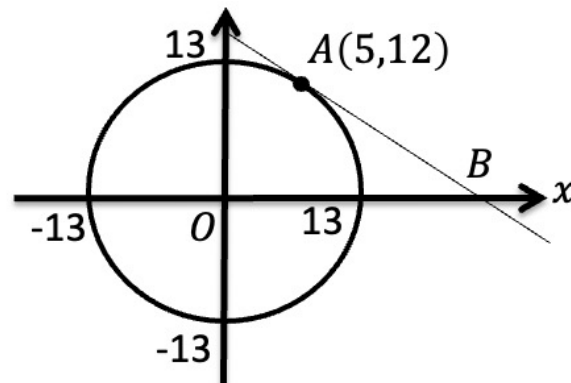
## Purposeful Practice

- 1 The diagram shows the circle with equation  $x^2 + y^2 = 25$ . What is the equation of the tangent at the following points?



- a  $A(0,5)$ :
- b  $B(4,3)$ :
- c  $C(-3,4)$ :

2



The tangent to the above circle at the point  $A(5,12)$  intersects the  $x$  axis at the point  $B$ .

- a Find the equation of the tangent to the circle at the point  $A$ .

- b Find the area of triangle  $OAB$ .



The line  $l$  is tangent at the point  $P(x, y)$  to the circle with equation  $x^2 + y^2 = 1$ . The gradient of  $l$  is  $-\frac{1}{2}$ . Determine the point  $P(x, y)$ .

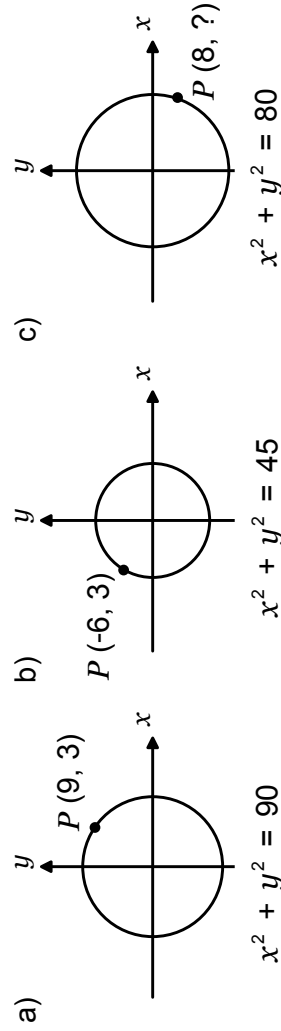
## Purposeful Practice

### WARM -UP

1. Work out the gradient of the line from the origin to each of these points:  
 a) (5, 8)      b) (-3, 7)      c)  $(-2, \sqrt{3})$       d)  $(-2\sqrt{2}, 4)$
2. In each case, the point P lies on the circle C. Work out  $a$ ,  $b$  and  $c$ .  
 a) C:  $x^2 + y^2 = 100$       b) C:  $x^2 + y^2 = 4^2$       c) C:  $x^2 + y^2 - 25 = 0$   
 P:  $(-6, a)$       P:  $(2, b)$       P:  $(-2, c)$   
 $a > 0$        $b > 0$        $c < 0$
3. Work out the gradient of a line **perpendicular** to a line with gradient:  
 a) -4      b)  $\frac{2}{5}$       c)  $-\frac{3}{4}$       d)  $\frac{1}{5}$
4. In each case, the point P lies on a line with gradient  $m$ . Work out the equation of the line in the form  $y = mx + c$ .  
 a) P(2, 5),  $m = 3$       b) P(-3, 4),  $m = -1$       c) P(-3, -2),  $m = \frac{1}{4}$

### QUESTIONS

1. Work out the equation of the tangents to these circles at the given point P. Give your answers in the form  $y = mx + c$ .

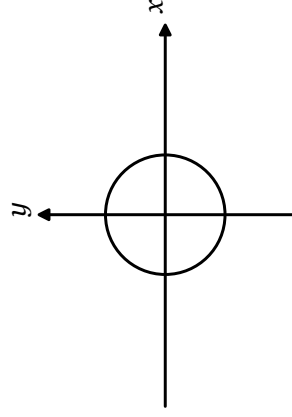


2. Circle C has equation  $x^2 + y^2 = 400$ .  
 Point P( $a$ ,  $b$ ) lies on C such that  $a = -12$  and  $b < 0$ .  
 a) Work out the value of  $b$ .  
 b) Work out the equation of the tangent to C at P in the form  $y = mx + c$ .

## Purposeful Practice

3. Circle C has equation  $x^2 + y^2 = 72$ .  
Two points on C have a tangent with a gradient of 0.

Work out the coordinates of these two points.

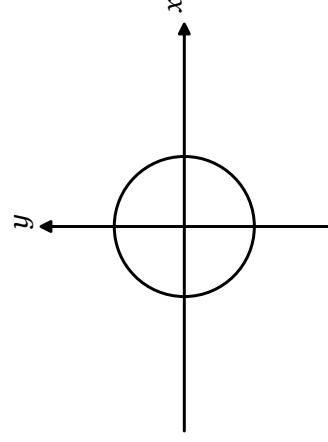


4. Circle C is centred on the origin and has a radius of 15 units.  
Point  $P(a, b)$  lies on C, such that  $a = 12$  and  $b > 0$ .  
Line T is the tangent to C at P.

- Work out the equation of T.
- Work out the coordinates of the point at which T meets the  $y$ -axis.
- Work out the coordinates of the point at which T meets the  $x$ -axis.
- Work out the area of the triangle formed by the  $x$ -axis, the  $y$ -axis and T.

5. Circle C has equation  $x^2 + y^2 = 32$ .  
Two different lines are both tangents to C and have a gradient of 1.  
Work out the coordinates of the points where the tangents meet C.

6. A circle is centred on the origin.  
The points  $P(-15, -20)$  and  $Q(-24, c)$  lie on the circle, where  $c > 0$ .  
Work out the equation of the tangent to the circle at Q.

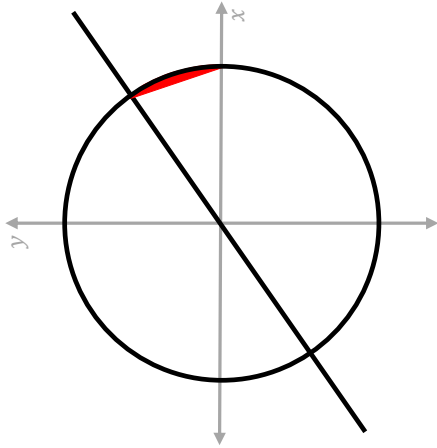


7. Circle C has equation  $x^2 + y^2 = 50$ .  
O is the origin  $(0, 0)$ .  
 $P(5, s)$  is a point on the circle.  
 $Q(1, t)$  is a point on the circle.  
 $s > 0$  and  $t < 0$ .

- Work out the values of  $s$  and  $t$ .
- Work out the equations of the tangents to C at P and Q.
- Work out the coordinates of the point R, where the two tangents meet.
- Given that  $\hat{POQ} = 127^\circ$ , work out  $\hat{PRQ}$ .

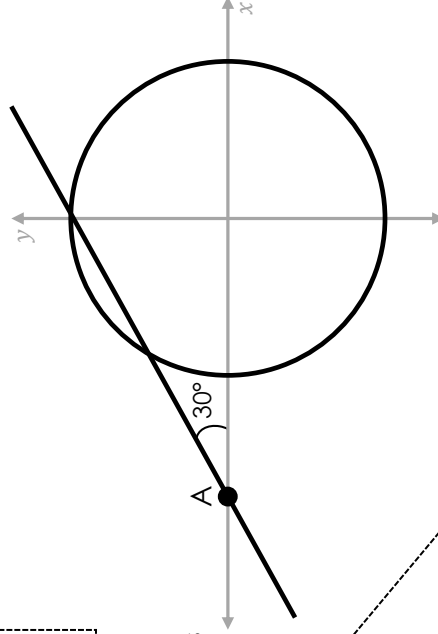
# Purposeful Practice

8 fiendishly  
difficult circle  
questions for  
GCSE

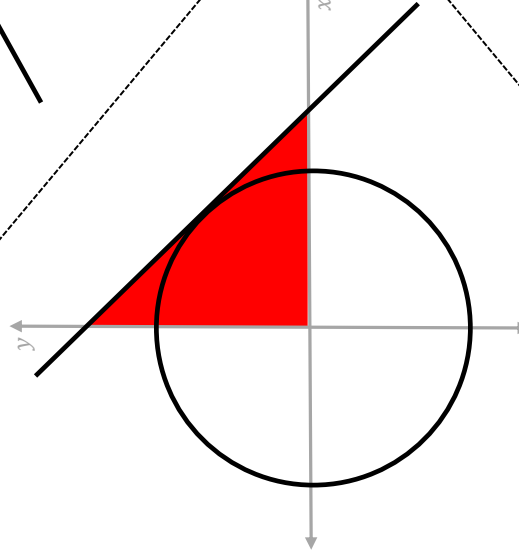


The equation of the circle is  $x^2 + y^2 = 36$ . The equation of the straight line is  $y = \frac{\sqrt{3}}{3}x$ . What is the area of the shaded segment?

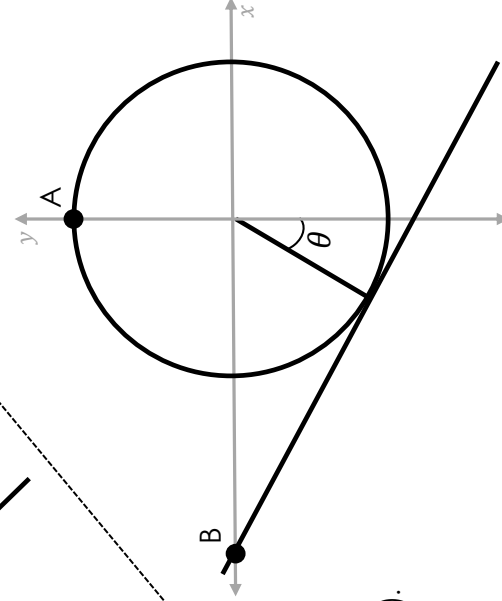
The point A has coordinates  $(-6, 0)$ . The straight line shown makes an angle of  $30^\circ$  with the  $x$ -axis. What is the equation of the circle?



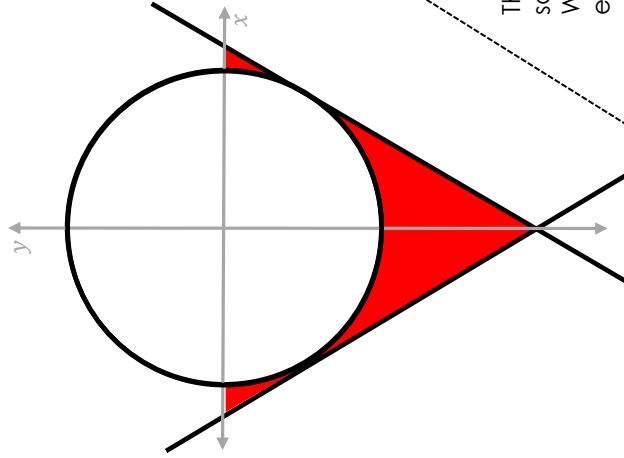
A tangent meets the circle at  $(5, 12)$ . What is the area of the shaded triangle?



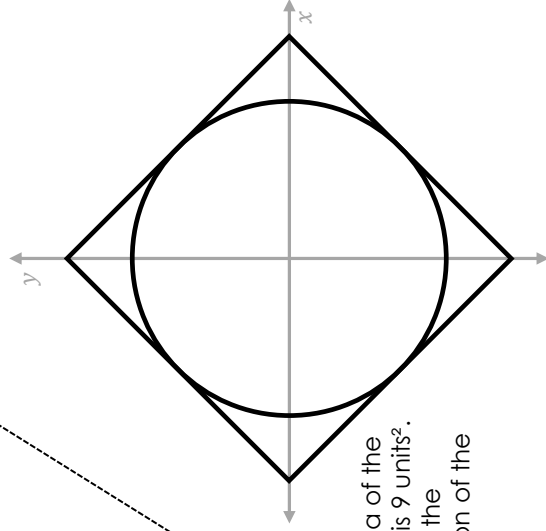
The point A has coordinates  $(0, \frac{5}{3})$ . The point B has coordinates  $(-\frac{8}{3}, 0)$ . Calculate the angle labelled  $\theta$ .



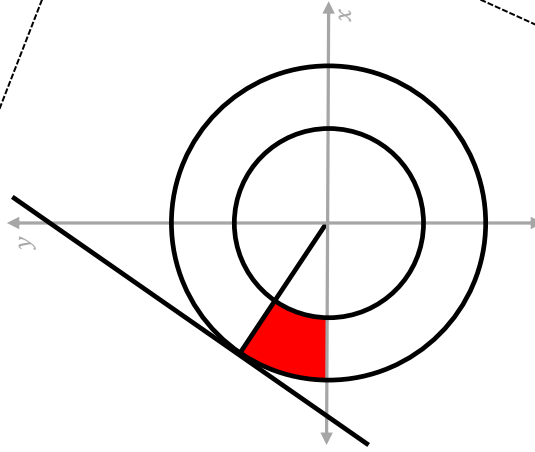
## Purposeful Practice



One tangent meets the circle at  $(6, -8)$ , and another meets the circle at  $(-6, -8)$ . What is the total area of the shaded sections?



The area of the square is 9 units<sup>2</sup>. What is the equation of the circle?



Two concentric circles are shown, one is an enlargement of the other by a scale factor 3 and has the equation  $x^2 + y^2 = 144$ .

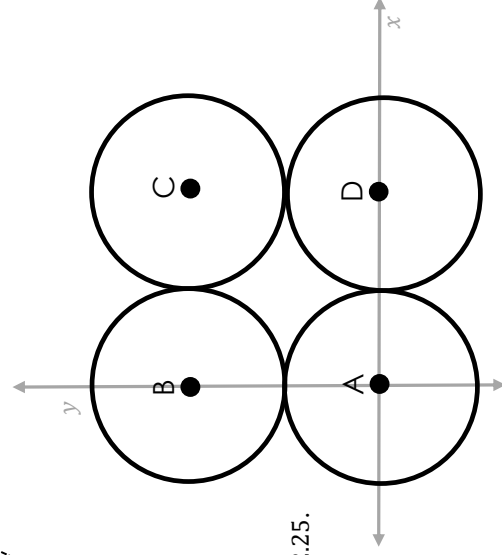
The tangent shown has the equation  $y = \sqrt{3}x + 24$ .

Calculate the area of the shaded section.

Four congruent circles are shown, with centres A, B, C, and D marked.

One circle has the equation  $y^2 + x^2 = 2.25$ .

Calculate the distance A to C





## 2 Advanced Sequences

## Fluency Practice

**irrational sequences: what comes next?**

Look at these sequences, can you work out the term-to-term rule?  
What will be the next number in the sequence?

a  $\sqrt{2}$ , 2,  $2\sqrt{2}$ , 4, ... ?

b  $\sqrt{3}$ , 6,  $12\sqrt{3}$ , 72, ... ?

c  $\sqrt{2}$ ,  $\sqrt{8}$ ,  $\sqrt{18}$ ,  $\sqrt{32}$ , ... ?

d  $\sqrt{75}$ ,  $\sqrt{48}$ ,  $\sqrt{27}$ ,  $\sqrt{12}$ , ... ?

e 9,  $3\sqrt{3}$ , 3,  $\sqrt{3}$ , ... ?

f  $2\sqrt{5}$ , -10,  $10\sqrt{5}$ , -50, ... ?

g  $\sqrt{3}-2$ ,  $3-2\sqrt{3}$ ,  $3\sqrt{3}-6$ ,  $9-6\sqrt{3}$ , ... ?

h  $\sqrt{2}+1$ ,  $2\sqrt{2}+3$ ,  $5\sqrt{2}+7$ ,  $12\sqrt{2}+17$ , ... ?

extension:

Make up your own sequence. Can you partner figure out your term-to-term rule?

## Fluency Practice

geometric series: what are the constant ratios?

|     |         |     |       |     |      |     |           |     |        |
|-----|---------|-----|-------|-----|------|-----|-----------|-----|--------|
| (1) | 9       | (2) | —     | (3) | —    | (4) | —         | (5) | —      |
|     | —       |     | 21    |     | —    |     | —         |     | 48     |
|     | 576     |     | —     |     | 80   |     | 324       |     | —      |
|     | —       |     | 189   |     | —    |     | —         |     | —      |
|     | —       |     | —     |     | 1280 |     | 26,244    |     | 10,368 |
|     | 294,912 |     | —     |     | —    |     | —         |     | —      |
|     | ...     |     | 5,103 |     | —    |     | 2,125,764 |     | —      |
|     |         |     | ...   |     | ...  |     | ...       |     |        |

|     |       |     |     |     |      |     |           |      |     |
|-----|-------|-----|-----|-----|------|-----|-----------|------|-----|
| (6) | 4     | (7) | —   | (8) | —    | (9) | 4,782,969 | (10) | —   |
|     | —     |     | 27  |     | 3000 |     | —         |      | 12  |
|     | —     |     | —   |     | —    |     | —         |      | —   |
|     | 5,324 |     | —   |     | —    |     | 177,147   |      | —   |
|     | —     |     | 729 |     | 81   |     | —         |      | —   |
|     | —     |     | —   |     | —    |     | —         |      | 192 |
|     | ...   |     | ... |     | ...  |     | ...       |      | ... |

what are the first terms (a) and the ratios (r) of these geometric series?

|      |        |      |           |      |           |      |         |      |        |
|------|--------|------|-----------|------|-----------|------|---------|------|--------|
| (10) | —      | (11) | —         | (12) | —         | (13) | —       | (14) | —      |
|      | 35     |      | 441       |      | 1221      |      | 200     |      | 111    |
|      | —      |      | —         |      | —         |      | —       |      | —      |
|      | —      |      | —         |      | —         |      | —       |      | —      |
|      | —      |      | —         |      | 1,625,151 |      | —       |      | —      |
|      | 21,875 |      | 1,058,841 |      | ...       |      | 0.00125 |      | —      |
|      | ...    |      | ...       |      |           |      | ...     |      | 26,973 |
|      |        |      |           |      |           |      |         |      | ...    |

## Fluency Practice

geometric series: what are the missing terms?

(1) 24  
—  
54  
—  
...

(2) —  
30  
—  
480  
...

(3) —  
—  
72  
—  
32  
...

(4) —  
—  
1200  
—  
1875  
...

(5) —  
8  
—  
—  
1

(6) 25  
—  
—  
102.4  
...

(7) —  
9.9  
—  
11.979  
—  
...

(8) —  
150  
—  
216  
—  
...

(9) 192  
—  
—  
81  
...

(10) 250  
—  
—  
54  
...

what are 'a' and 'r' for these geometric series?

(10)  $ar = 4$   
 $ar^4 = 4000$

(11)  $ar^2 = 144$   
 $ar^6 = 729$

(12)  $ar^3 = 1792$   
 $ar^6 = 917504$

(13)  $ar^2 = 1728$   
 $ar^6 = 729$

(14)  $ar = 1.6$   
 $ar^7 = 0.0001024$

## Purposeful Practice

geometric series: find possible values of  $n$  and the corresponding constant ratios

(1)  $\frac{n-1}{2n}$   
 $\frac{3n+9}{\quad}$   
 $\frac{\quad}{\quad}$   
 $\dots$

(2)  $\frac{n-2}{2n}$   
 $\frac{3n+18}{\quad}$   
 $\frac{\quad}{\quad}$   
 $\dots$

(3)  $\frac{2n-6}{n}$   
 $\frac{n-4}{\quad}$   
 $\frac{\quad}{\quad}$   
 $\dots$

(4)  $\frac{n-2}{2n+2}$   
 $\frac{3n+88}{\quad}$   
 $\frac{\quad}{\quad}$   
 $\dots$

(5)  $\frac{n}{10+n}$   
 $\frac{35+2n}{\quad}$   
 $\frac{\quad}{\quad}$   
 $\dots$

(6)  $\frac{n-2}{2n-2}$   
 $\frac{3n+6}{\quad}$   
 $\frac{\quad}{\quad}$   
 $\dots$

(7)  $\frac{n-1}{n+1}$   
 $\frac{3n-1}{\quad}$   
 $\frac{\quad}{\quad}$   
 $\dots$

(8)  $\frac{n+1}{2n-1}$   
 $\frac{5n-7}{\quad}$   
 $\frac{\quad}{\quad}$   
 $\dots$

(9)  $\frac{2n+2}{3n-1}$   
 $\frac{4n-3}{\quad}$   
 $\frac{\quad}{\quad}$   
 $\dots$

(10)  $\frac{n-2}{2n}$   
 $\frac{3n+20}{\quad}$   
 $\frac{\quad}{\quad}$   
 $\dots$

(11)  $\frac{n+1}{2n+7}$   
 $\frac{3n-3}{\quad}$   
 $\frac{\quad}{\quad}$   
 $\dots$

(12)  $\frac{2n-3}{2-3n}$   
 $\frac{3n+8}{\quad}$   
 $\frac{\quad}{\quad}$   
 $\dots$

(13)  $\frac{3n-2}{4n-2}$   
 $\frac{5n-1}{\quad}$   
 $\frac{\quad}{\quad}$   
 $\dots$

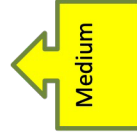
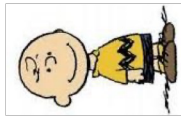
(14)  $\frac{n-4}{3n-4}$   
 $\frac{10n-4}{\quad}$   
 $\frac{\quad}{\quad}$   
 $\dots$

(15)  $\frac{3n-2}{3n+3}$   
 $\frac{4n}{\quad}$   
 $\frac{\quad}{\quad}$   
 $\dots$

# Fluency Practice

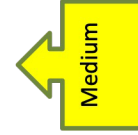
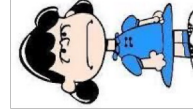
1. Below is the first number in a geometric sequence. Write the first 4 terms of 2 possible sequences that could start with that number:

- |       |                  |                     |
|-------|------------------|---------------------|
| a. 2  | f. 0.4           | k. $ab$             |
| b. 4  | g. -4            | l. $\frac{a^2b}{2}$ |
| c. 8  | h. $\frac{1}{4}$ | m. $\sqrt{a}$       |
| d. 10 | i. -0.3          | n. $ab^2c^3$        |
| e. 7  | j. $\sqrt{2}$    | o. $\frac{\pi}{2}$  |



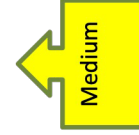
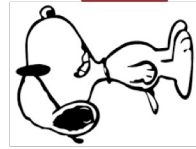
2. Calculate the first 5 terms of these sequences:

- |                 |                                               |                                    |
|-----------------|-----------------------------------------------|------------------------------------|
| a. $2^n$        | f. $3 \times 3^n$                             | k. $a \times 3^{bn}$               |
| b. $2^{n+1}$    | g. $0.4 \times 3^{(n-1)}$                     | l. $\frac{h}{3^n}$                 |
| c. $(-2)^n$     | h. $\frac{3}{5} \times 1.1^n$                 | m. $n \times a^n p^2$              |
| d. $-(2)^n$     | i. $9.1 \times 10^{\left(\frac{1}{n}\right)}$ | n. $m \times 0.03^{(b+n)}$         |
| e. $0.2^{(2n)}$ | j. $0.03 \times 0.9^{(3n)}$                   | o. $\frac{1}{f} \times ad^{(n-3)}$ |



3. Work out  $T(n)$  for the following geometric series:

- |                      |                                                            |                                                                       |
|----------------------|------------------------------------------------------------|-----------------------------------------------------------------------|
| a. 4, 8, 16, 32      | f. -2, 4, -8, 16                                           | k. $ab, ab^2, ab^3, ab^4$                                             |
| b. 4, 2, 1, 0.5      | g. 4, -4, 4, -4                                            | l. $\frac{a}{b^2}, \frac{a^2}{b^3}, \frac{a^3}{b^4}, \frac{a^4}{b^5}$ |
| c. 5, 15, 45, 135    | h. $\frac{1}{2}, \frac{1}{8}, \frac{1}{32}, \frac{1}{128}$ | m. $-a^2, a^2, -a^2, a^2$                                             |
| d. 3, 18, 108, 648   | i. $\sqrt{2}, 2, 2\sqrt{2}, 4$                             | n. $-ab, ab^2, -ab^3, ab^4$                                           |
| e. -5, -10, -20, -40 | j. $5, 5\sqrt{5}, 25, 25\sqrt{5}$                          | o. $a^2b^2, a^2b, a^2, \frac{a^2}{b}$                                 |



# Fluency Practice

| Geometric Sequences                                                                                 |                                                                            |                                                                                                                                                                                                    |                                                                            |
|-----------------------------------------------------------------------------------------------------|----------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|----------------------------------------------------------------------------|
| <b>(a)</b>                                                                                          | <b>(b)</b>                                                                 | <b>(c)</b>                                                                                                                                                                                         | <b>(d)</b>                                                                 |
| Find the next two terms in the sequence 7, 14, 28, 56, ...                                          | Find the next two terms in the sequence 40, 20, 10, 5, ....                | Find the first four terms of the sequence with first term 2 and common ratio 3                                                                                                                     | Find the first term and common ratio for the sequence: 3, 15, 75, 375, ... |
| <b>(e)</b>                                                                                          | <b>(f)</b>                                                                 | <b>(g)</b>                                                                                                                                                                                         | <b>(h)</b>                                                                 |
| Find the first term and common ratio for the sequence: 160, 80, 40, 20, ...                         | Find the next two terms in the sequence 2, -4, 8, -16, ....                | Find the first four terms of the sequence with first term 120 and common ratio 0.5                                                                                                                 | Find the first term and common ratio for the sequence: 4, -8, 16, -32, ... |
| <b>(i)</b>                                                                                          | <b>(j)</b>                                                                 | <b>(k)</b>                                                                                                                                                                                         | <b>(l)</b>                                                                 |
| Find the first four terms of the sequence with first term 5 and common ratio -2                     | Find the first four terms of the sequence with nth term $6 \times 3^{n-1}$ | Find the nth term of the sequence with first term 10 and common ratio 4                                                                                                                            | Find the nth term of the sequence with first term 250 and common ratio 0.2 |
| <b>(m)</b>                                                                                          | <b>(n)</b>                                                                 | <b>(p)</b>                                                                                                                                                                                         |                                                                            |
| Find the first four terms of the sequence with nth term $400 \times \left(\frac{1}{2}\right)^{n-1}$ | Find the nth term of the sequence with first term 8 and common ratio -5    | A tree starts with four branches. Every month each branch splits into two. How many branches will the tree have after 5 months?<br>Find a formula for the number of branches $b$ after $n$ months. |                                                                            |

## Fluency Practice

| Q  | Find the value(s) of $x$ where...                                                             | Answers |
|----|-----------------------------------------------------------------------------------------------|---------|
| 1  | $x - 1$ , $2x + 5$ , and $4x + 8$ are three consecutive terms of an arithmetic sequence       |         |
| 2  | $x - 1$ , $2x + 5$ , and $4x + 8$ are three consecutive terms of a geometric sequence         |         |
| 3  | $x - 4$ , $x + 2$ and $3x + 1$ are three consecutive terms of an arithmetic sequence          |         |
| 4  | $x - 4$ , $x + 2$ and $3x + 1$ are three consecutive terms of a geometric sequence            |         |
| 5  | $x - 3$ , $x + 1$ and $4x - 2$ are three consecutive terms of an arithmetic sequence          |         |
| 6  | $x - 3$ , $x + 1$ and $4x - 2$ are three consecutive terms of a geometric sequence            |         |
| 7  | $x - 1$ , $x + 4$ , $2x + 7$ are three consecutive terms of an arithmetic sequence            |         |
| 8  | $x - 1$ , $x + 4$ , $2x + 7$ are three consecutive terms of a geometric sequence              |         |
| 9  | $\sqrt{x} - 1$ , $1$ and $\sqrt{x} + 1$ are three consecutive terms of an arithmetic sequence |         |
| 10 | $\sqrt{x} - 1$ , $1$ and $\sqrt{x} + 1$ are three consecutive terms of a geometric sequence   |         |



## Fluency Practice

By finding the first and second differences, decide whether each of these sequences is quadratic.

- (a) 1, 5, 11, 19, 29, 41
- (b) 2, 5, 8, 11, 14, 17
- (c) 0, 8, 22, 41, 68, 98
- (d) 2, 9, 20, 35, 54, 77
- (e) 4, 1, 0, 1, 4, 9
- (f) 6, 17, 36, 65, 98, 141
- (g) 18, 37, 62, 93, 130, 173
- (h) 3, 9, 23, 43, 75, 113
- (i) -10, -4, 12, 38, 74, 120
- (j) 17, 39, 69, 107, 153, 207

Generate the first five terms of each of these quadratic sequences.

- (a)  $n^2 + 10$       (b)  $n^2 - 1$
- (c)  $n^2 + n$       (d)  $n^2 + 2n + 1$
- (e)  $n^2 - 3n$       (f)  $n^2 - n - 2$
- (g)  $2n^2 + 5$       (h)  $3n^2 - 7$
- (i)  $2n^2 + n - 5$       (j)  $4n^2 + 3n - 1$

Find the 10<sup>th</sup> and 50<sup>th</sup> term of the following quadratic sequences.

- (a)  $n^2 + 5$       (b)  $n^2 - 2$
- (c)  $n^2 - n$       (d)  $n^2 + 2n$
- (e)  $n^2 - 3n + 1$       (f)  $n^2 - n - 2$
- (g)  $4n^2 + 1$       (h)  $3n^2$
- (i)  $2n^2 + n - 1$       (j)  $5n^2 + 3n$

## Fluency Practice

**QUADRATIC SEQUENCES**      Substitute to complete each term.

|                 | 1 <sup>st</sup> | 2 <sup>nd</sup> | 3 <sup>rd</sup> | 4 <sup>th</sup> | 5 <sup>th</sup> | 10 <sup>th</sup> |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|------------------|
| $n^2$           |                 |                 |                 | 16              |                 |                  |
| $n^2 - 5$       |                 |                 |                 | 11              | 20              |                  |
| $n^2 + n$       |                 | 6               | 12              |                 |                 |                  |
| $n^2 + 2n$      | 3               |                 |                 |                 | 35              |                  |
| $n^2 - n$       |                 |                 |                 | 12              | 20              |                  |
| $n^2 + n + 3$   |                 |                 | 15              | 23              |                 |                  |
| $n^2 + 4n$      | 5               |                 |                 |                 | 45              |                  |
| $n^2 - 3n$      |                 |                 |                 | 4               |                 |                  |
| $n^2 + 2n + 6$  |                 | 14              | 21              |                 |                 |                  |
| $n^2 - n + 5$   |                 | 7               |                 |                 | 25              |                  |
| $n^2 + 3n - 4$  |                 |                 | 14              | 24              |                 |                  |
| $n^2 - 2n + 5$  | 4               | 5               |                 |                 |                 |                  |
| $2n^2 + 3n$     |                 |                 |                 | 44              | 65              |                  |
| $2n^2 - 4n$     |                 |                 |                 | 16              |                 |                  |
| $2n^2 + 2n + 3$ |                 |                 | 27              |                 | 63              |                  |
| $4n^2 - 3n - 5$ |                 | 5               |                 | 47              |                 |                  |

## Fluency Practice

Find the  $n$ th term of these sequences:

- (a) 8, 11, 16, 23, ...
- (b)  $-4, -1, 4, 11, \dots$
- (c) 6, 12, 22, 36, ...
- (d)  $-1, 5, 15, 29, \dots$
- (e) 4, 13, 28, 49, ...
- (f) 2, 14, 34, 62, ...

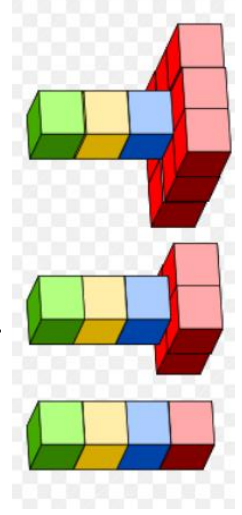
Find the  $n$ th term of these sequences:

- (a) 3, 7, 13, 21, ...
- (b) 2, 7, 14, 23, ...
- (c) 9, 18, 31, 48, ...
- (d) 1, 5, 13, 25, ...
- (e) 7, 23, 49, 85, ...
- (f) 7, 15, 29, 49, ...

Find the  $n$ th term of these sequences:

- (a) 10.5, 12, 14.5, 18, ...
- (b) 5.5, 9, 13.5, 19, ...
- (c) 1.5, 5, 11.5, 21, ...
- (d) 3.5, 10, 19.5, 32, ...
- (e) 19, 16, 11, 4,  $-5, \dots$
- (f) 5, 6, 5, 2,  $-3, \dots$

Here are some patterns made from cubes. Find an expression for the  $n$ th term of this sequence.



## Quadratic Sequences

Quadratic sequences take the form

$$an^2 + bn + c$$

For each of the following quadratic sequences, identify the values of a, b and c:

1.  $2n^2 + 3n + 4$

4.  $n^2 + n - 2$

2.  $4n^2 + 8n + 5$

5.  $\frac{n^2}{2} - 4$

3.  $5n^2 - 11n$

6.  $-n^2$

Find the first 3 terms for these quadratic sequences:

7.  $n^2 + 3$

10.  $4n^2 - 30$

8.  $2n^2 + 2$

11.  $n^2 + n - 9$

9.  $3n^2$

12.  $2n^2 - 2n - 4$

13.  $\frac{n^2}{2} + 3n + 9$

14.  $n(n + 3)$

15.  $(n + 3)(2n + 6)$

Find the 10<sup>th</sup> and 100<sup>th</sup> term of these sequences:

16.  $n^2 + 3$

18.  $-8n^2$

17.  $2n^2 - 4n$

19.  $\frac{n^2}{3} + 20$

Which of these terms are **not** in the sequences provided in **bold**:

20. {12, 39, 55, 103} ( **$n^2 + 3$** )

22. {108, 116, 128, 162} ( **$2n^2$** )

21. {15, 35, 63, 82} ( **$n^2 - 1$** )

23. {1, 38, 56, 119} ( **$n^2 + 10n$** )

Match the sequences to the descriptions below (assume n is always a whole number):

24.  $n^2 + 1$

A: Always even

25.  $n^2 + n$

B: Always odd

26.  $n^2 \div 3$

C: Sometimes even, sometimes odd

27.  $2n^2$

D: Sometimes an integer

28.  $n^2 + 3n + \frac{3}{5}$

E: Never an integer

Find the nth term of these quadratic sequences:

29. 1, 4, 9, 16

34. 0, 6, 16, 30

30. -3, 0, 5, 12

35. 8, 14, 24, 38

31. 3, 12, 27, 48

36. 11, 20, 35, 56

32. 2, 5, 10, 17

37. 0.5, 2, 4.5, 8

33. 8, 11, 16, 23

38. 9.3, 10.3, 12, 14.3

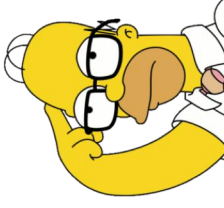
39. 3, 7, 13, 21

40. 5, 13, 25, 41

41. 7, 16, 27, 40

42. 0, 1, 4, 9

43. -5, 8, 27, 52

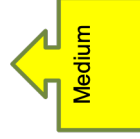


Section 1

Section 2

Section 3

Section 4



## Fluency Practice

Find the  $n$ th term for each sequence and the specified term.

$-1, 5, 15, 29, 47$

$10^{\text{th}}$  term

$5, 14, 29, 50, 77$

$20^{\text{th}}$  term

$3, 6, 11, 18, 27$

$50^{\text{th}}$  term

$5, 11, 21, 35, 53$

$25^{\text{th}}$  term

$6, 17, 34, 57, 86$

$10^{\text{th}}$  term

$3, 18, 41, 72, 111$

$20^{\text{th}}$  term

## Fluency Practice

find the next two terms in these **quadratic** sequences

(1)  $6, 7, 10, 15, 22, 31, ?, ?$

(2)  $11, 15, 21, 29, 39, 51, ?, ?$

(3)  $-18, -14, -8, 0, 10, 22, ?, ?$

(4)  $-7, -8, -7, -4, 1, 8, ?, ?$

(5)  $-10, -13, -14, -13, -10, -5, ?, ?$

## Purposeful Practice

two of the terms in these **quadratic** sequences are wrong

(1) 5 , 0 , -1 , -4 , -1 , 0 , 5 , 12 , 21 , 40

(2) 10 , 6 , 4 , 4 , 6 , 12 , 16 , 24 , 32 , 46

(3) 5 , 1 , -1 , -3 , -1 , 5 , 11 , 19 , 29 , 41

(4) 17 , 11 , 3 , -1 , -3 , -3 , -1 , 3 , 11 , 17

(5) -26 , -22 , -20 , -14 , -6 , 4 , 14 , 30 , 46

## Purposeful Practice

fit a quadratic rule to these sequences

(1)    1    2    3    4    5    6    .....     $n$   
          2    3    6    11    18    27    .....

(6)    1    2    3    4    5    6    .....     $n$   
          9    18    31    48    69    94    .....

(2)    1    2    3    4    5    6    .....     $n$   
          3    6    11    18    27    38    .....

(7)    1    2    3    4    5    6    .....     $n$   
          31    48    69    94    123    156    .....

(3)    1    2    3    4    5    6    .....     $n$   
          3    2    3    6    11    18    .....

(8)    1    2    3    4    5    6    .....     $n$   
          4    9    18    31    48    69    .....

(4)    1    2    3    4    5    6    .....     $n$   
          6    11    18    27    38    51    .....

(9)    1    2    3    4    5    6    .....     $n$   
          6    3    4    9    18    31    .....

(5)    1    2    3    4    5    6    .....     $n$   
          11    18    27    38    51    66    .....

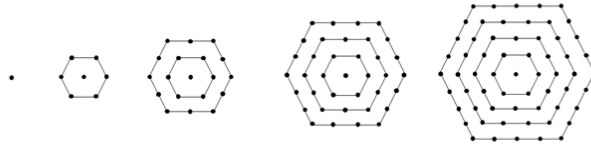
(10)    1    2    3    4    5    6    .....     $n$   
          24    13    6    3    4    9    .....



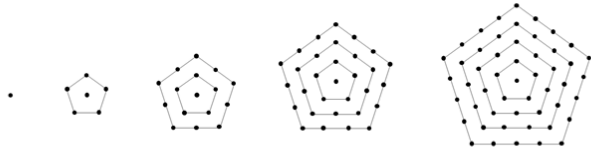
# Purposeful Practice

find a rule for the numbers of dots/circles in these quadratic sequences

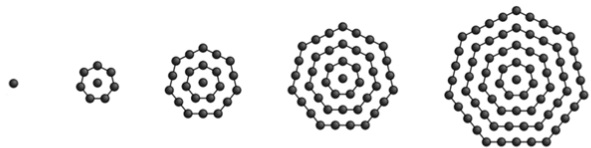
1) centred hexagonal numbers



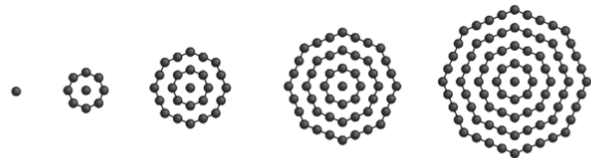
2) centred pentagonal numbers



3) centred heptagonal numbers



4) centred octagonal numbers



5) corner hexagonal numbers



6) corner pentagonal numbers



7) corner heptagonal numbers



8) corner octagonal numbers



# Purposeful Practice

The  $n$ th term of a sequence is in the form  $an^2 + bn$   
 The 2<sup>nd</sup> term in the sequence is 14.  
 The 4<sup>th</sup> term in the sequence is 44.  
 Find  $a$  &  $b$ .

1) Form two equations  
 in  $a$  &  $b$ .

2) Solve the simultaneous equations.

$$\begin{array}{rcl}
 a(2)^2 + b(2) = 14 & & 16a + 4b = 44 \\
 4a + 2b = 14 & \xrightarrow{\times 2} & 8a + 4b = 28 \\
 \hline
 a(4)^2 + b(4) = 44 & & 16a + 4b = 44 \\
 16a + 4b = 44 & & \hline
 & & 8a = 16 \\
 & & a = 2 \\
 & & \hline
 & & 4(2) + 2b = 14 \\
 & & 2b = 6 \\
 & & b = 3
 \end{array}$$

$n$ th term =  $2n^2 + 3n$

## Finding the $n$ th Term of a Quadratic Sequence

①

The  $n$ th term of a sequence is in the form  $an^2 + bn$   
 The 2<sup>nd</sup> term in the sequence is 18.  
 The 4<sup>th</sup> term in the sequence is 52.  
 Find  $a$  &  $b$ .

How can we  
 check our answer?

②

A sequence is in the form  $an^2 + bn$   
 The 3<sup>rd</sup> term in the sequence is 30.  
 The 6<sup>th</sup> term in the sequence is 114.  
 Find the  $n$ th term

③

A sequence is in the form  $an^2 + bn$   
 The 2<sup>nd</sup> term in the sequence is 22.  
 The 3<sup>rd</sup> term in the sequence is 45.  
 Find the  $n$ th term

④

The 4<sup>th</sup> term in a sequence is 64.  
 The 5<sup>th</sup> term in a sequence is 105.  
 Find the 3<sup>rd</sup> term in the sequence.

### Purposeful Practice

- (a) A quadratic sequence begins 3, 15, \_ , 51, 75,...  
What is the the third term?
- (b) The quadratic sequence beginning  $-3, 3, 11, 21, 33...$   
has  $n$ th term  $n^2 + 3n - 7$ .  
Find the  $n$ th term of the quadratic sequence that begins  
3, 11, 21, 33...
- (c) The first term of a quadratic sequence is 10, the fifth term is 25,  
and the ninth term 45. Find the thirteenth term of the sequence.

# Fluency Practice

Question 1: Find the next two terms for each quadratic sequence

- (a) 4, 6, 10, 16, 24 ... .. (b) 1, 2, 4, 7, 11 ... .. (c) 2, 5, 10, 17, 26 ... ..  
 (d) 3, 9, 19, 33, 51 ... .. (e) 50, 48, 44, 38, 30 ... .. (f) 3, 14, 29, 48, 71 ... ..

Question 2: List the first 5 terms of the sequences with  $n^{\text{th}}$  term:

- (a)  $n^2$  (b)  $n^2 + 1$  (c)  $n^2 + 4$  (d)  $n^2 - 2$  (e)  $2n^2$   
 (f)  $5n^2$  (g)  $\frac{1}{2}n^2$  (h)  $\frac{1}{4}n^2$  (i)  $3n^2 + 10$  (j)  $\frac{3}{5}n^2$

Question 3: The quadratic  $n^{\text{th}}$  term of the sequence below is  $n^2$   
 1, 4, 9, 16, 25, 36, 49 ...

Find the  $n^{\text{th}}$  term of each of these sequences

- (a) 4, 7, 12, 19, 28, 39, 52 ... (b) 51, 54, 59, 66, 75, 86, 99 ... (c) -5, -2, 3, 10, 19, 30...  
 (d) 3, 12, 27, 48, 75, 108 ... (e) 20, 80, 180, 320, 500, 720 ... (f) 0.2, 0.8, 1.8, 3.2, 5 ...  
 (g) 3, 9, 19, 33, 51, 73, 99 ... (h) 2.5, 4, 6.5, 10, 14.5, 20 ...

Question 4: For each  $n^{\text{th}}$  term, work out the first five terms of the sequence.

- (a)  $n^2 + n$  (b)  $n^2 + 2n$  (c)  $n^2 - n$  (d)  $n^2 - 3n$   
 (e)  $n^2 + n + 2$  (f)  $n^2 - 2n + 5$  (g)  $n^2 + 4n - 10$  (h)  $2n^2 + n$   
 (i)  $3n^2 - n + 6$  (j)  $10n^2 + 5n - 7$

Question 5: For each  $n^{\text{th}}$  term, work out the first five terms of the sequence.

- (a)  $-n^2$  (b)  $-2n^2$  (c)  $-4n^2 + 2$  (d)  $-n^2 + 3n$   
 (e)  $50 - n^2$  (f)  $6n - n^2$  (g)  $-n^2 - 7n - 2$

## Fluency Practice

Question 6: For each  $n^{\text{th}}$  term, work out the first five terms of the sequence.

- (a)  $n(n+1)$       (b)  $n(n+3)$       (c)  $(n+1)(n+5)$       (d)  $n(n-2)$   
 (e)  $(n-3)(n+1)$       (f)  $(n-8)(n-3)$

Question 7: Work out the  $n^{\text{th}}$  term for each quadratic sequence

- (a) 7, 12, 19, 28, 39 ...      (b) 7, 16, 31, 52, 79 ...      (c) 6, 13, 24, 39, 58 ...  
 (d) 3, 13, 27, 45, 67 ...      (e) 9, 20, 35, 54, 77 ...      (f) 9, 24, 45, 72, 105 ...  
 (g) -6, -1, 6, 15, 26 ...      (h) -5, -4, -1, 4, 11 ...      (i) 7, 10, 17, 28, 43 ...  
 (j) 2.5, 5, 8.5, 13, 18.5 ...      (k) -0.5, 1, 4.5, 10, 17.5 ...

Question 8: Calculate the 10<sup>th</sup> term of each sequence in question 7

Question 9: Work out the  $n^{\text{th}}$  term for each quadratic sequence

- (a) 3, 1, -3, -9, -17 ...      (b) -4, -12, -24, -40, -60 ...      (c) 6, 5, 2, -3, -10 ...  
 (d) 100, 96, 90, 82, 72 ...      (e) -17, -30, -49, -74, -105 ...      (f) 6, 5.5, 4.5, 3, 1 ...

Question 10: Calculate the 10<sup>th</sup> term of each sequence in question 9

Question 11: A sequence has an  $n^{\text{th}}$  term of  $n^2 + n - 20$   
 Work out which term in the sequence has a value of 52.

Question 12: A sequence has an  $n^{\text{th}}$  term of  $n^2 + 2n - 5$   
 Work out which term in the sequence has a value of 58.

Question 13: A sequence has an  $n^{\text{th}}$  term of  $n^2 - 6n + 7$   
 Work out which term in the sequence has a value of 23.

Apply

Question 1: The first 5 terms of a quadratic sequence are: 4, 10, 18, 28, 40  
 Work out the difference between the 10<sup>th</sup> and 20<sup>th</sup> terms.

## Purposeful Practice

**Question 2:** Below are patterns of tiles.  
The number of tiles in each form quadratic sequences.  
Find the number of tiles in pattern  $n$  for each.

(a)

Pattern 1      Pattern 2      Pattern 3

(b)

Pattern 1      Pattern 2      Pattern 3

(c)

Pattern 1      Pattern 2      Pattern 3

(d)

Pattern 1      Pattern 2      Pattern 3

**Question 3:** Here is a pattern made from tiles.  
How many tiles are needed to make Pattern 20?

Pattern 1      Pattern 2      Pattern 3

**Question 4:** The first 4 terms of a sequence are: 400, 390, 375, 355 ...  
Which term is the first to be negative?

**Question 5:** The  $n^{\text{th}}$  term of a quadratic sequence is  $n^2 + 4n$   
Two consecutive terms have a difference of 25.  
Work out the two terms.

**Question 6:** Prove every term in the sequence  $n^2 - 8n + 21$  is positive

## Fluency Practice

| Q  |                                                                                                                           | Answers |
|----|---------------------------------------------------------------------------------------------------------------------------|---------|
| 1  | The $n^{\text{th}}$ term of a quadratic sequence is $n^2 + n - 20$ . Work out which term has a value of 52.               |         |
| 2  | The $n^{\text{th}}$ term of a quadratic sequence is $n^2 + n - 20$ . Work out the value of the 52 <sup>nd</sup> term.     |         |
| 3  | The $n^{\text{th}}$ term of a quadratic sequence is $n^2 + 2n - 5$ . Work out which term has a value of 58.               |         |
| 4  | The $n^{\text{th}}$ term of a quadratic sequence is $n^2 + 2n - 5$ . Work out the value of the 58 <sup>th</sup> term.     |         |
| 5  | The $n^{\text{th}}$ term of a quadratic sequence is $n^2 - 6n + 7$ . Work out which term has a value of 23.               |         |
| 6  | The $n^{\text{th}}$ term of a quadratic sequence is $n^2 - 6n + 7$ . Work out the value of the 23 <sup>rd</sup> term.     |         |
| 7  | The $n^{\text{th}}$ term of a quadratic sequence is $2n^2 - 6n + 7$ . Work out which term has a value of 7.               |         |
| 8  | The $n^{\text{th}}$ term of a quadratic sequence is $2n^2 - 6n + 7$ . Work out the value of the 7 <sup>th</sup> term.     |         |
| 9  | The $n^{\text{th}}$ term of a quadratic sequence is $3n^2 - 29n + 75$ . Work out which term has a value of 7.             |         |
| 10 | The $n^{\text{th}}$ term of a quadratic sequence is $3n^2 - 29n + 75$ . Work out the value of the 7 <sup>th</sup> term.   |         |
| 11 | The $n^{\text{th}}$ term of a quadratic sequence is $an^2 + bn + c$ . Work out which term has a value of $25a + 5b + c$ . |         |
| 12 | The $n^{\text{th}}$ term of a quadratic sequence is $an^2 + bn + c$ . Work out the value of the 10 <sup>th</sup> term     |         |

## Fluency Practice

| Using the Nth Term of Quadratic Sequences                                          |                                                                         |                                                                                         |                                                                         |
|------------------------------------------------------------------------------------|-------------------------------------------------------------------------|-----------------------------------------------------------------------------------------|-------------------------------------------------------------------------|
| $u_n = n^2 + 3n - 5$                                                               | $u_n = 3n^2 - n + 1$                                                    | $u_n = n^2 - 2n$                                                                        | $u_n = n^2 + an - b$                                                    |
| <b>(a)</b>                                                                         | <b>(a)</b>                                                              | <b>(a)</b>                                                                              | <b>(a)</b>                                                              |
| Find the value of $u_4$                                                            | Find the value of $u_6$ .                                               | Find the 9 <sup>th</sup> term of the sequence.                                          | Find the value of $u_5$ in terms of $a$ and $b$ .                       |
| <b>(b)</b>                                                                         | <b>(b)</b>                                                              | <b>(b)</b>                                                                              | <b>(b)</b>                                                              |
| Find the difference between the 6 <sup>th</sup> term and the 7 <sup>th</sup> term. | Find the sum of the 9 <sup>th</sup> term and the 10 <sup>th</sup> term. | Find an expression for the $(n + 1)^{th}$ term.                                         | Find the value of $u_7$ in terms of $a$ and $b$ .                       |
| <b>(c)</b>                                                                         | <b>(c)</b>                                                              | <b>(c)</b>                                                                              | <b>(c)</b>                                                              |
| A term of the sequence is 65<br>Find the value of $n$ .                            | A term of the sequence is 103<br>Find the value of $n$ .                | Find an expression for the difference between the $n^{th}$ and the $(n + 1)^{th}$ term. | Given that $u_5 = 25$ and $u_7 = 70$ , find the values of $a$ and $b$ . |



## Fluency Practice

| Using the Nth Term of Sequences                                      |                                                                     |                                                                     |                                                                       |
|----------------------------------------------------------------------|---------------------------------------------------------------------|---------------------------------------------------------------------|-----------------------------------------------------------------------|
| $u_n = \frac{6n}{n+7}$                                               | $u_n = \frac{4n+3}{n+1}$                                            | $u_n = \frac{10-3n}{2+n}$                                           | $u_n = \frac{4n^2}{n^2+8}$                                            |
| <b>(a)</b>                                                           | <b>(a)</b>                                                          | <b>(a)</b>                                                          | <b>(a)</b>                                                            |
| Find the value of $u_8$ as a fraction in its simplest form.          | Find the value of $u_9$ .                                           | Find the $6^{th}$ term.                                             | Find the value of $u_5$ as a mixed number.                            |
| <b>(b)</b>                                                           | <b>(b)</b>                                                          | <b>(b)</b>                                                          | <b>(b)</b>                                                            |
| A term of the sequence is $\frac{11}{3}$<br>Find the value of $n$ .  | A term of the sequence is $\frac{15}{4}$<br>Find the value of $n$ . | A term of the sequence is $-\frac{7}{5}$<br>Find the value of $n$ . | Find the term in the sequence closest to 3.8                          |
| <b>(c)</b>                                                           | <b>(c)</b>                                                          | <b>(c)</b>                                                          | <b>(c)</b>                                                            |
| Find the difference between the $5^{th}$ term and the $9^{th}$ term. | Find the sum of the $4^{th}$ term and the $14^{th}$ term.           | Find $2u_{10} - u_{16}$                                             | Find the difference between the $8^{th}$ term and the $10^{th}$ term. |
| <b>(d)</b>                                                           | <b>(d)</b>                                                          | <b>(d)</b>                                                          | <b>(d)</b>                                                            |
| Find the first term in the sequence that is greater than 4           | Find the first term in the sequence that is greater than 3.9        | Find the first term in the sequence that is negative.               | Find the smallest value of $n$ for which $u_n > \frac{7}{2}$          |

## 3 Iterations

## Fluency Practice

- 1 Show that the equation  $x^3 + 2.5x^2 - 1 = 0$  has a solution between  $-1$  and  $0$ .

?

?

- 2 Show that the equation  $2x^3 + 5x^2 - 5x - 1 = 0$  has a solution between  $-3.5$  and  $-3$ .

?

?

## Fluency Practice

- 3 Show that the equation  $x - \sqrt{x} - 1 = 0$  has a solution between 2.6 and 2.7.

?

?

- 4 Show that the equation  $2x^3 - 7x^2 + 5x = 8$  has a solution between 3 and 3.5.

?

?

## Fluency Practice

**5** [CCEA GCSE Summer 2021 M8.2 Q2 Edited]

A solution to the equation  $4x - x^3 - 2 = 0$  lies between  $x = 0$  and  $x = 1$

Find this solution correct to 1 decimal place.

| $x$ | $4x - x^3 - 2$                      |
|-----|-------------------------------------|
| 0   | $4 \times 0 - 0^3 - 2 = -2$         |
| 0.1 | $4 \times 0.1 - 0.1^3 - 2 = -1.601$ |
| 0.2 | ?                                   |
| 0.3 | ?                                   |
| 0.4 | ?                                   |
| 0.5 | ?                                   |
| 0.6 | ?                                   |
| 0.7 | ?                                   |
| 0.8 | $4 \times 0.8 - 0.8^3 - 2 = 0.688$  |
| 0.9 | $4 \times 0.9 - 0.9^3 - 2 = 0.871$  |
| 1   | $4 \times 1 - 1^3 - 2 = 1$          |

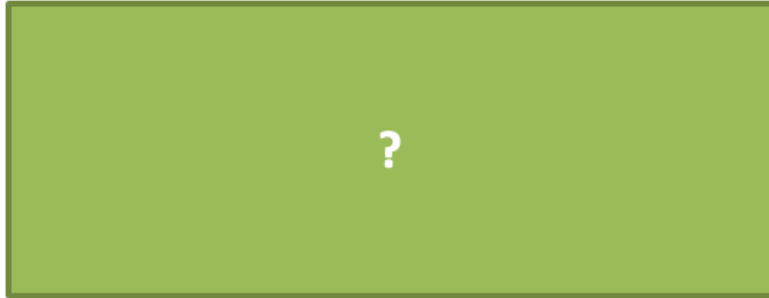
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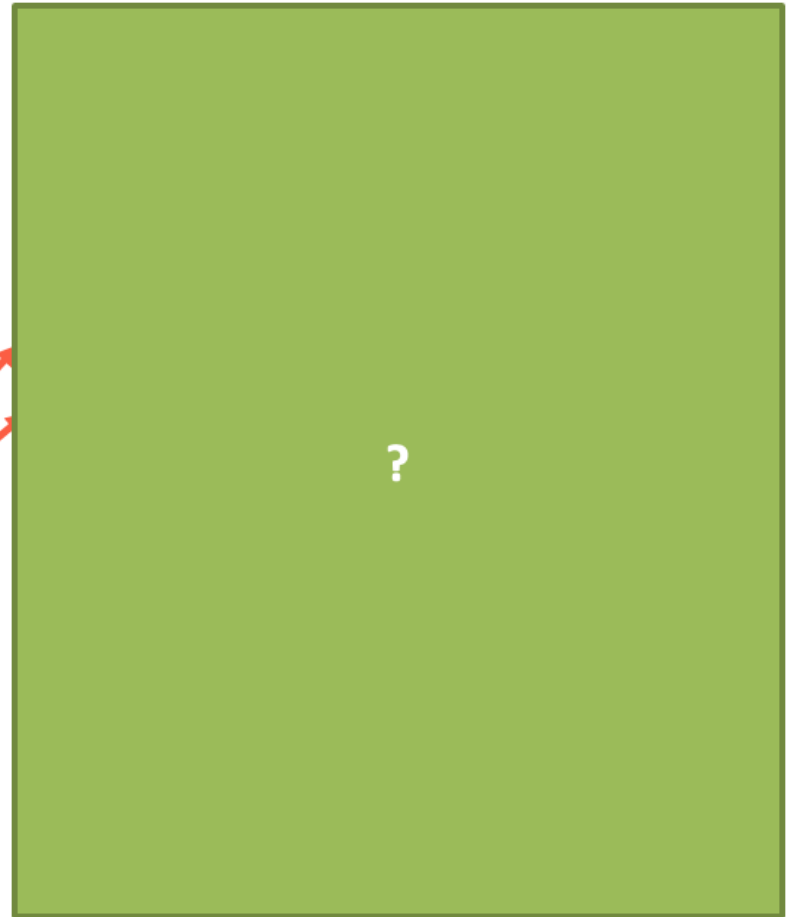
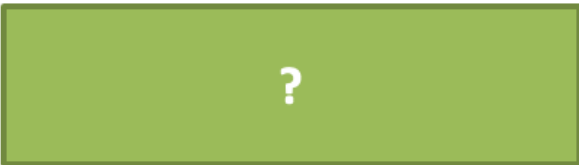
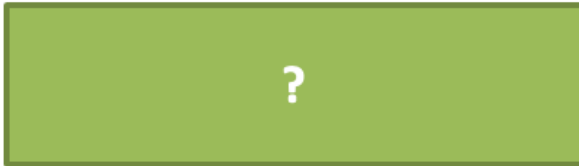
## Fluency Practice

6 [OCR GCSE Nov 2021 6H Q15]

a Show that the equation  $x^3 - 5x - 1 = 0$  has a solution between  $x = 2$  and  $x = 3$ .



b Find this solution correct to 1 decimal place.



## Fluency Practice

- 1 Show that the equation

$$x^2 - 9x + 4 = 0$$

can be rearranged to give

$$x = \sqrt{9x - 4}.$$

?

- 2 Show that the equation

$$2x^3 - 8x^2 + 10x - 4 = 0$$

can be rearranged to give

$$x = \frac{4 + 8x^2 - 2x^3}{10}.$$

?

- 3 Show that the equation

$$2x^3 + 4x^2 - 3x - 2 = 0$$

can be rearranged to give

$$x = \sqrt[3]{\frac{2 + 3x - 4x^2}{2}}.$$

?

## Fluency Practice

4 We wish to solve the equation

$$3x^3 - 10x^2 + 9x - 2 = 0$$

This can be rearranged to

$$x = \frac{-3x^3 + 10x^2 + 2}{9}$$

Starting with  $x_0 = 1$ , use the iteration formula

$$x_{n+1} = \frac{-3(x_n)^3 + 10(x_n)^2 + 2}{9}$$

to find the value of  $x_2$ .

Give your answer correct to 3 decimal places.



5

We wish to solve the equation

$$x^3 - 5x^2 + 3x + 9 = 0$$

This can be rearranged to

$$x = \sqrt[3]{5x^2 - 3x - 9}$$

Starting with  $x_0 = 2$ , use the iteration formula

$$x_{n+1} = \sqrt[3]{5(x_n)^2 - 3x_n - 9}$$

to find the value of  $x_2$ .

Give your answer correct to 3 decimal places.





## Fluency Practice

**6** We wish to solve the equation

$$3x^3 + 10x^2 + 9x + 2 = 0$$

**a** Show that this can be rearranged to

$$x = \frac{-3x^3 - 10x^2 - 2}{9}$$



**b** Starting with  $x_0 = -2$ , use the iteration formula

$$x_{n+1} = \frac{-3(x_n)^3 - 10(x_n)^2 - 2}{9}$$

to find the value of  $x_1$ .

Give your answer correct to 3 decimal places.



## Fluency Practice

7

We wish to solve the equation

$$3x^3 - 10x^2 + 9x - 2 = 0$$

a

Show that this can be rearranged to

$$x = \sqrt[3]{\frac{10x^2 - 9x + 2}{3}}.$$

b

Starting with  $x_0 = 0$ , use the iteration formula

$$x_{n+1} = \sqrt[3]{\frac{10(x_n)^2 - 9x_n + 2}{3}}$$

to find the value of  $x_1$ .

Give your answer correct to 3 decimal places.

?

?

## Fluency Practice

8

*[Edexcel GCSE Nov 2017 H3 Q15]*

a

Show that the equation  $x^3 + 7x - 5 = 0$  has a solution between  $x = 0$  and  $x = 1$ .

(2 marks)

?

b

Show that the equation  $x^3 + 7x - 5 = 0$  can be arranged to give  $x = \frac{5}{x^2 + 7}$ .

(2 marks)

?

## Fluency Practice

8

- c Starting with  $x_0 = 1$ , use the iteration formula  $x_{n+1} = \frac{5}{x_n^2 + 7}$  three times to find an estimate for the solution of  $x^3 + 7x - 5 = 0$ .

(3 marks)

?

- d By substituting your answer to part (c) into  $x^3 + 7x - 5$ , Comment on the accuracy of your estimate for the solution to  $x^3 + 7x - 5 = 0$ .

(2 marks)

?

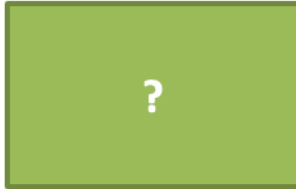
## Fluency Practice



[Edexcel GCSE Nov 2021 H2 Q16]

a

Use the iteration formula  $x_{n+1} = \sqrt[3]{10 - 2x_n}$  to find the values of  $x_1$ ,  $x_2$  and  $x_3$ .  
Start with  $x_0 = 2$



[3 marks]

b

The values of  $x_1$ ,  $x_2$  and  $x_3$  found in part (a) are estimate of the solution of an equation of the form  $x^3 + ax + b = 0$  where  $a$  and  $b$  are integers.

Find the value of  $a$  and the value of  $b$ .



[1 mark]

# Purposeful Practice

## Solving Quadratic Equations using Iteration

Quadratic Equation:  $x^2 - 2x - 1 = 0$

(rearrange)

Iterative Formula:  $x_{n+1} = \sqrt{2x_n + 1}$

Initial Value:  $x_1 = 3$

|          |            |
|----------|------------|
| $x_2$    | 2.64575... |
| $x_3$    | 2.50828... |
| $x_4$    | 2.45287... |
| $x_5$    | 2.43017... |
| $x_6$    | 2.42081... |
| $x_7$    | 2.41694... |
| $x_8$    | 2.41534... |
| $x_9$    | 2.41468... |
| $x_{10}$ | 2.41440... |
| $x_{11}$ | 2.41429... |
| $x_{12}$ | 2.41424... |

$x = 2.4$   
Both solutions  
are the same  
(to 1 dp).

$x = 2.41$   
Both solutions  
are the same  
(to 2 dp).

**A** For this quadratic equation find the solutions,  
to 1 dp & to 2 decimal places.

$$x^2 - 3x - 1 = 0$$

Iterative Formula:  $x_{n+1} = \sqrt{3x_n + 1}$

Initial Value:  $x_1 = 4$

|          |          |
|----------|----------|
| $x_2$    |          |
| $x_3$    | 3.437... |
| $x_4$    |          |
| $x_5$    |          |
| $x_6$    |          |
| $x_7$    |          |
| $x_8$    |          |
| $x_9$    | 3.303... |
| $x_{10}$ |          |
| $x_{11}$ |          |
| $x_{12}$ |          |

**B** Find a solution for  
this equation to 2 dp & to 3 dp.

$$x^2 - 5x + 2 = 0$$

Iterative Formula:  $x_{n+1} = \sqrt{5x_n - 2}$

Initial Value:  $x_1 = 5$

|          |           |
|----------|-----------|
| $x_2$    |           |
| $x_3$    |           |
| $x_4$    |           |
| $x_5$    | 4.5991... |
| $x_6$    |           |
| $x_7$    |           |
| $x_8$    |           |
| $x_9$    |           |
| $x_{10}$ |           |
| $x_{11}$ |           |
| $x_{12}$ |           |
| $x_{13}$ |           |

**C** Find a solution for  
this equation to 3 dp.  
Choose your own initial value.

$$x^2 - 4x - 7 = 0$$

**D** Find a solution for  
this equation to 1 dp.

$$x^2 + 4x - 13 = 0$$

Why do only some initial values work?

**E** Find a solution for  
this cubic equation to 3 dp.  
 $x^3 - 5x + 3 = 0$   
(Rearrange so  $x^3$  is by itself,  
then take the cube root of both sides).

## Purposeful Practice

- 1 The sequence  $a_n$  is defined such that  $a_{n+1} = a_n + 3$  and  $a_0 = 4$ .
  - (a) Find the values of  $a_1$ ,  $a_2$  and  $a_3$ .
  - (b) What type of sequence is  $a_n$ ?
- 2 The sequence  $u_n$  is defined such that  $u_{n+1} = 2u_n - 3$  and  $u_1 = -2$ .
  - (a) Find the values of  $u_2$ ,  $u_3$  and  $u_4$ .
  - (b) Find the value of  $u_0$ .
- 3 Given that  $x_{n+1} = 3 - \frac{x_n}{4}$  and  $x_1 = 1$ , find the values of
  - (a)  $x_2$ ,  $x_3$  and  $x_4$ ,
  - (b)  $x_0$ .

Where appropriate, give each value to three decimal places.
- 4 Given that  $x_{n+1} = x_n^2 - \frac{7}{8}$  and  $x_0 = 0$ ,
  - (a) write down the smallest term in the sequence.
  - (b) Find the values of  $x_1$ ,  $x_2$  and  $x_3$ , giving each value to four significant figures.
- 5 A deadly disease infects a population of plants.  
 The number of plants infected with the disease  $n$  days after the first infection,  $p_n$ , is given by
 
$$p_{n+1} = 4p_n + 6$$
 and  $p_0 = 3$ .
  - (a) Interpret the meaning of  $p_0$  in this context.
  - (b) Find the number of plants infected with the disease 2 days after the first infection.  
 There are 20000 plants in the population.
  - (c) Show that all the plants in the population will be infected with the disease after 6 days.
- 6 Markov invests  $\pounds P$  into a bank account.  
 He is paid  $r\%$  per annum compound interest and takes out  $\pounds D$  a year from this account.  
 The amount of money in the account after  $n$  years is  $A_n$ .
  - (a) Show that
 
$$A_{n+1} = A_n \left( 1 + \frac{r}{100} \right) - D$$

Markov invests  $\pounds 1000$  into the account and the interest rate is 2%.  
 After 3 years, Markov has  $\pounds 143$  in the savings account.

  - (b) How much money does Markov take out of the account each year? Give your answer to the nearest pound.

## Purposeful Practice

- 7 Here is a table of values for the curve with equation  $y = f(x)$ .

|        |     |    |    |    |    |
|--------|-----|----|----|----|----|
| $x$    | -3  | -2 | -1 | 0  | 1  |
| $f(x)$ | -10 | -3 | 2  | 15 | 82 |

The equation  $f(x) = 0$  has at least one solution in the interval  $x = a$  and  $x = b$ .

Using the table, write down the value of  $a$  and  $b$ . Explain your reasoning.

- 8 Show that the equation  $x^3 - 2x^2 - 7x + 1 = 0$  has a solution between  $x = 3$  and  $x = 4$ .
- 9 Show that the equation  $5x^3 + x^2 - 8x + 4 = 0$  has a solution between  $x = -2$  and  $x = -1$ .
- 10 (a) Show that the equation  $-3x^3 + 2x^2 + 4x + 5 = 0$  has a solution  $\alpha$  between  $x = 1.5$  and  $x = 2$ .

(b) Show that the equation  $-3x^3 + 4x + 5 = 0$  can be re-arranged to give  $x = \left(\frac{4x+5}{3}\right)^{\frac{1}{3}}$ .

(c) Starting with  $x_0 = 1.5$ , use the iterative formula

$$x_{n+1} = \left(\frac{4x_n + 5}{3}\right)^{\frac{1}{3}}$$

three times to find an estimate for  $\alpha$ .

Give your answer to three decimal places.

(d) The actual value of  $\alpha = 1.5516$  to four decimal places.

(i) Find the percentage error in your estimate in (c).

(ii) Explain how you could make your approximation in (c) more accurate.



## Fluency Practice

### iteration 1: roots & intervals

1. Show that there is a root of the equation  $x^3 + x - 3 = 0$  in the interval  $(1, 2)$ .

2. Show that there is a root of the equation  $x^2 - 5x + 2 = 0$  in the interval  $(0.4, 0.5)$ .

3. Here is a table of values for the continuous function  $f(x)$ :

|        |     |     |     |     |      |      |
|--------|-----|-----|-----|-----|------|------|
| $x$    | 1   | 2   | 3   | 4   | 5    | 6    |
| $f(x)$ | 1.9 | 1.6 | 1.1 | 0.4 | -0.5 | -1.6 |

State two consecutive values of  $x$  between which there is a solution to the equation  $f(x) = 0$ .

4. Here is a table of values for the continuous function  $g(x)$ :

|        |      |      |      |       |       |       |      |
|--------|------|------|------|-------|-------|-------|------|
| $x$    | -3   | -2   | -1   | 0     | 1     | 2     | 3    |
| $g(x)$ | 2.33 | 3.52 | 2.16 | -0.24 | -2.20 | -2.20 | 1.24 |

State **two pairs** of consecutive values of  $x$  between which there are solutions to the equation  $g(x) = 0$ .

5.  $f(x)$  is a continuous function. Choose the correct statement for each of the following:

(a)  $f(3) = 2.34$   
 $f(4) = -0.01$

(b)  $f(6) = -4$   
 $f(7) = -1.52$

(c)  $f(-1) = 0.62$   
 $f(0.5) = 0.74$

(d)  $f(7) < 0$   
 $f(8) > 0$

#### Statements

① There is a root of  $f(x) = 0$  in the interval.

② There is **not** a root of  $f(x) = 0$  in the interval.

③ It is not possible to tell if there is a root of  $f(x) = 0$  in the interval.

6.  $f(x) = x^2 - 4x - 3$ .

(a) Show that there is a root of the equation  $f(x) = 0$  in the interval  $(4, 5)$ .

(b) Work out  $f(4.5)$ .

(c) State an interval that is smaller than  $(4, 5)$  and contains the root.

7. (a)  $a = 2.2$  correct to 1 decimal place. State the lower and upper bounds of  $a$ .

(b) Use the bounds as an interval to prove that  $a$  is root of the equation  $x^3 - x - 9 = 0$ , correct to 1 decimal place.

## Fluency Practice

### Iteration 2: recurrence relations

1. Given the recurrence relation  $x_{n+1} = 3x_n - 2$  and  $x_0 = 2$ , work out  $x_1, x_2, x_3$  and  $x_4$ .
2. Given the recurrence relation  $x_{n+1} = (x_n)^2 - 5$  and  $x_0 = 3$ , work out  $x_1, x_2, x_3$  and  $x_4$ .
3. Given that  $P_{n+1} = 1.05P_n$  and  $P_0 = 2000$ , work out  $P_4$  correct to 3 significant figures.
4. Given that  $U_{n+1} = 1.02(U_n - 50)$  and  $U_0 = 1500$ , work out  $U_3$  correct to 3 significant figures.
5. Match the recurrence relation rules to the correct sequence of values ( $x_0, x_1, x_2, x_3$ ):

A:  $x_{n+1} = 3x_n - 1$

P: 3, 5, 7, 9

B:  $x_{n+1} = x_n + 2$

Q: 2, 5, 14, 41

C:  $x_{n+1} = (x_n)^2 + 1$

R: 2, 5, 8, 11

D:  $x_{n+1} = x_n + 3$

S: 3, 6, 15, 42

E:  $x_{n+1} = 3(x_n - 1)$

T: 2, 5, 26, 677

6. An approximate solution to the equation  $x^3 - 8x + 4 = 0$  can be found using the iterative formula:

$$x_{n+1} = \frac{x_n^3 + 4}{8}, \quad x_0 = 0$$

Work out  $x_1, x_2, x_3$  and  $x_4$ . Round each value to 4 decimal places where necessary.

7. (a) Show that the equation  $x^3 - 5x - 2 = 0$  has a root between  $x = -1$  and  $x = 0$ .

(b) Starting with  $x_0 = 0$ , use the iterative formula:

$$x_{n+1} = \frac{x_n^3 - 2}{5}$$

three times to find an estimate for the root of  $x^3 - 5x - 2 = 0$ . Round your answer to 2 decimal places.

8. (a) Show that the equation  $x^3 + x^2 = 10x - 6$  has a root in the interval  $(0, 1)$ .

(b) Starting with  $x_0 = 1$ , use the iterative formula

$$x_{n+1} = \frac{x_n^3 + x_n^2 + 6}{10}$$

three times to find an estimate for the root of  $x^3 + x^2 = 10x - 6$ . Round your answer to 3 significant figures.

## Fluency Practice

### iteration 3: rearrangement

1. Show that the equation  $x^3 - 8x - 5 = 0$  can be rearranged to each of these forms:

(a)  $x = \sqrt[3]{8x + 5}$       (b)  $x = \frac{x^3 - 5}{8}$

(c)  $x = \frac{5}{x^2 - 8}$       (d)  $x = \pm \sqrt{\frac{8x + 5}{x}}$

2. (a) Show that the equation  $x^3 - 4x - 5 = 0$  has a root in the interval  $(2, 3)$ .

(b) Show that the equation  $x^3 - 4x - 5 = 0$  can be rearranged to give  $x = \sqrt[3]{4x + 5}$ .

- (c) Starting with  $x_0 = 2$ , use the iteration formula

$$x_{n+1} = \sqrt[3]{4x_n + 5}$$

three times to find an estimate for the root of  $x^3 - 4x - 5 = 0$ . Round your estimate to 3 significant figures.

3. (a) Show that the equation  $x^3 + 3x - 2 = 0$  has a root between  $x = 0$  and  $x = 1$ .

(b) Show that the equation  $x^3 + 3x - 2 = 0$  can be rearranged to give  $x = \frac{2}{x^2 + 3}$ .

- (c) Starting with  $x_0 = 0.5$ , use the iteration formula

$$x_{n+1} = \frac{2}{x_n^2 + 3}$$

twice to find an estimate for the root of  $x^3 + 3x - 2 = 0$ . Round your answer to 2 decimal places.

4. (a) Show that the equation  $x^3 - 4x - 2 = 0$  has a root in the interval  $(-1, 0)$ .

(b) Show that the equation  $x^3 - 4x - 2 = 0$  can be rearranged to give  $x = \frac{x^3 - 2}{4}$ .

- (c) Starting with  $x_0 = -0.5$ , use the iteration formula

$$x_{n+1} = \frac{x_n^3 - 2}{4}$$

three times to find an estimate for the root of  $x^3 - 4x - 2 = 0$ . Round your answer to 3 significant figures.

## Fluency Practice

### iteration 4: review

1. The expected number of birds in a colony  $t$  years from now is  $P_t$ , where

$$P_0 = 600$$

$$P_{t+1} = 1.025P_t$$

Work out the expected number of birds in the colony 3 years from now.

2. The value of an investment  $t$  years from now is  $£V_t$ , where

$$V_0 = 20,000$$

$$V_{t+1} = 1.02V_t + 2000$$

Work out the value of the investment 2 years from now.

3. The expected value of a car  $n$  years from now is  $£c_n$ , where

$$c_0 = 18,000$$

$$c_{n+1} = 0.85c_n - 500$$

Work out the expected value of the car 2 years from now.

4. (a) Show that the equation  $x^3 - 6x - 6 = 0$  has a root in the interval  $(2, 3)$ .

(b) Show that the equation  $x^3 - 6x - 6 = 0$  can be rearranged to give  $x = \sqrt{\frac{6x+6}{x}}$ .

- (c) Starting with  $x_0 = 3$ , use the iteration formula

$$x_{n+1} = \sqrt{\frac{6x_n+6}{x_n}}$$

three times to find an estimate for the root of  $x^3 - 6x - 6 = 0$ . Round your answer to 2 decimal places.

5. (a) Show that the equation  $x^3 + 2x + 8 = 0$  has a root between  $x = -2$  and  $x = -1$ .

(b) Show that the equation  $x^3 + 2x + 8 = 0$  can be rearranged to give  $x = \sqrt[3]{-2x - 8}$ .

- (c) Starting with  $x_0 = -1.5$ , use the iteration formula

$$x_{n+1} = \sqrt[3]{-2x_n - 8}$$

three times to find an estimate for the root of  $x^3 + 2x + 8 = 0$ . Round your answer to 2 decimal places.

## Purposeful Practice

### solving cubic equations by iteration

example

find a solution to  $x^3 - 10x = 30$  between 4 and 5  
give your answer correct to 3 decimal places

$$x^3 - 10x = 30$$

$$x = \sqrt[3]{(30 + 10x)}$$

$$x_{n+1} = \sqrt[3]{(30 + 10x_n)}$$
 the iteration equation

do  $\sqrt[3]{(30 + 10\text{Ans})}$  enter the smaller number, 4, then =

press the = button

carries forward the previous answer,  $x_n$ , to obtain the next one,  $x_{n+1}$

repeatedly...

until you have a good reason to stop

the initial value,  $x_1$ , is 4

round this number to 3 decimal places

4.150638865  
takes 12 steps on a calculator

4.151

use an iteration method to try to find a solution

(1)  $x^3 + 2x = 40$  between 3 and 4

(2)  $x^3 + 10x = 51$  between 2 and 3

(3)  $x^3 + 2x = 1$  between 0 and 1

(4)  $x^3 + 4x = 10$  between 1 and 2

(5)  $x^3 + 10x = 25$  between 1 and 2

what happens?

explore further...

## Purposeful Practice

### solving cubic equations by iteration

example: another rearrangement

find a solution to  $x^3 - 4x = 10$  between 2 and 3  
give your answer correct to 3 decimal places

$$x^3 - 4x = 10$$

$$x^3 = 4x + 10$$

$$x^2 = 4 + \frac{10}{x}$$

$$x_{n+1} = \sqrt{4 + 10/x_n}$$

the iteration equation  
enter the 'seed' value  
of  $x_1 = 2$  [ 2 = ]

$$\text{do } \sqrt{4 + 10/\text{Ans}}$$

(15 steps on a calculator)

2.760817834

2.761

what happens if you  
enter a 'seed' value of

$x_1 = -1$  or

$x_1 = -2$ ? why?

what happens for seed  
values less than  $-2$ ?

- (6) a race: use two rearrangements  
to try to find a solution to:

$x^3 - 3x = 2$  between 1 and 3  
(although the solution is obvious)

$$(i) \quad x_{n+1} = \sqrt[3]{2 + 3x_n}$$

$$(ii) \quad x_{n+1} = \sqrt{3 + 2/x_n}$$

which iteration equation is faster - needs  
fewer steps (iterations)?

- (7) find a solution to:

$x^3 - 5x = 15$  between 3 and 4 using:

$$(i) \quad x_{n+1} = \sqrt[3]{15 + 5x_n}$$

$$(ii) \quad x_{n+1} = \sqrt{5 + 15/x_n}$$

which iteration equation is faster?

- (8) find a solution to:

$x^3 - 5x = 2$  between 2 and 3 using two  
different iteration equations

## Purposeful Practice

### a different type of cubic equation

(9) find a solution to  $x^3 + 3x^2 = 4$  with  $x_1 = 3$

$$x^3 + 3x^2 = 4$$

$$x^2(x + 3) = 4$$

$$x^2 = \frac{4}{x + 3}$$

$$x_{n+1} = \sqrt{\frac{4}{x_n + 3}}$$

what happens for  $x_1 = -4$  ?

try to find a solution to  $x^3 + 3x^2 = 4$  with  $x_1 = 3$  using:

$$x_{n+1} = \sqrt[3]{4 - 3x_n^2}$$

what happens ?

### using different iteration equations

(10) find a solution to  $x^3 - 8x = 3$  with  $x_1 = 2$

$$x^3 - 8x = 3$$

$$x^3 - 3 = 8x$$

$$8x = x^3 - 3$$

$$x_{n+1} = \frac{x_n^3 - 3}{8}$$

try other  $x_1$  values

find the (small) range of  $x_1$  values for which the sequence converges (tends to a limit)

find a solution to  $x^3 - 8x = 3$  with  $x_1 = 2$  using

$$x_{n+1} = \sqrt[3]{3 + 8x_n}$$

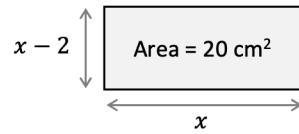
$$x_{n+1} = \sqrt{8 + 3/x_n}$$

$$x_{n+1} = -\sqrt{8 + 3/x_n}$$

# Purposeful Practice

## Forming & Solving Equations using Iteration

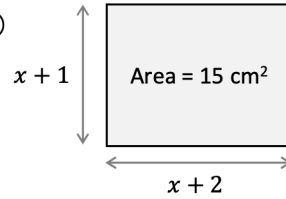
①



- 1) Form a quadratic equation ( $= 0$ ).
- 2) Rearrange to form an iterative function.  
(Use a square root to form  $x_{n+1} =$ )
- 3) Use the function to find the value of  $x$  (to 2 dp)

How can we check our answer?

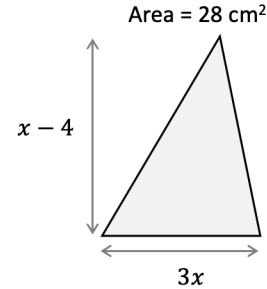
②



Use iteration to find the value of  $x$  to 2 dp

Why do we need to be careful choosing an initial value?

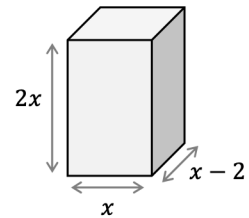
③



Use iteration to find the value of  $x$  to 3 dp

④

Volume  
= 132 cm<sup>3</sup>



Use iteration to find the value of  $x$  to 3 dp



## 4 Algebraic Proof

# Fluency Practice

## Prove It!

a)  $2n + 4(n + 1) + 4$   
is always even

b)  $2(n + 1) + 4(n + 2)$   
is always even

c)  $5(n + 3) - 3(n - 1)$   
is always even

d)  $4(2n - 3) - 3(2n - 1)$   
is always odd

A number ( $n$ )  
An even number  $2(n)$   
An odd number  $2(n) + 1$

e)  $(n + 1)^2 - n^2 + 1$   
is always even

f)  $(n + 2)^2 - n^2$   
is always even

g)  $n^2 + (n + 1)^2$   
is always odd

h)  $n^2 + (n - 3)^2$   
is always odd

i)  $(2n + 1)^2 + 3$   
is divisible by 4

j)  $(n + 1)^2 + (n + 2)^2$   
is always odd

k)  $(n + 3)^2 - (n + 1)^2$   
is divisible by 4

l)  $(n - 4)^2 - (n - 2)^2$   
is even

m)  $(2n + 1)^3$   
is always odd

n)  $(4n + 1)^2 - (4n - 1)^2$   
is divisible by 8

o)  $(2n - 1)^2 - 3(n - 2)^2 - 2(n - 10)$   
is a square number

# Fluency Practice

1.  $n$  is a positive integer. Write each expression in the correct box.

$6n$

$n + 2$

$2n - 3$

$5n$

$4n + 8$

$2n + 1$

$(2n)^2$

$4(n + 1)$

$2n^2 + 4n - 3$

$n^2 + 1$

Always

an even number

Sometimes

an even number

Never

an even number

|    | Example                                                                                                             | Your turn                                                                                                            |
|----|---------------------------------------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------|
| 2. | $n$ is a positive integer. Prove algebraically that $6(n + 3) - n + 2$ is <b>always</b> a <b>multiple of 5</b> .    | $n$ is a positive integer. Prove algebraically that $n(2n + 1) + n^2 + 2n$ is <b>always</b> a <b>multiple of 3</b> . |
| 3. | $n$ is a positive integer. Prove algebraically that $(n + 2)^2 - n^2$ is <b>always</b> a <b>multiple of 4</b> .     | $n$ is a positive integer. Prove algebraically that $(4n + 1)^2 - 1$ is <b>always</b> a <b>multiple of 8</b> .       |
| 4. | $n$ is a positive integer. Prove algebraically that $(4n + 3)^2 + 2(n + 1)$ is <b>always</b> an <b>odd number</b> . | $n$ is a positive integer. Prove algebraically that $(n + 1)(n + 2) - n^2$ is <b>never</b> a <b>multiple of 3</b> .  |

## Fluency Practice

expanding quadratic brackets ( $n$  is a positive integer)

proofs

(1) prove that  $(n + 6)(n + 3)$  is always 10 greater than  $(n + 8)(n + 1)$

(2) prove that  $(n + 1)(n + 2) - n(n + 3)$  is always 2

(3) prove that  $n(n + 1) + (n + 1)(n + 2)$  is double a square number

(4) prove that  $(n - 3)(n - 2)$  is always 2 greater than  $(n - 4)(n - 1)$

(5) prove that  $(3n - 1)(3n + 1) + 1$  is  $9 \times$  a square number

(6) prove that  $(2n + 1)(2n + 3) + 1$  is an even square number

(7) prove that  $(3n + 2)^2$  is always 1 more than  $(3n + 1)(3n + 3)$

(8) prove that  $(3n + 1)(3n + 2) - 3n(3n + 3)$  is always 2

(9) prove that  $(2n + 1)^2 + (n + 3)^2$  is always a multiple of 5

# Fluency Practice

- $n$  is a positive integer. Find a counter-example to show that these statements are **false**:
  - $3n + 1$  is always even.
  - $6n - 1$  is always prime.

|    | Example                                                                                                                                                                   | Your turn                                                                                          |
|----|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------|
| 2. | Prove algebraically that for any three consecutive integers, the difference between the square of the middle number and the product of the two outer numbers is always 1. | Prove algebraically that the sum of any pair of consecutive odd numbers is always a multiple of 4. |
| 3. | Prove algebraically that the sum of any two odd numbers is an even number.                                                                                                | Prove algebraically that the difference between any two odd numbers is an even number.             |

- $n$  is a positive integer. Prove algebraically that  $(n + 3)^2 - (n + 1)^2$  is a multiple of 4.
- Prove that the difference between the squares of two consecutive odd numbers is always a multiple of 8.

## Fluency Practice

| Algebraic Proof with Multiples                              |                                                                          |                                                                               |
|-------------------------------------------------------------|--------------------------------------------------------------------------|-------------------------------------------------------------------------------|
| <b>(a)</b>                                                  | <b>(b)</b>                                                               | <b>(c)</b>                                                                    |
| Show that $4(x + 3) + x - 2$ is always a multiple of 5      | Show that $3(7 + 2x) - 9$ is always a multiple of 6                      | Show that $4(x + 7) + 2(1 - x)$ is always a multiple of 2                     |
| <b>(d)</b>                                                  | <b>(e)</b>                                                               | <b>(f)</b>                                                                    |
| Show that $7(2x - 1) - 5(x - 2)$ is always a multiple of 3  | Show that $(x + 7)(x - 2) - x^2 - 1$ is always a multiple of 5           | Show that $(x + 8)(x + 1) - x(x + 5)$ is always a multiple of 4               |
| <b>(g)</b>                                                  | <b>(h)</b>                                                               | <b>(i)</b>                                                                    |
| Show that $(x + 5)^2 + (x - 3)^2$ is always a multiple of 2 | Show that $(3x + 5)(2x - 1) + (2x + 1)(x + 1)$ is always a multiple of 4 | Show that $(3x + 2)^2 - (x + 4)(3x - 2) + 4(x - 3)$ is always a multiple of 6 |

## Fluency Practice

- (a) Show that  $4(n + 3) - n$  is a multiple of 3 for all integer values of  $n$
- (b) Show that  $(n + 2)^2 + 3n^2$  is a multiple of 4 for all integer values of  $n$
- (c) Show that  $(3n - 1)^2 - (2n + 1)^2$  is a multiple of 5 for all integer values of  $n$
- (d) Show that  $(2n + 1)(4n - 3) - (n + 2)^2 - n$  is a multiple of 7 for all integer values of  $n$

- (a) Show that the sum of three consecutive integers is always a multiple of 3
- (b) Show that the sum of three consecutive even numbers is always a multiple of 6
- (c) Show that the product of two consecutive even numbers is always a multiple of 4

- (a) Prove algebraically that the sum of three consecutive square numbers is never a multiple of 3
- (b) Prove algebraically that the sum of the squares of any two odd numbers is never a multiple of 4
- (c) Prove algebraically that the product of two consecutive odd numbers is never a multiple of 4

- (a) Prove algebraically that the product of three consecutive even numbers is always a multiple of 8
- (b) Prove algebraically that the sum of the cubes of two consecutive even numbers is always a multiple of 8
- (c) Prove algebraically that the product of the squares of two odd numbers is always one more than a multiple of 4

## Fluency Practice

Write down algebraic expressions for:

- (a) two different odd numbers
- (b) two consecutive numbers
- (c) two consecutive even numbers
- (d) two different odd numbers squared

- (a) Prove algebraically that the sum of any two odd numbers is always even.
- (b) Prove algebraically that the product of an odd number and an even number is always even
- (c) Prove algebraically that the sum of two consecutive numbers is always odd
- (d) Prove algebraically that the product of two consecutive even numbers is always even

- (a) Prove algebraically that the mean of two consecutive odd numbers is always even
- (b) Prove algebraically that the difference between an odd number and an even number is always odd
- (c) Prove algebraically that the mean of three consecutive odd numbers is always equal to the middle number

- (a) Prove algebraically that the sum of the squares of two even numbers is always a multiple of 4
- (b) Prove algebraically that the difference between the squares of two odd numbers is always a multiple of 4
- (c) Prove algebraically that the sum of the squares of two consecutive numbers is always odd

- (a) Show for all integers values of  $n$  that  $(n + 3)^2 - n(n - 6) + 2$  is always odd
- (b) Show for all integer values of  $n$  that  $(3n + 5)^2 + (3 - n)^2$  is always even



## Fluency Practice

- (a) Prove that the difference of two odd numbers is even.
- (b) Prove that  $n^2 - 2 - (n - 2)^2$  is always an even number.
- (c) Prove that the product of two odd numbers is always odd.
- (d) Prove that the square of an even number is always even.
- (e) Prove that the difference between any two consecutive odd numbers is always two.
- (f) Prove that the mean of three consecutive integers is always the middle number.
- (g) Prove that the difference between the squares of any two consecutive numbers is always odd.
- (h) Prove that  $7(n + 8) + 5(n - 4)$  is always a multiple of 12.
- (i) Prove that  $(m + 2)^2 - m^2 - 12$  is always a multiple of 4.
- (j) Prove that the sum of three consecutive odd numbers is always a multiple of three.
- (k) Prove that the sum of the squares of any two positive odd integers is always even.
- (l) Prove that  $(3n + 1)^2 - (3n - 1)^2$  is always a multiple of 12 for all positive integer values of  $n$ .
- (m) Prove that the sum of four consecutive integers is not divisible by four.

## Fluency Practice

| Q  | Prove algebraically that:                                                                            | Answers. Even $2n/2k$ Odd $2n + 1/2k + 1$ |
|----|------------------------------------------------------------------------------------------------------|-------------------------------------------|
| 1  | the product of two consecutive integers is always even                                               |                                           |
| 2  | the sum of two consecutive integers is always odd                                                    |                                           |
| 3  | the difference between two even numbers is even                                                      |                                           |
| 4  | the difference between two odd numbers is even                                                       |                                           |
| 5  | the sum of the squares of any two even numbers is a multiple of 4                                    |                                           |
| 6  | the sum of the squares of any two odd numbers is 2 more than a multiple of 4                         |                                           |
| 7  | the sum of three consecutive even integers is even                                                   |                                           |
| 8  | the sum of three consecutive odd integers is odd                                                     |                                           |
| 9  | the product of three consecutive even integers is even                                               |                                           |
| 10 | the product of three consecutive odd integers is odd                                                 |                                           |
| 11 | the sum of the squares of any three consecutive even numbers is always 4 less than a multiple of 12  |                                           |
| 12 | the sum of the squares of any three consecutive odd numbers is always 11 more than a multiple of 12  |                                           |
| 13 | $(2n + 1)^2 - (2n + 1)$ is an even number for all positive integer values of $n$                     |                                           |
| 14 | $(3n + 2)^2 - (n - 1)^2$ is an odd number for all positive integer values of $n$                     |                                           |
| 15 | $n^2 - 2 - (n - 2)^2$ is always an even number for all positive integer values of $n$ greater than 1 |                                           |
| 16 | $n^2 - 1 - (n - 4)^2$ is always an odd number for all positive integer values of $n$ greater than 1  |                                           |

## Purposeful Practice

### triangular number proofs

(1)  $1 + 3 = 4$   
 $3 + 6 = 9$   
 $6 + 10 = 16$   
 $\dots$   
 general rule? proof?

(2)  $3 - 1 = 2$   
 $6 - 3 = 3$   
 $10 - 6 = 4$   
 $\dots$   
 general rule? proof?

(3)  $1 \times 3 = 3 \times 1$   
 $2 \times 6 = 4 \times 3$   
 $3 \times 10 = 5 \times 6$   
 $\dots$   
 general rule? proof?

(4)  $8 \times 1 + 1 = 3 \times 3$   
 $8 \times 3 + 1 = 5 \times 5$   
 $8 \times 6 + 1 = 7 \times 7$   
 $\dots$   
 general rule? proof?

(5)  $3 - 2 \times 1 = 1$   
 $10 - 2 \times 3 = 4$   
 $21 - 2 \times 6 = 9$   
 $\dots$   
 general rule? proof?

(6)  $3 \times 3 + 1 = 10$   
 $3 \times 6 + 3 = 21$   
 $3 \times 10 + 6 = 36$   
 $\dots$   
 general rule? proof?

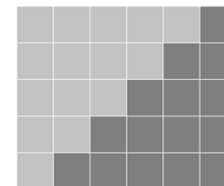
(7)  $3 \times 3 + 6 = 15$   
 $3 \times 6 + 10 = 28$   
 $3 \times 10 + 15 = 45$   
 $\dots$   
 general rule? proof?

(8)  $9 \times 1 + 1 = 10$   
 $9 \times 3 + 1 = 28$   
 $9 \times 6 + 1 = 55$   
 $\dots$   
 general rule? proof?

(9)  $1^2 + 3^2 = 10$   
 $3^2 + 6^2 = 45$   
 $6^2 + 10^2 = 136$   
 $\dots$   
 general rule? proof?

(10)  $3^2 - 1^2 = 8$   
 $6^2 - 3^2 = 27$   
 $10^2 - 6^2 = 64$   
 $\dots$   
 general rule? proof?

$$T_n = \frac{1}{2} n (n + 1)$$



| $n$ | $T_n$ |
|-----|-------|
| 1   | 1     |
| 2   | 3     |
| 3   | 6     |
| 4   | 10    |
| 5   | 15    |
| 6   | 21    |
| 7   | 28    |
| 8   | 36    |
| 9   | 45    |
| 10  | 55    |
| 11  | 66    |
| 12  | 78    |
| 13  | 91    |
| 14  | 105   |
| 15  | 120   |
| 16  | 136   |
| 17  | 153   |
| 18  | 171   |
| 19  | 190   |
| 20  | 210   |
| 21  | 231   |
| 22  | 253   |
| 23  | 276   |
| 24  | 300   |
| 25  | 325   |
| 26  | 351   |
| 27  | 378   |
| 28  | 406   |
| 29  | 435   |

## Purposeful Practice

consecutive numbers – setting up and using an equations to solve a problem

find three consecutive numbers that add up to 45

find four consecutive numbers that add up to 38

find five consecutive numbers that add up to 60

any number is  $n$ ; the  
next number is  $n + 1$

any even number is  
 $2n$ ; the next even  
number is  $2n + 2$  etc

find three consecutive even numbers that add up to 42

find four consecutive even numbers that add up to 60

find five consecutive even numbers that add up to 60

find three consecutive odd numbers that add up to 57

find four consecutive odd numbers that add up to 48

find even numbers of consecutive odd numbers that add up to 72

any odd number is  
 $2n + 1$ ; the next odd  
number is  $2n + 3$  etc

any multiple of 4 is  
 $4n$ ; the next multiple  
of 4 is  $4n + 4$  etc

find three consecutive multiples of 3 that add up to 63

find four consecutive multiples of 5 that add up to 150

find five consecutive multiples of 2 that add up to 94

## Purposeful Practice

- 1 Write down algebraic expressions for
  - (a) an even number
  - (b) an odd number
  - (c) a multiple of four
  - (d) a positive number which leaves a remainder of 1 upon division by 5
  - (e) the sum of two consecutive even numbers
  - (f) the sum of two even numbers
  - (g) the product of two odd numbers
  - (e) the cube of a multiple of 6
  - (h) the sum of the squares of two consecutive multiples of 3
- 2 Prove algebraically that
  - (a) the sum of two consecutive numbers is odd
  - (b) the product of two even numbers is even
  - (c) the sum of any three consecutive numbers is a multiple of 3
- 3
  - (a) Prove algebraically that the sum of the squares of two consecutive odd numbers is **not** a multiple of 4.
  - (b) What is the remainder when the sum of the squares of two consecutive odd numbers is divided by 4? Is the number odd or even?
- 4 Johnny has an even number. He adds 3 to this number and then squares it.  
The result is 81.  
What even number did Johnny start with?
- 5 Prove algebraically that  $(5n + 1)^2 - (5n - 1)^2$  is both a multiple of 4 and a multiple of 5 for positive integers  $n$ .
- 6 Anne has a number which is not a multiple of 3.
  - (a) Explain why it must be of the form  $(3n + 1)$  or  $(3n + 2)$ , where  $n$  is an integer.
  - (b) Hence, by considering these two different cases, show that the square of Anne's number leaves a remainder of 1 when divided by 3.
- 7
  - (a) Prove that if  $n$  is an even integer, then  $3n^2 + n + 14$  is even.
  - (b) Prove that if  $n$  is an odd integer, then  $3n^2 + n + 14$  is even.
  - (c) What can you deduce about the number  $3n^2 + n + 14$  for all integers  $n$ ?

## Fluency Practice

### Negative.... when?

$x$  = any number

Can these expressions have a negative value?

For each expression, test values & *limits* for  $x$ .

(hint: factorising the expression, or a part of it may help)

$$x^2$$

$$x^2 - 4$$

$$x^2 + x$$

$$x^2 + 2x$$

$$x^2 - 3x$$

$$x^2 + 2x + 1$$

$$x^2 + 2x + 1$$

## Fluency Practice

Question 1: Prove the following

(a)  $(n + 7)^2 - (n + 1)^2 = 12(n + 4)$

(b)  $(n + 1)^2 - (n - 3)^2 = 8(n - 1)$

(c)  $(n + 1)^2 + (n + 5)^2 - (n + 9)^2 = (n + 5)(n - 11)$

Question 2: Prove the following

(a)  $(n + 4)^2 - (n + 2)^2$  is always a multiple of 4 for all positive integer values of  $n$ .

(b)  $(n + 10)^2 - (n + 2)^2$  is always a multiple of 16 for all positive integer values of  $n$ .

(c)  $(2n + 3)^3 - (2n + 1)$  is always even for all positive integer values of  $n$ .

(d)  $(5n + 2)^2 - (5n - 1)^2$  is always a multiple of 3 for all positive integer values of  $n$ .

(e)  $(2n + 9)^2 - (2n + 5)^2$  is always a multiple of 4 for all positive integer values of  $n$ .

(f)  $(n + 2)^2 - (n - 2)^2 + 3$  is always odd for all positive integer values of  $n$ .

Question 3:  $n$  is a positive integer.

(a) Write an expression for an even number.

(b) Write an expression for an odd number.

Question 4: Prove the following

(a) The sum of any three consecutive integers is divisible by 3.

(b) The sum of any three consecutive even numbers is always a multiple of 6.

(c) The sum of two consecutive odd numbers is even.

(d) The sum of three consecutive odd numbers is always a multiple of 3.

(e) The sum of four consecutive odd numbers is always a multiple of 8.

(f) The sum of two consecutive integers is always odd.

(g) The sum of four consecutive integers is **not** a multiple of 4

# Purposeful Practice

Question 5: Prove the following

- (a) Prove the product of two odd numbers is always odd.
- (b) Prove the product of two even consecutive numbers is always a multiple of 4.
- (c) The difference between the squares of any two consecutive integers is equal to the sum of the two integers.
- (d) Prove the sum of the squares of any two consecutive even numbers is always a multiple of 4.
- (e) Prove that when any odd integer is squared, the result is always one more than a multiple of 8.

Apply

Question 1: The first five terms of a linear sequence are 2, 7, 12, 17, 22

- (a) Find the  $n$ th term of the sequence

A new sequence is generated by squaring each term of the linear sequence and then adding 1.

- (b) Prove that all terms in the new sequence are divisible by 5.

Question 2: The first two terms of a fibonacci sequence are  $a$  and  $b$ .

- (a) Show the 4th term of the sequence is  $a+2b$
- (b) Prove that the sum of the first 10 terms is equal to 11 times the 7th term.

Question 3: Cara writes down a 3-digit number where the first digit is greater than the last. e.g. 681

She then reverses the number to give 186.

Cara then subtracts this number from her starting number.  $681 - 186 = 495$

She then reverses her answer to give 594.

Cara then adds these number  $495 + 594 = 1089$ .

|       |      |
|-------|------|
| 681   | 594  |
| -186  | +495 |
| <hr/> |      |
| 495   | 1089 |

Cara repeats this several times and always gets 1089 as her answer.  
Prove algebraically that the answer is always 1089.



## Fluency Practice

|                                         |       |       |
|-----------------------------------------|-------|-------|
| $8(ax + 2) \equiv 24x + 16$             | $a =$ |       |
| $5(2x + a) \equiv 10x + 45$             | $a =$ |       |
| $3(9x + 1) \equiv 27x + a$              | $a =$ |       |
| $6(5 - x) \equiv 30 - ax$               | $a =$ |       |
| $4(7x + 2) + a \equiv 28x + 10$         | $a =$ |       |
| $5(3x + 4) + ax \equiv 19x + 20$        | $a =$ |       |
| $2x + 9 + ax - b \equiv 10x + 6$        | $a =$ | $b =$ |
| $a(7x - 3) \equiv 35x - 15$             | $a =$ |       |
| $2(ax + 5) + 3(2x + b) \equiv 18x + 13$ | $a =$ | $b =$ |
| $9(ax + b) + 4(x - 2) \equiv 13x + 37$  | $a =$ | $b =$ |

# Fluency Practice

| Algebraic Proof                                                                                                                  |                                                                                                                                    |                                                                                                                                           |
|----------------------------------------------------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------------------------------------------------------------|
| <b>(a)</b>                                                                                                                       | <b>(b)</b>                                                                                                                         | <b>(c)</b>                                                                                                                                |
| <p>Show that</p> $3x(x + 5) + 2x(x - 5) \equiv 5x(x + 1)$                                                                        | <p>Show that</p> $(x + 6)(x - 2) + 12 \equiv x(x + 4)$                                                                             | <p>Show that</p> $(x - 4)^2 + 6x - 16 \equiv x(x - 2)$                                                                                    |
| <b>(d)</b>                                                                                                                       | <b>(e)</b>                                                                                                                         | <b>(f)</b>                                                                                                                                |
| <p>Show that</p> $3(8 - x) + 2(5x - 6) \equiv ax + b$ <p>where <math>a</math> and <math>b</math> are integers to be found</p>    | <p>Show that</p> $(x + 5)(x - 3) - x(x - 8) \equiv ax + b$ <p>where <math>a</math> and <math>b</math> are integers to be found</p> | <p>Show that</p> $(x + 6)^2 + 4(x - 9) \equiv x(x + a)$ <p>where <math>a</math> is an integer to be found</p>                             |
| <b>(g)</b>                                                                                                                       | <b>(h)</b>                                                                                                                         | <b>(i)</b>                                                                                                                                |
| <p>Show that</p> $(2x + 5)(x - 1) + 3(5 - x) = ax^2 + b$ <p>where <math>a</math> and <math>b</math> are integers to be found</p> | <p>Show that</p> $(x + 4)^2 + (x + 2)(x - 8) = ax(x + b)$ <p>where <math>a</math> and <math>b</math> are integers to be found</p>  | <p>Show that</p> $(3x + 4)^2 - (5x + 8)(x + 2) \equiv ax(bx + c)$ <p>where <math>a</math> and <math>b</math> are integers to be found</p> |

Fluency Practice

|    |                                                      |
|----|------------------------------------------------------|
| 1. | $2(3x + 8) + 6(x - 5) \equiv ax + b$                 |
| 2. | $10x(x - 3) - 3x(2x - 2) \equiv cx^2 + dx$           |
| 3. | $6(2x + 1) + 2(4x - 1) + 10 \equiv ex + f$           |
| 4. | $6(x + 3) + g(2x + 10) \equiv 14x + h$               |
| 5. | $(x + 7)(x - 6) + 3 \equiv x^2 + jx + k$             |
| 6. | $(x + 3)(x + n) + mx^2 + 5x \equiv 4x^2 + 16x + p$   |
| 7. | $(x + 5)(x - 2) + qx(2x + 8) \equiv 9x^2 + 35x - 10$ |
| 8. | $(x + r)(x - s) + 2x + 15 \equiv x^2 + 8x - 25$      |

Answers

|     |     |     |     |     |     |     |     |
|-----|-----|-----|-----|-----|-----|-----|-----|
| $a$ | $b$ | $c$ | $d$ | $e$ | $f$ | $g$ | $h$ |
|     |     |     |     |     |     |     |     |

|     |     |     |     |     |     |     |     |
|-----|-----|-----|-----|-----|-----|-----|-----|
| $j$ | $k$ | $m$ | $n$ | $p$ | $q$ | $r$ | $s$ |
|     |     |     |     |     |     |     |     |

## Fluency Practice

|    |                                                         |                         |
|----|---------------------------------------------------------|-------------------------|
| 1  | $(2x + 3)(x - 5) + 7 \equiv ax^2 + bx + c$              | $a =$<br>$b =$<br>$c =$ |
| 2  | $(x + 1)(x + 4)(x - 1) \equiv x^3 + ax^2 + bx + c$      | $a =$<br>$b =$<br>$c =$ |
| 3  | $2x^2 + 12x - 9 \equiv a(x + b)^2 + c$                  | $a =$<br>$b =$<br>$c =$ |
| 4  | $(ax + 1)(x + c) + b \equiv 5x^2 + 11x + 8$             | $a =$<br>$b =$<br>$c =$ |
| 5  | $3(ax + 1) - (6x + b) \equiv 21x - 8$                   | $a =$<br>$b =$          |
| 6  | $(ax + 5)(x - b) + 4x \equiv 2x^2 + cx - 10$            | $a =$<br>$b =$<br>$c =$ |
| 7  | $4(3x + a) + 5(ax + b) \equiv 22x + 28$                 | $a =$<br>$b =$          |
| 8  | $4x^2 + ax + 3 \equiv b(x + 1)^2 + c$                   | $a =$<br>$b =$<br>$c =$ |
| 9  | $\frac{ax^2 + x - b}{x + 2} \equiv x - 1$               | $a =$<br>$b =$          |
| 10 | $2(5x + 1)(3x + b) + ax(2x + 3) \equiv 44x^2 + 37x + c$ | $a =$<br>$b =$<br>$c =$ |

# Fluency Practice

Find the value of each letter in the identities below. Circle the answers at the bottom as you find them.

|                                                                                 |                                                                                                    |                                                                                                        |                                                                                          |
|---------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------|
| $8(ax + 2) \equiv 24x + 16$<br><br>$a = \square$                                | $5(2x + b) \equiv 10x + 45$<br><br>$b = \square$                                                   | $c(9x + d) \equiv 18x + 26$<br><br>$c = \square, d = \square$                                          | $e(2 + fx) \equiv 24x - 8$<br><br>$e = \square, f = \square$                             |
| $6(3x + 5) + g \equiv 18x + 44$<br><br>$g = \square$                            | $4(5x - 3) + hx \equiv 13x - 12$<br><br>$h = \square$                                              | $i(8x - 3) \equiv 4x + j$<br><br>$i = \square, j = \square$                                            | $12(6x + k) - lx \equiv 71x + 4$<br><br>$k = \square, l = \square$                       |
| $(x + m)(x - 2) \equiv x^2 + 5x - 14$<br><br>$m = \square$                      | $(x + 4)(x - 7) \equiv x^2 + nx - 28$<br><br>$n = \square$                                         | $(x + \sigma)(x + 20) \equiv x^2 + px - 160$<br><br>$\sigma = \square, p = \square$                    | $(2x + 1)(x - 5) \equiv 2x^2 + rx + q$<br><br>$q = \square, r = \square$                 |
| $(x - s)^2 + 10 \equiv x^2 - 10x + 35$<br><br>$s = \square$                     | $(x + 3)^2 - 3 \equiv x^2 + tx + t$<br><br>$t = \square$                                           | $(x + u)^2 - v \equiv x^2 + 22x + 71$<br><br>$u = \square, v = \square$                                | $x^2 - 5x + 6 \equiv (x + w)^2 - y$<br><br>$w = \square, y = \square$                    |
| $\frac{ax - 8}{4x + \beta} \equiv 4$<br><br>$\alpha = \square, \beta = \square$ | $\frac{x^2 + \gamma x + 44}{x + 11} \equiv x + \delta$<br><br>$\gamma = \square, \delta = \square$ | $(x + 5)^2(x - 2) \equiv x^3 + \epsilon x^2 + 5x + \zeta$<br><br>$\epsilon = \square, \zeta = \square$ | $(x - \eta)(x + 8) \equiv (x + \theta)^2 - 81$<br><br>$\eta = \square, \theta = \square$ |

**Answers** (One of the questions has multiple solutions. Which one? What are the other possible solutions?)

|    |    |    |    |    |    |    |    |    |               |               |               |                 |                 |     |    |
|----|----|----|----|----|----|----|----|----|---------------|---------------|---------------|-----------------|-----------------|-----|----|
| 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10            | 11            | 12            | 13              | 14              | 15  | 16 |
| -1 | -2 | -3 | -4 | -5 | -6 | -7 | -8 | -9 | $\frac{1}{2}$ | $\frac{1}{3}$ | $\frac{1}{4}$ | $-1\frac{1}{2}$ | $-2\frac{1}{2}$ | -50 | 50 |

# Fluency Practice

Find the value of each letter in the identities below. Circle the answers at the bottom as you find them.

|                                                                                                          |                                                                                                                     |                                                                                                                         |                                                                                                                                    |
|----------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------------------------------|
| $\frac{ax+12}{4x+6} \equiv 2$<br><br>$a = \square$                                                       | $\frac{7x+b}{21x-12} \equiv \frac{3x+5}{9x+15}$<br><br>$b = \square$                                                | $\frac{cx-10}{18x-12} \equiv \frac{35x-5}{42x+d}$<br><br>$c = \square, d = \square$                                     | $\frac{12-9x}{6x+e} \equiv f$<br><br>$e = \square, f = \square$                                                                    |
| $\frac{x}{3} - \frac{x}{g} \equiv \frac{x}{12}$<br><br>$g = \square$                                     | $\frac{x+1}{7} - \frac{x}{2} \equiv \frac{hx+20}{140}$<br><br>$h = \square$                                         | $\frac{2x+i}{5} + \frac{x+5}{3} \equiv \frac{jx+22}{15}$<br><br>$i = \square, j = \square$                              | $\frac{kx-8}{6} - \frac{4x-11}{8} \equiv \frac{l}{12}$<br><br>$k = \square, l = \square$                                           |
| $\frac{x^2+mx}{x^2+7x+10} \equiv \frac{x}{x+2}$<br><br>$m = \square$                                     | $\frac{x^2-x-12}{x^2-9x+20} \equiv \frac{x+3}{x+n}$<br><br>$n = \square$                                            | $\frac{x+4}{x^2-\sigma} \equiv \frac{p}{x-4}$<br><br>$\sigma = \square, p = \square$                                    | $\frac{x^2+qx+r}{x^2+x-20} \equiv \frac{x^2+3x-18}{x^2+11x+30}$<br><br>$q = \square, r = \square$                                  |
| $\frac{3x^2-sx-5}{x-5} \equiv 3x+1$<br><br>$s = \square$                                                 | $\frac{3x^2+tx}{2x^2-18} \equiv \frac{3x}{2x-6}$<br><br>$t = \square$                                               | $\frac{x^2+ux-18}{2x^2-x-6} \equiv \frac{x+9}{vx+3}$<br><br>$u = \square, v = \square$                                  | $\frac{wx^2+12x-1}{26x^2+11x-1} \equiv \frac{11x^2+yx-1}{22x^2+9x-1}$<br><br>$w = \square, y = \square$                            |
| $\frac{5}{x+3} - \frac{4}{x+\alpha} = \frac{x-27}{x^2+\beta}$<br><br>$\alpha = \square, \beta = \square$ | $\frac{\gamma}{x+4} - \frac{x-44}{x^2-16} \equiv \frac{\delta}{10x-40}$<br><br>$\gamma = \square, \delta = \square$ | $\frac{6\zeta}{5-x} + \frac{3}{x+7} \equiv \frac{29-x}{14\epsilon-2x-x^2}$<br><br>$\epsilon = \square, \zeta = \square$ | $\frac{1}{x} + \frac{1}{x+1} + \frac{\eta}{x+2} \equiv \frac{12\theta x+2}{x^3+3x^2+2x}$<br><br>$\eta = \square, \theta = \square$ |

Answers

|    |    |    |    |    |    |    |    |    |               |               |               |                 |                 |     |    |
|----|----|----|----|----|----|----|----|----|---------------|---------------|---------------|-----------------|-----------------|-----|----|
| 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10            | 11            | 12            | 13              | 14              | 15  | 16 |
| -1 | -2 | -3 | -4 | -5 | -6 | -7 | -8 | -9 | $\frac{1}{2}$ | $\frac{1}{3}$ | $\frac{1}{4}$ | $-1\frac{1}{2}$ | $-2\frac{1}{2}$ | -50 | 50 |

## Fluency Practice

### versions

find the values of the letters in these (equivalent) 'families' of quadratic expressions

(1)  $(x + 2)(x + 6) + 3$

$$(x + 7)(x + a) + b$$

$$(x + c)^2 - d$$

$$(x + 1)(x + 15) - ex$$

$$(x + f)(x + g)$$

(2)  $(x - 8)(x + 1) + 14$

$$(x - 2)(x - a) - b$$

$$(x - 4)(x - c) - d$$

$$(x - 3)(x - 2) - ex$$

$$(x - f)(x - g)$$

(3)  $(x - 2)(x - 10) + 15$

$$(x + 3)(x - a) + b$$

$$(x - c)^2 - d$$

$$(x + 5)(x + 7) - ex$$

$$(x - f)(x - g)$$

(4)  $(x - 3)(x - 5) - 35$

$$(x - 11)(x + a) + b$$

$$(x - c)^2 - d$$

$$x^2 - 4(2x + e)$$

$$(x + f)(x - g)$$

(5)  $(2x + 3)(x + 8) + 11$

$$(2x + 7)(x + a) - b$$

$$(x + 9)(2x + c) + d$$

$$(x - 5)(2x - 7) + ex$$

$$(2x + f)(x + g)$$

(6)  $(2x + 1)(x + 2) - 14$

$$(2x - 1)(x + a) - b$$

$$(x + 6)(2x - c) + d$$

$$2(x - 3)(x + 2) + ex$$

$$(2x - f)(x + g)$$

# Prove it!

Prove that  $(a + b)^2$  is not equal to  $a^2 + b^2$

Show that  $5a + 2a + 4 + 3$  is always a multiple of 7

$n - 1 + n + 1$  will be even. Why?

The sum of three consecutive numbers is 3 times the middle number

(hint - call your first number  $a$ , your second number  $a + 1$ )

The sum of two consecutive odd numbers is always even

For any three consecutive numbers, the difference between the square of the middle number and the product of the largest and smallest is always 1

Show that  $5(2x + 5) + 11(x - 2) = 3(7x + 1)$



## Purposeful Practice

from exercises in algebra part three by HF Browne (1936)

solve the equations

$$\frac{2}{x} + \frac{3}{y} = 4$$

$$\frac{5}{x} + \frac{6}{y} = 7$$

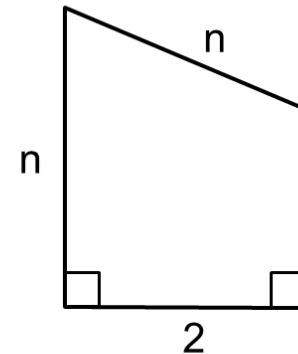
find two numbers such that  
the sum of their squares is  
five times their sum  
and  
the difference of their squares  
is nine times their difference

the perimeter of a  
right-angled triangle is 40 cm  
and the area is 60 cm<sup>2</sup>

find the lengths of the sides

the area of the  
right trapezium  
is 4

find n



# Fluency Practice

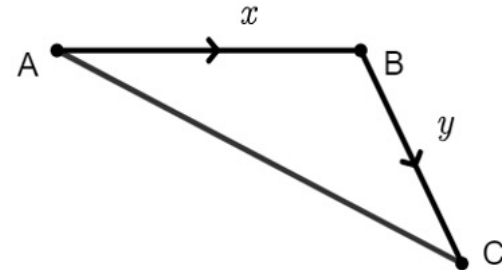
| Number and Algebra Proof Revision                                            |                                                                                                                                    |                                                                                          |                                                                           |
|------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------|---------------------------------------------------------------------------|
| <b>(a)</b>                                                                   | <b>(b)</b>                                                                                                                         | <b>(c)</b>                                                                               | <b>(d)</b>                                                                |
| Show that $8\frac{1}{2} - 3\frac{2}{3} = 4\frac{5}{6}$                       | Show that $\sqrt{80}$ can be written in the form $k\sqrt{5}$ where $k$ is an integer to be found                                   | Show that $0.\dot{7}\dot{5} = \frac{25}{33}$                                             | Show that $3\frac{5}{8} \div 1\frac{5}{6} = 1\frac{43}{44}$               |
| <b>(e)</b>                                                                   | <b>(f)</b>                                                                                                                         | <b>(g)</b>                                                                               | <b>(h)</b>                                                                |
| Show that $0.3\dot{1} = \frac{14}{45}$                                       | Show that $0.4\dot{2}\dot{7} = \frac{47}{110}$                                                                                     | Show that $(7 - 5\sqrt{3})^2 = a + b\sqrt{3}$ where $a$ and $b$ are integers to be found | Show that the product of an even number and an odd number is always even. |
| <b>(i)</b>                                                                   | <b>(j)</b>                                                                                                                         | <b>(k)</b>                                                                               |                                                                           |
| Show that the sum of three consecutive odd numbers is always a multiple of 3 | Show that $\frac{3\sqrt{12}}{2-\sqrt{3}}$ can be written in the form $c + d\sqrt{3}$ , where $c$ and $d$ are integers to be found. | Show that $(3n + 4)(n - 3) + n(n - 3)$ is a multiple of 4 for all integer values of $n$  |                                                                           |

## 5 Advanced Vectors

## Fluency Practice

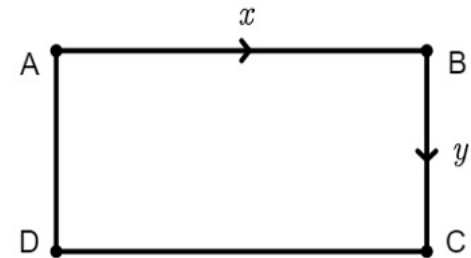
1. ABC is a triangle. Find, in terms of  $x$  and  $y$  :

- (a)  $\overrightarrow{BA}$  (b)  $\overrightarrow{CB}$   
 (c)  $\overrightarrow{AC}$  (d)  $\overrightarrow{CA}$



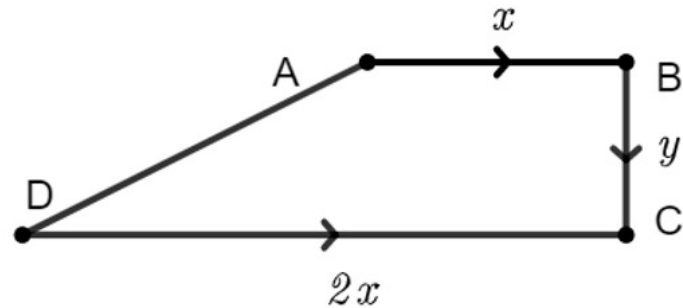
2. ABCD is a rectangle. Find, in terms of  $x$  and  $y$  :

- (a)  $\overrightarrow{DA}$  (b)  $\overrightarrow{AC}$   
 (c)  $\overrightarrow{CA}$  (d)  $\overrightarrow{BD}$



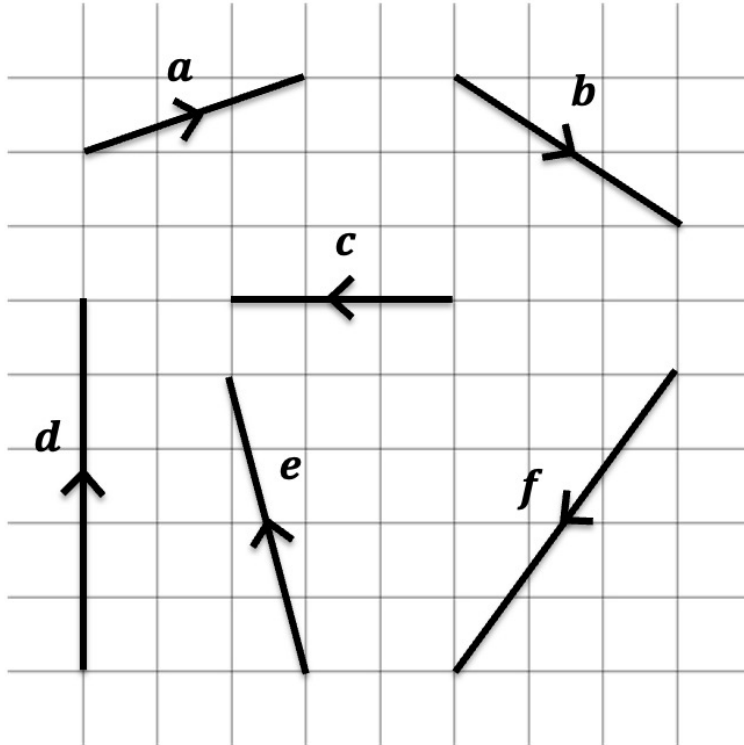
3. ABCD is a trapezium. Find, in terms of  $x$  and  $y$  :

- (a)  $\overrightarrow{AC}$  (b)  $\overrightarrow{DB}$   
 (c)  $\overrightarrow{AD}$  (d)  $\overrightarrow{DA}$



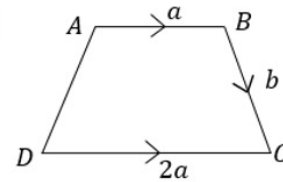
# Fluency Practice

1 State the value of each vector.



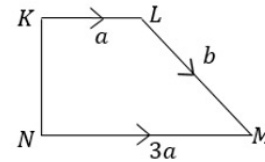
$$\begin{array}{lll} a = \boxed{?} & b = \boxed{?} & c = \boxed{?} \\ d = \boxed{?} & e = \boxed{?} & f = \boxed{?} \end{array}$$

2



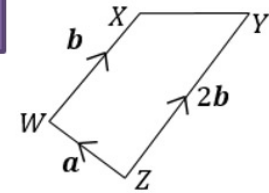
- $\overrightarrow{BA} = \boxed{?}$
- $\overrightarrow{AC} = \boxed{?}$
- $\overrightarrow{DB} = \boxed{?}$
- $\overrightarrow{AD} = \boxed{?}$

3



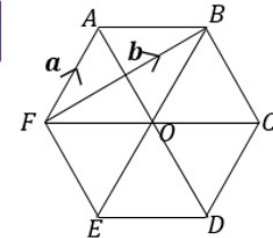
- $\overrightarrow{MK} = \boxed{?}$
- $\overrightarrow{NL} = \boxed{?}$
- $\overrightarrow{NK} = \boxed{?}$
- $\overrightarrow{KN} = \boxed{?}$

4



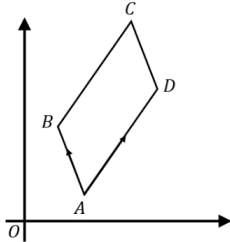
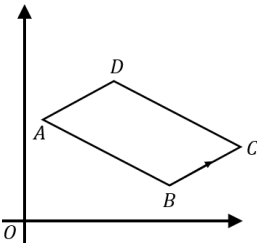
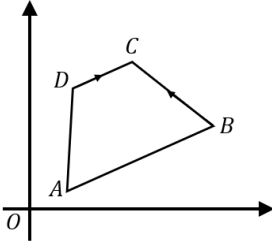
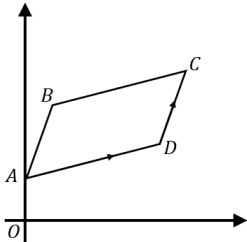
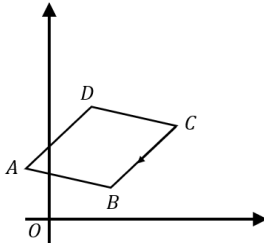
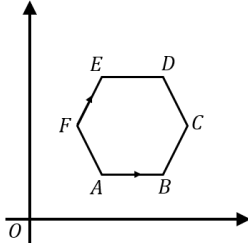
- $\overrightarrow{ZX} = \boxed{?}$
- $\overrightarrow{YW} = \boxed{?}$
- $\overrightarrow{XY} = \boxed{?}$
- $\overrightarrow{XZ} = \boxed{?}$

5



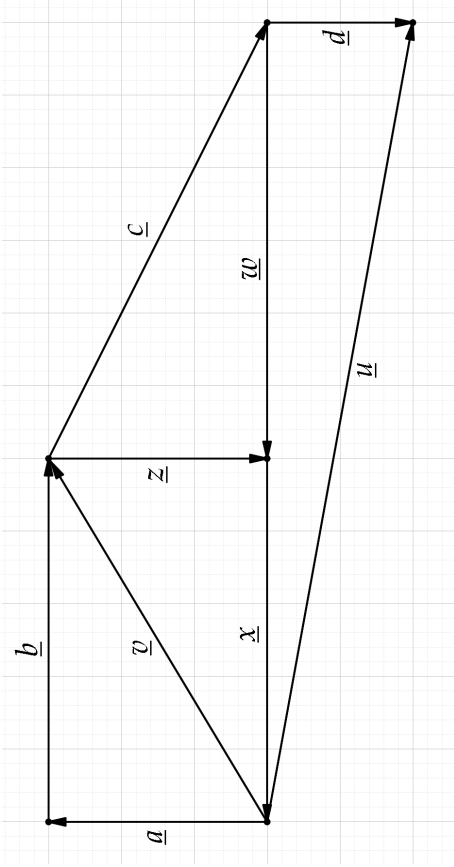
- $\overrightarrow{AB} = \boxed{?}$
- $\overrightarrow{FO} = \boxed{?}$
- $\overrightarrow{AO} = \boxed{?}$
- $\overrightarrow{FD} = \boxed{?}$

## Purposeful Practice

| Vectors and Coordinates                                                                                                                                                                                                                                                           |                                                                                                                                                                                                                                                     |                                                                                                                                                                                                                                                                                                                                     |
|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| <b>(a)</b><br>$ABCD$ is a parallelogram.<br>$\overrightarrow{AB} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$ and $\overrightarrow{AD} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$<br>Given that the coordinates of $A$ are $(3, 1)$ , find the coordinates of points $B$ , $C$ and $D$ . | <b>(c)</b><br>$ABCD$ is a parallelogram. $\overrightarrow{BC} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$<br>The coordinates of $A$ are $(2, 7)$ and of $B$ are $(8, 3)$ . Find the coordinates of $C$ and $D$ , and the vector $\overrightarrow{DC}$ . | <b>(e)</b><br>$ABCD$ is a trapezium. $\overrightarrow{AB} = 2\overrightarrow{DC}$ .<br>$\overrightarrow{DC} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $\overrightarrow{BC} = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$<br>The coordinates of $D$ are $(2, 8)$ . Find the coordinates of $A$ , and the vector $\overrightarrow{AD}$ . |
|                                                                                                                                                                                                  |                                                                                                                                                                   |                                                                                                                                                                                                                                                  |
| <b>(b)</b><br>$ABCD$ is a parallelogram.<br>$\overrightarrow{AD} = \begin{pmatrix} 6 \\ 1 \end{pmatrix}$ and $\overrightarrow{DC} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$<br>Given that the coordinates of $A$ are $(0, 1)$ , find the coordinates of points $B$ , $C$ and $D$ .  | <b>(d)</b><br>$ABCD$ is a rhombus. $\overrightarrow{CB} = \begin{pmatrix} -3 \\ -4 \end{pmatrix}$<br>The coordinates of $A$ are $(-1, 4)$ and of $B$ are $(3, 1)$ . Find the coordinates of $C$ and $D$ , and the vector $\overrightarrow{DC}$ .    | <b>(f)</b><br>$ABCDEF$ is a regular hexagon.<br>$\overrightarrow{AB} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}$ and $\overrightarrow{FE} = \begin{pmatrix} 2 \\ 2\sqrt{3} \end{pmatrix}$<br>$\overrightarrow{FC} = 2\overrightarrow{AB}$ . The coordinates of $A$ are $(5, 2)$ . Find the coordinates of $B$ , $C$ and $D$ .           |
|                                                                                                                                                                                                 |                                                                                                                                                                  |                                                                                                                                                                                                                                                 |

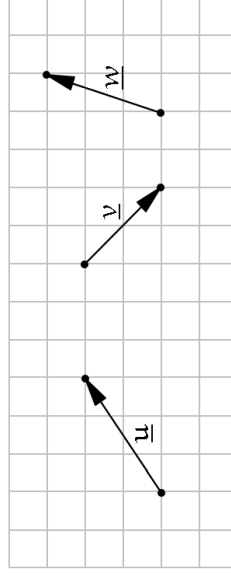
# Fluency Practice

1. Based on the picture, determine whether each of the given statements is true or false.



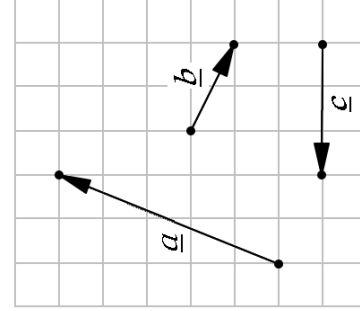
- |                                                                    |                                                                                                    |                                                                     |
|--------------------------------------------------------------------|----------------------------------------------------------------------------------------------------|---------------------------------------------------------------------|
| a) $\underline{v} = \underline{a} + \underline{b}$                 | f) $\underline{w} + \underline{x} + \underline{u} = \underline{d}$                                 | k) $\underline{x} + \underline{w} + \underline{v} = -\underline{c}$ |
| b) $\underline{v} = \underline{z} + \underline{x}$                 | g) $\underline{d} + \underline{u} = \underline{x} + \underline{w}$                                 | l) $\underline{v} + \underline{c} + \underline{d} = -\underline{u}$ |
| c) $\underline{v} = \underline{x} - \underline{z}$                 | h) $\underline{w} - \underline{c} = \underline{x} - \underline{v}$                                 | m) $\underline{a} + \underline{z} = \underline{0}$                  |
| d) $\underline{c} = \underline{d} - \underline{u} + \underline{v}$ | i) $\underline{a} + \underline{b} + \underline{z} + \underline{x} = \underline{0}$                 | n) $\underline{a} - \underline{b} = -\underline{v}$                 |
| e) $\underline{a} = \underline{z}$                                 | j) $\underline{a} + \underline{b} + \underline{c} + \underline{w} + \underline{x} = \underline{0}$ | o) $\underline{z} - \underline{w} = \underline{c}$                  |

2. Given the vectors  $\underline{u}$ ,  $\underline{v}$  and  $\underline{w}$ , draw each of the following vectors.



- $\underline{u} + \underline{v}$
- $\underline{v} - \underline{w}$
- $2\underline{u} - \underline{v} + \underline{w}$

3. Given the vectors  $\underline{a}$ ,  $\underline{b}$  and  $\underline{c}$ , draw each of the following vectors.



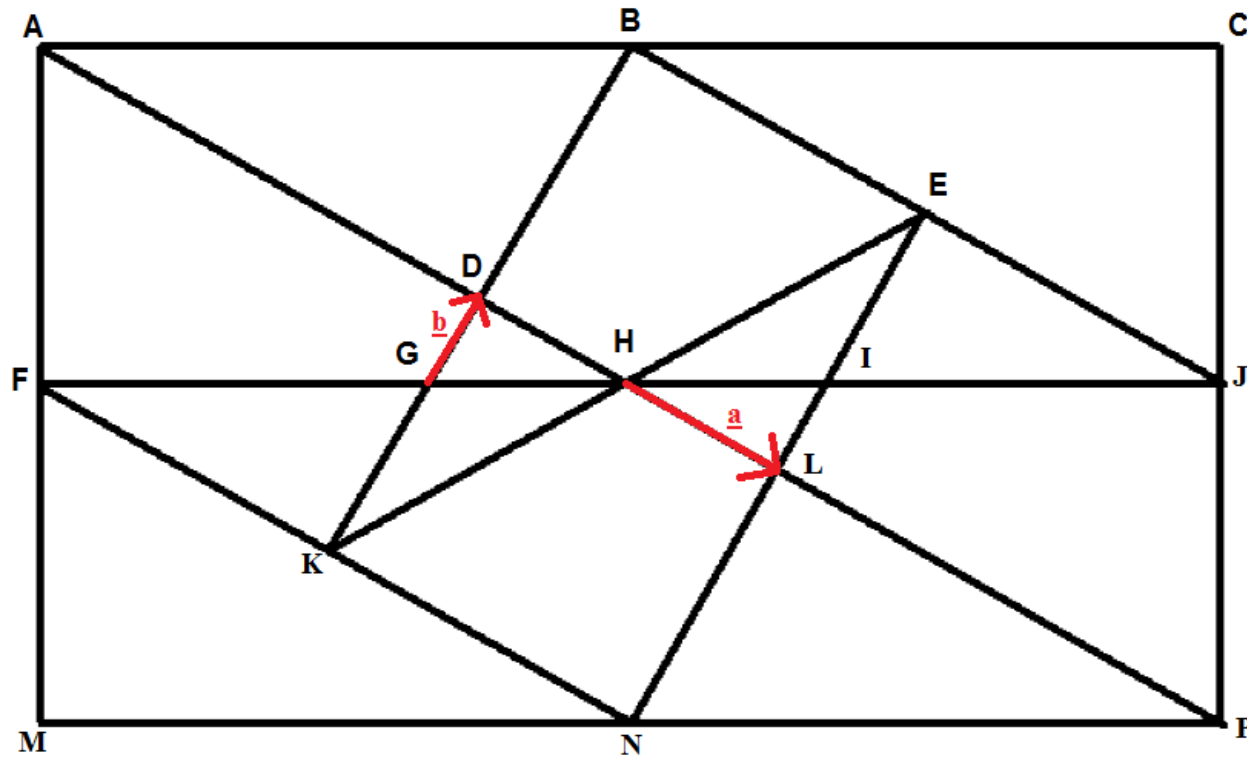
- $\underline{a} + \underline{b}$
- $3\underline{b}$
- $\underline{b} - \underline{a}$
- $\underline{a} + \underline{b} + \underline{c}$
- $\underline{a} - 2\underline{b} + \underline{c}$
- $-2\underline{c}$
- $-2\underline{a} + 3\underline{b} - \underline{c}$

## Purposeful Practice

4. Suppose that  $M$  is the midpoint of line segment  $AB$ . Let  $O$  be a point in the plane of  $AB$ . Let  $\underline{a}$  denote the vector pointing from  $O$  to  $A$ , let  $\underline{b}$  denote the vector pointing from  $O$  to  $B$ , and  $\underline{m}$  denote the vector pointing from  $O$  to  $M$ . Prove that  $\underline{m} = \frac{\underline{a} + \underline{b}}{2}$ . (Hint: Let  $\underline{x}$  be the vector pointing from  $A$  to  $M$ . Express first  $\underline{x}$  in terms of  $\underline{a}$  and  $\underline{b}$ .  $\underline{m} = \underline{a} + \underline{x}$ ).



## Purposeful Practice

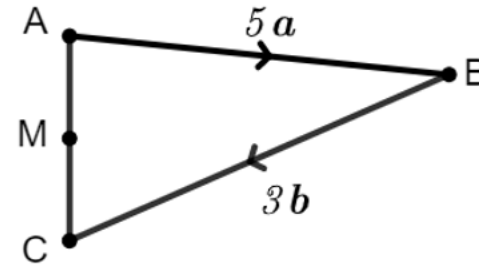


If  $HL = \mathbf{a}$  and  $GD = \mathbf{b}$   
 how many other  
 vectors can you define  
 in terms of  $\mathbf{a}$  and  $\mathbf{b}$ ?

## Fluency Practice

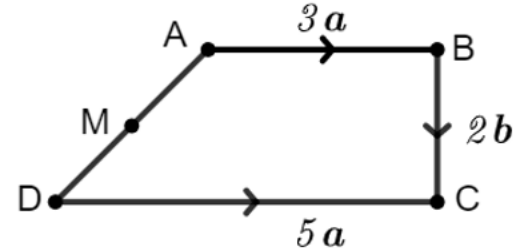
1. ABC is a triangle. M is the midpoint of AC. Find, in terms of  $\mathbf{a}$  and  $\mathbf{b}$  :

(a)  $\overrightarrow{AC}$  (b)  $\overrightarrow{CA}$   
 (c)  $\overrightarrow{AM}$  (d)  $\overrightarrow{BM}$



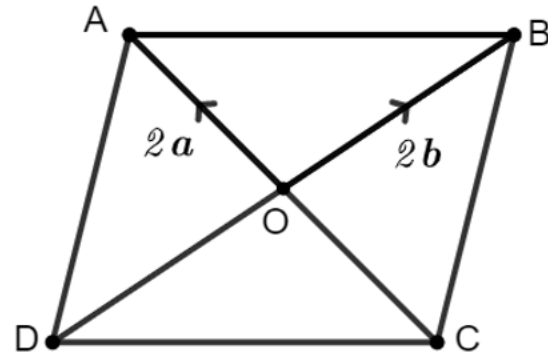
2. ABCD is a trapezium. M is the midpoint of AD. Find, in terms of  $\mathbf{a}$  and  $\mathbf{b}$  :

(a)  $\overrightarrow{DA}$  (b)  $\overrightarrow{DM}$   
 (c)  $\overrightarrow{AD}$  (d)  $\overrightarrow{MC}$

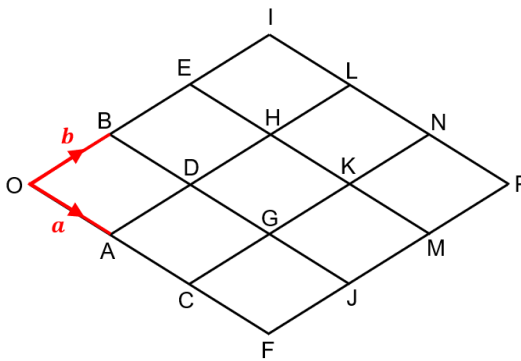
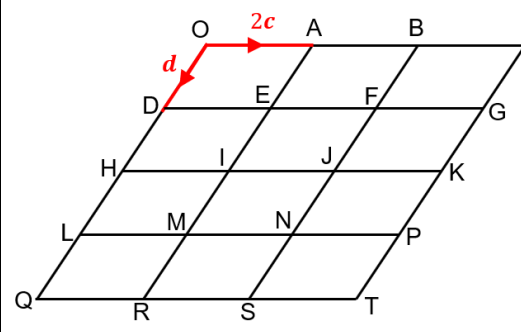
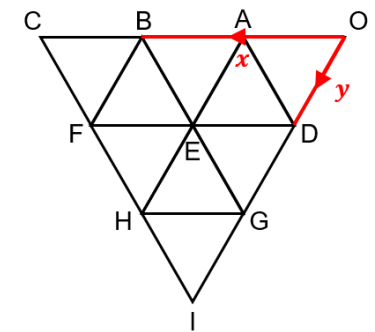


3. ABCD is a parallelogram. Find, in terms of  $\mathbf{a}$  and  $\mathbf{b}$  :

(a)  $\overrightarrow{OC}$  (b)  $\overrightarrow{AC}$   
 (c)  $\overrightarrow{AB}$  (d)  $\overrightarrow{AD}$



# Fluency Practice

| Defining Vectors                                                                                                                                                                                                                                                                                                                                          |                                                                                                                                                                                                                                                                                                                                                                    |                                                                                                                                                                                                                                                                                                                                                                         |
|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (a)                                                                                                                                                                                                                                                                                                                                                       | (b)                                                                                                                                                                                                                                                                                                                                                                | (c)                                                                                                                                                                                                                                                                                                                                                                     |
| <p>The diagram is made up of nine congruent rhombuses.</p> <p><math>\overrightarrow{OA} = \mathbf{a}</math> and <math>\overrightarrow{OB} = \mathbf{b}</math>.</p>  <p>Define the following vectors in terms of <math>\mathbf{a}</math> and <math>\mathbf{b}</math>.</p> | <p>The diagram is made up of twelve congruent parallelograms.</p> <p><math>\overrightarrow{OA} = 2\mathbf{c}</math> and <math>\overrightarrow{OD} = \mathbf{d}</math>.</p>  <p>Define the following vectors in terms of <math>\mathbf{c}</math> and <math>\mathbf{d}</math>.</p> | <p>The diagram is made up of nine congruent equilateral triangles.</p> <p><math>\overrightarrow{OB} = \mathbf{x}</math> and <math>\overrightarrow{OD} = \mathbf{y}</math>.</p>  <p>Define the following vectors in terms of <math>\mathbf{x}</math> and <math>\mathbf{y}</math>.</p> |
| <p>(a) <math>\overrightarrow{OE}</math></p> <p>(b) <math>\overrightarrow{OF}</math></p> <p>(c) <math>\overrightarrow{GJ}</math></p> <p>(d) <math>\overrightarrow{MP}</math></p> <p>(e) <math>\overrightarrow{AO}</math></p> <p>(f) <math>\overrightarrow{LD}</math></p>                                                                                   | <p>(a) <math>\overrightarrow{EG}</math></p> <p>(b) <math>\overrightarrow{BS}</math></p> <p>(c) <math>\overrightarrow{KJ}</math></p> <p>(d) <math>\overrightarrow{RE}</math></p> <p>(e) <math>\overrightarrow{OF}</math></p> <p>(f) <math>\overrightarrow{JT}</math></p>                                                                                            | <p>(a) <math>\overrightarrow{FD}</math></p> <p>(b) <math>\overrightarrow{HA}</math></p> <p>(c) <math>\overrightarrow{GH}</math></p> <p>(d) <math>\overrightarrow{OC}</math></p> <p>(e) <math>\overrightarrow{DB}</math></p> <p>(f) <math>\overrightarrow{OE}</math></p>                                                                                                 |
| <p>(g) <math>\overrightarrow{OH}</math></p> <p>(h) <math>\overrightarrow{DP}</math></p> <p>(i) <math>\overrightarrow{IM}</math></p> <p>(j) <math>\overrightarrow{MD}</math></p> <p>(k) <math>\overrightarrow{CB}</math></p> <p>(l) <math>\overrightarrow{NO}</math></p>                                                                                   | <p>(g) <math>\overrightarrow{MJ}</math></p> <p>(h) <math>\overrightarrow{QF}</math></p> <p>(i) <math>\overrightarrow{IL}</math></p> <p>(j) <math>\overrightarrow{HB}</math></p> <p>(k) <math>\overrightarrow{TE}</math></p> <p>(l) <math>\overrightarrow{KD}</math></p>                                                                                            | <p>(g) <math>\overrightarrow{FO}</math></p> <p>(h) <math>\overrightarrow{CE}</math></p> <p>(i) <math>\overrightarrow{HI}</math></p> <p>(j) <math>\overrightarrow{BG}</math></p> <p>(k) <math>\overrightarrow{HD}</math></p> <p>(l) <math>\overrightarrow{IF}</math></p>                                                                                                 |

| Vectors and Midpoints                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                   |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                             |
|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| <p><b>(a)</b></p> <p>In the triangle <math>OAB</math>, <math>\overrightarrow{OA} = 2\mathbf{a}</math> and <math>\overrightarrow{OB} = 4\mathbf{b}</math>. <math>C</math> is the midpoint of the line <math>AB</math>.</p> <p>Express the following in terms of <math>\mathbf{a}</math> and <math>\mathbf{b}</math>:</p> <div style="display: flex; justify-content: space-between;"> <div> <p>(a) <math>\overrightarrow{AB}</math></p> <p>(c) <math>\overrightarrow{AC}</math></p> <p>(e) <math>\overrightarrow{OC}</math></p> </div> <div> <p>(b) <math>\overrightarrow{BA}</math></p> <p>(d) <math>\overrightarrow{BC}</math></p> <p>(f) <math>\overrightarrow{CO}</math></p> </div> </div>                                                                                                                           | <p><b>(b)</b></p> <p><math>OABC</math> is a trapezium, where <math>\overrightarrow{OA} = \mathbf{a}</math> and <math>\overrightarrow{AB} = \mathbf{b}</math>. <math>D</math> is the midpoint of <math>BC</math> and <math>\overrightarrow{OC} = 2\overrightarrow{AB}</math>.</p> <p>Express the following in terms of <math>\mathbf{a}</math> and <math>\mathbf{b}</math>:</p> <div style="display: flex; justify-content: space-between;"> <div> <p>(a) <math>\overrightarrow{OC}</math></p> <p>(c) <math>\overrightarrow{BC}</math></p> <p>(e) <math>\overrightarrow{AD}</math></p> </div> <div> <p>(b) <math>\overrightarrow{CB}</math></p> <p>(d) <math>\overrightarrow{BD}</math></p> <p>(f) <math>\overrightarrow{DO}</math></p> </div> </div>                                                                                                                                                                                                        |
| <p><b>(c)</b></p> <p>In the parallelogram <math>OABC</math>, <math>\overrightarrow{OA} = \mathbf{a}</math> and <math>\overrightarrow{OC} = \mathbf{c}</math>. <math>X</math> is the midpoint of the line <math>OB</math>.</p> <p>Express the following in terms of <math>\mathbf{a}</math> and <math>\mathbf{c}</math>:</p> <div style="display: flex; justify-content: space-between;"> <div> <p>(a) <math>\overrightarrow{CB}</math></p> <p>(c) <math>\overrightarrow{OB}</math></p> <p>(e) <math>\overrightarrow{XC}</math></p> </div> <div> <p>(b) <math>\overrightarrow{BA}</math></p> <p>(d) <math>\overrightarrow{XB}</math></p> <p>(f) <math>\overrightarrow{AX}</math></p> </div> </div> <p>What do the answers to (e) and (f) tell us about the points <math>C</math>, <math>X</math> and <math>A</math>?</p> | <p><b>(d)</b></p> <p><math>OABC</math> is a quadrilateral. <math>\overrightarrow{OX} = \mathbf{a}</math>, <math>\overrightarrow{OC} = \mathbf{c}</math> and <math>\overrightarrow{CB} = \mathbf{b}</math>. <math>X</math> is the midpoint of <math>OA</math> and <math>Y</math> is the midpoint of <math>AB</math>.</p> <p>Express the following in terms of <math>\mathbf{a}</math>, <math>\mathbf{b}</math> and <math>\mathbf{c}</math>:</p> <div style="display: flex; justify-content: space-between;"> <div> <p>(a) <math>\overrightarrow{OA}</math></p> <p>(c) <math>\overrightarrow{CA}</math></p> <p>(e) <math>\overrightarrow{AY}</math></p> </div> <div> <p>(b) <math>\overrightarrow{OB}</math></p> <p>(d) <math>\overrightarrow{AB}</math></p> <p>(f) <math>\overrightarrow{XY}</math></p> </div> </div> <p>What do the answers to (b) and (f) tell us about vectors <math>\overrightarrow{OB}</math> and <math>\overrightarrow{XY}</math>?</p> |

# Fluency Practice

## Vectors Around Shapes

$$\overrightarrow{WY} =$$

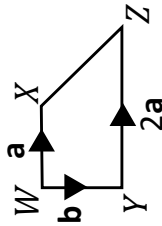
$$\overrightarrow{WZ} =$$

$$\overrightarrow{XZ} =$$

1)  $\overrightarrow{WX} =$

$$\overrightarrow{YZ} =$$

$$\overrightarrow{XY} =$$



$$\overrightarrow{WV} =$$

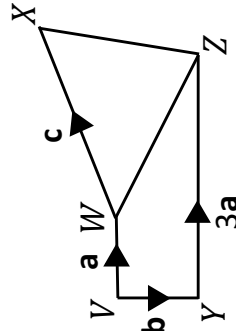
$$\overrightarrow{VZ} =$$

$$\overrightarrow{XZ} =$$

2)  $\overrightarrow{VX} =$

$$\overrightarrow{XV} =$$

$$\overrightarrow{WZ} =$$



$$\overrightarrow{VY} =$$

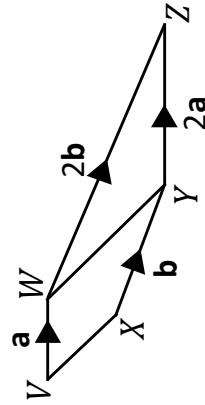
$$\overrightarrow{WY} =$$

$$\overrightarrow{XV} =$$

3)  $\overrightarrow{VZ} =$

$$\overrightarrow{XZ} =$$

$$\overrightarrow{YW} =$$



$$\overrightarrow{VX} =$$

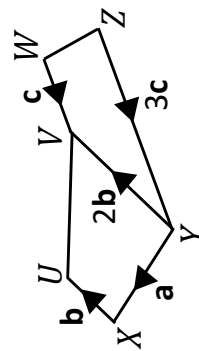
$$\overrightarrow{UY} =$$

$$\overrightarrow{UW} =$$

4)  $\overrightarrow{YZ} =$

$$\overrightarrow{ZW} =$$

$$\overrightarrow{UZ} =$$



$$\overrightarrow{WT} =$$

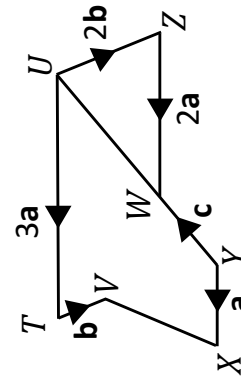
$$\overrightarrow{XZ} =$$

$$\overrightarrow{XV} =$$

5)  $\overrightarrow{VZ} =$

$$\overrightarrow{ZX} =$$

$$\overrightarrow{YT} =$$



$$\overrightarrow{WV} =$$

$$\overrightarrow{UV} =$$

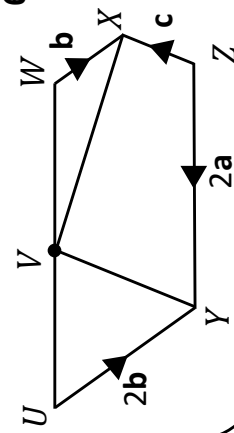
$$\overrightarrow{YV} =$$

6) (V is the midpoint of UW)

$$\overrightarrow{WU} =$$

$$\overrightarrow{UW} =$$

$$\overrightarrow{XV} =$$

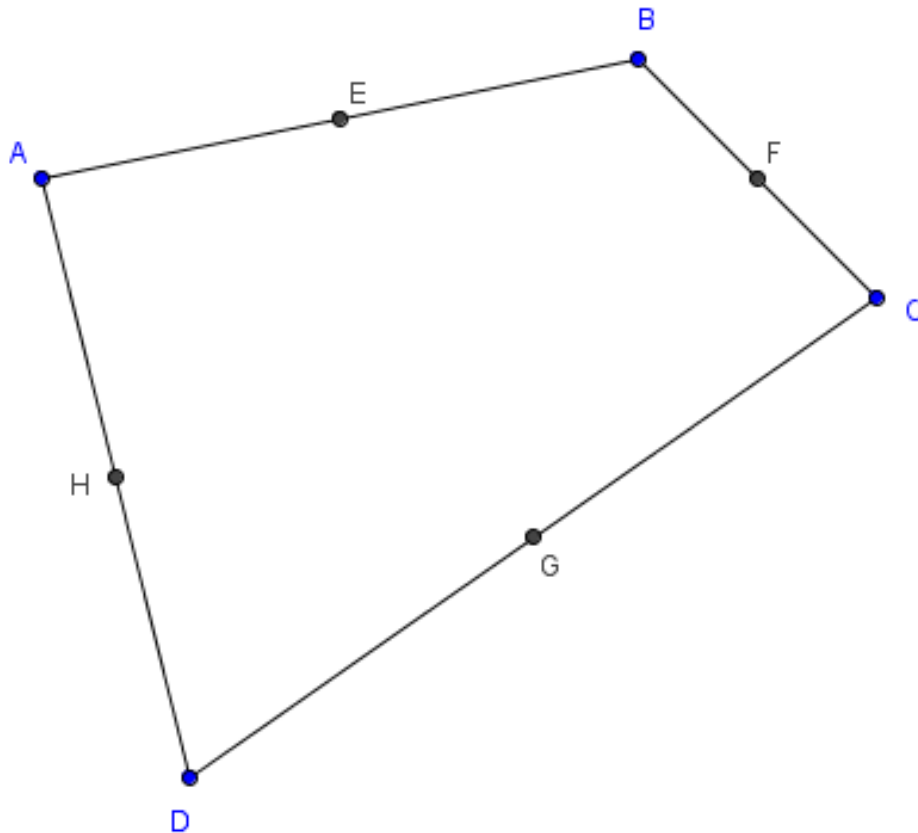


## Purposeful Practice

The diagram shows the quadrilateral ABCD.

The point E is the midpoint of A and B. The point F is the midpoint of B and C.

The point G is the midpoint of C and D. The point H is the midpoint of D and A.



$$\overrightarrow{AB} = \mathbf{a} \quad \overrightarrow{BC} = \mathbf{b} \quad \overrightarrow{CD} = \mathbf{c}$$

1. Express in terms of  $\mathbf{a}$ ,  $\mathbf{b}$  and/or  $\mathbf{c}$ :

$$\overrightarrow{DC} =$$

$$\overrightarrow{AD} =$$

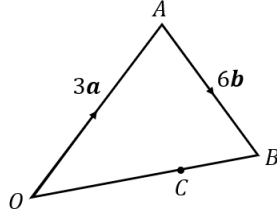
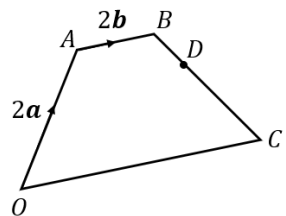
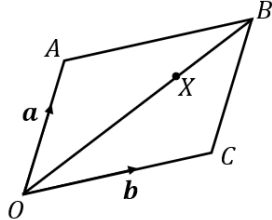
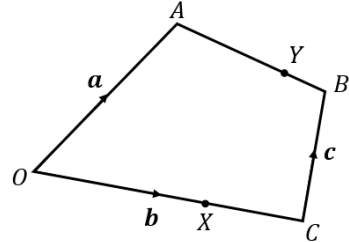
2. Express in terms of  $\mathbf{a}$ ,  $\mathbf{b}$  and/or  $\mathbf{c}$ :

$$\overrightarrow{HE} =$$

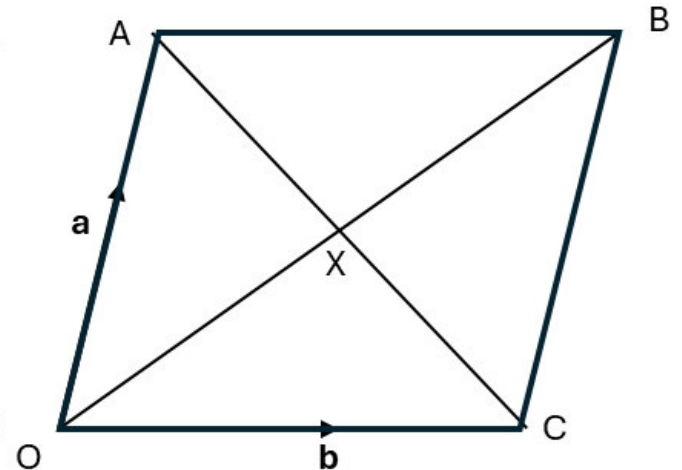
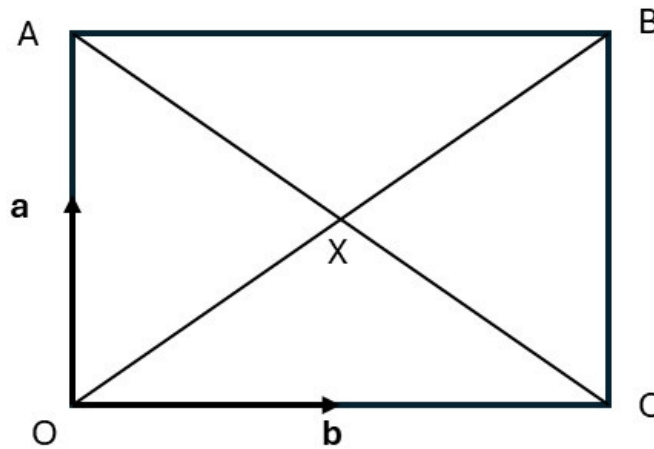
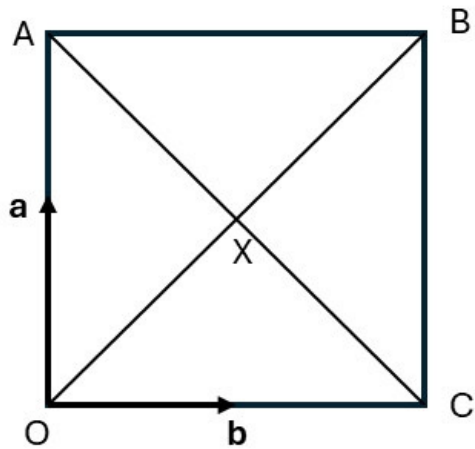
$$\overrightarrow{GF} =$$

What do you notice?

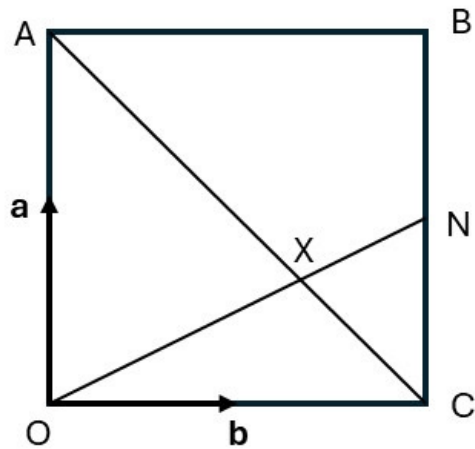
## Purposeful Practice

| Vectors and Ratio                                                                                                                                                                                                                                                                                                                        |                           |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                           |                           |
|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|---------------------------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|---------------------------|
| <b>(a)</b>                                                                                                                                                                                                                                                                                                                               |                           | <b>(b)</b>                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                |                           |
| <p>In the triangle <math>OAB</math>, <math>\overrightarrow{OA} = 3\mathbf{a}</math> and <math>\overrightarrow{AB} = 6\mathbf{b}</math>. The point <math>C</math> divides the line <math>OB</math> in the ratio <math>2 : 1</math>.</p> <p>Express the following in terms of <math>\mathbf{a}</math> and <math>\mathbf{b}</math>:</p>     |                           | <p><math>OABC</math> is a trapezium, where <math>\overrightarrow{OA} = 2\mathbf{a}</math> and <math>\overrightarrow{AB} = 2\mathbf{b}</math>. <math>\overrightarrow{OC} = 2\overrightarrow{AB}</math> and <math>D</math> splits the line <math>BC</math> in the ratio <math>1 : 3</math>.</p> <p>Express the following in terms of <math>\mathbf{a}</math> and <math>\mathbf{b}</math>:</p>                                                                                                               |                           |
|                                                                                                                                                                                                                                                         |                           |                                                                                                                                                                                                                                                                                                                                                                                                                        |                           |
| (a) $\overrightarrow{OB}$                                                                                                                                                                                                                                                                                                                | (b) $\overrightarrow{OC}$ | (a) $\overrightarrow{BO}$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                 | (b) $\overrightarrow{BC}$ |
| (c) $\overrightarrow{BC}$                                                                                                                                                                                                                                                                                                                | (d) $\overrightarrow{AC}$ | (c) $\overrightarrow{BD}$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                 | (d) $\overrightarrow{DO}$ |
| <b>(c)</b>                                                                                                                                                                                                                                                                                                                               |                           | <b>(d)</b>                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                |                           |
| <p>In the parallelogram <math>OABC</math>, <math>\overrightarrow{OA} = \mathbf{a}</math> and <math>\overrightarrow{OC} = \mathbf{b}</math>. The point <math>X</math> divides the line <math>OB</math> in the ratio <math>3 : 2</math>.</p> <p>Express the following in terms of <math>\mathbf{a}</math> and <math>\mathbf{b}</math>:</p> |                           | <p><math>OABC</math> is a quadrilateral. <math>\overrightarrow{OA} = \mathbf{a}</math>, <math>\overrightarrow{OC} = \mathbf{b}</math> and <math>\overrightarrow{CB} = \mathbf{c}</math>. The point <math>X</math> divides the line <math>OC</math> in the ratio <math>2 : 1</math>. The point <math>Y</math> divides the line <math>AB</math> in the ratio <math>3 : 1</math>.</p> <p>Express the following in terms of <math>\mathbf{a}</math>, <math>\mathbf{b}</math> and <math>\mathbf{c}</math>:</p> |                           |
|                                                                                                                                                                                                                                                         |                           |                                                                                                                                                                                                                                                                                                                                                                                                                        |                           |
| (a) $\overrightarrow{AB}$                                                                                                                                                                                                                                                                                                                | (b) $\overrightarrow{OB}$ | (a) $\overrightarrow{OX}$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                 | (b) $\overrightarrow{XC}$ |
| (c) $\overrightarrow{OX}$                                                                                                                                                                                                                                                                                                                | (d) $\overrightarrow{BX}$ | (c) $\overrightarrow{AB}$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                 | (d) $\overrightarrow{AY}$ |
| (e) $\overrightarrow{AX}$                                                                                                                                                                                                                                                                                                                | (f) $\overrightarrow{XC}$ | (e) $\overrightarrow{AX}$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                 | (f) $\overrightarrow{XY}$ |

## Purposeful Practice

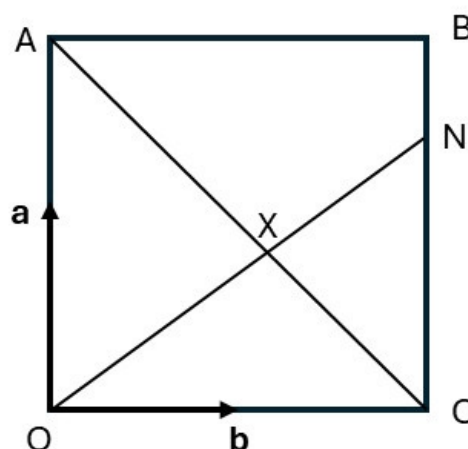


What is the ratio of  $OX:XB$  and  $AX:AC$

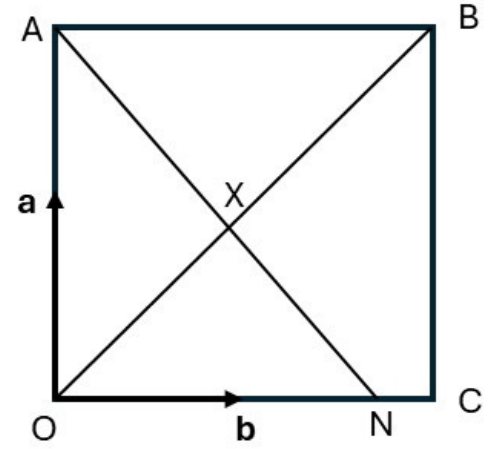


N is the midpoint of BC

What is the ratio of  $OX:ON$  and  $AX:AC$



Ratio of  $BN:NC$  is 1:3



Ratio of  $ON:OC$  is 4:5

What fraction of AN is AX?

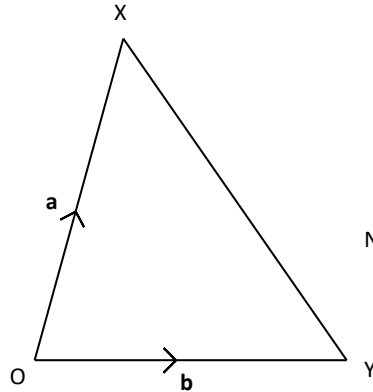


# Purposeful Practice

## Vectors Around Shapes

1)

M is the midpoint of line XY.



$$\overrightarrow{YX} =$$

$$\overrightarrow{YM} =$$

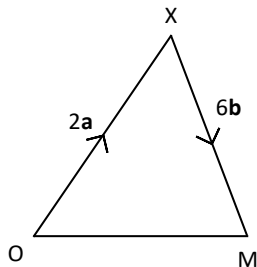
$$\overrightarrow{OM} =$$

N is the midpoint of line OM.

$$\overrightarrow{ON} =$$

2)

M divides the line OY in the ratio 2:3



$$\overrightarrow{OM} =$$

$$\overrightarrow{OY} =$$

$$\overrightarrow{MY} =$$

$$\overrightarrow{YM} =$$

$$\overrightarrow{YX} =$$

3)

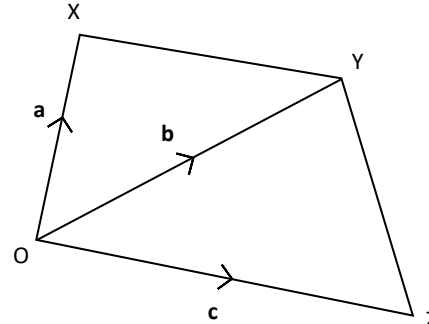
M is the midpoint of line XY.

N divides the line ZY in the ratio 1:2

$$\overrightarrow{XM} =$$

$$\overrightarrow{ZN} =$$

Express using **a**, **b** & **c**.



$$\overrightarrow{ZM} =$$

$$\overrightarrow{MN} =$$

4)

OZ = 2XY

M divides the line XZ in the ratio 1:2

N divides the line YZ in the ratio 1:2

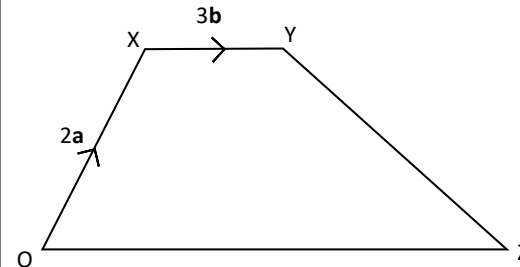
$$\overrightarrow{XZ} =$$

$$\overrightarrow{XM} =$$

$$\overrightarrow{YZ} =$$

$$\overrightarrow{YN} =$$

$$\overrightarrow{MN} =$$

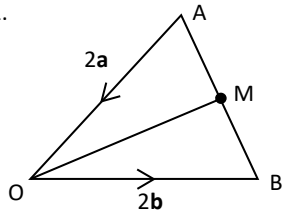


What statements can you make about the magnitude & direction of lines XY & MN?

# Purposeful Practice

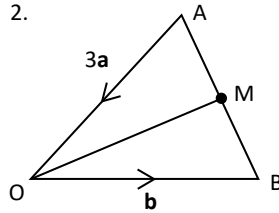
## Vector Proofs

1.



Prove  $\overrightarrow{AM} = \mathbf{a} + \mathbf{b}$

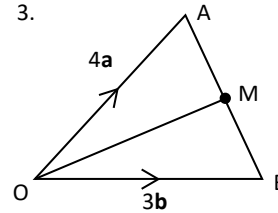
2.



Prove  $\overrightarrow{AM} = \frac{3}{2}\mathbf{a} + \frac{1}{2}\mathbf{b}$

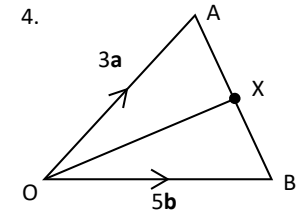
M is the midpoint of the line.

3.



Prove  $\overrightarrow{AM} = \frac{3}{2}\mathbf{b} - 2\mathbf{a}$

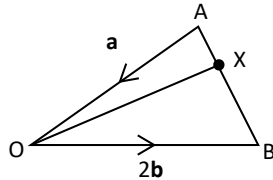
4.



$AX:AB = 1:2$

Prove  $\overrightarrow{AX} = 2.5\mathbf{b} - 1.5\mathbf{a}$

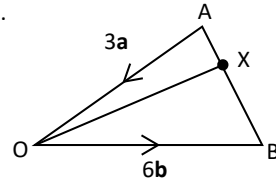
5.



$AX:AB = 1:3$

Prove  $\overrightarrow{AX} = \frac{1}{3}\mathbf{a} + \frac{2}{3}\mathbf{b}$

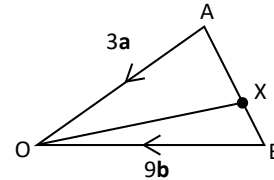
6.



$AX:XB = 1:2$

Prove  $\overrightarrow{AX} = \mathbf{a} + 2\mathbf{b}$

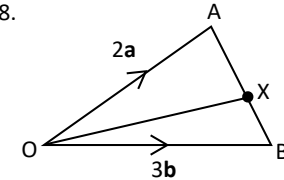
7.



$AX:XB = 2:1$

Prove  $\overrightarrow{OX} = -\mathbf{a} - 6\mathbf{b}$

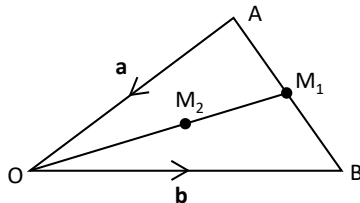
8.



$AX:XB = 3:2$

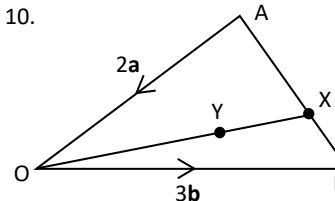
Prove  $\overrightarrow{AX} = 1.8\mathbf{b} - 1.2\mathbf{a}$

9.



Prove  $\overrightarrow{AM_2} = \frac{1}{4}\mathbf{b} - \frac{1}{4}\mathbf{a}$

10.



$AX:XB = 2:1$

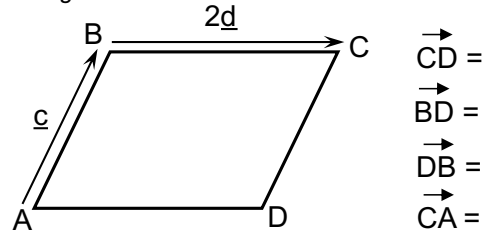
$OY:YX = 2:1$

Prove  $\overrightarrow{OY} = \frac{4}{3}\mathbf{b} - \frac{4}{9}\mathbf{a}$

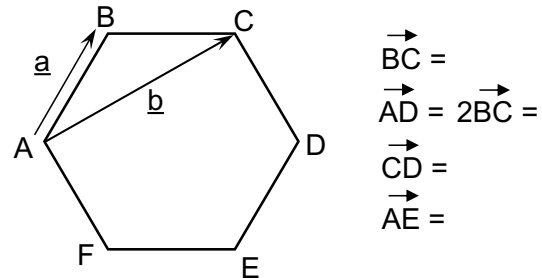
# Purposeful Practice

## vector questions (i)

a parallelogram

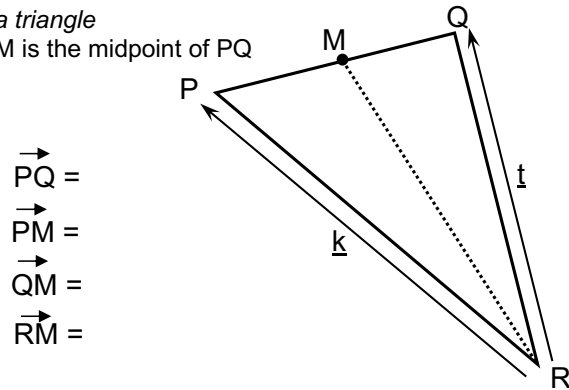


a regular hexagon

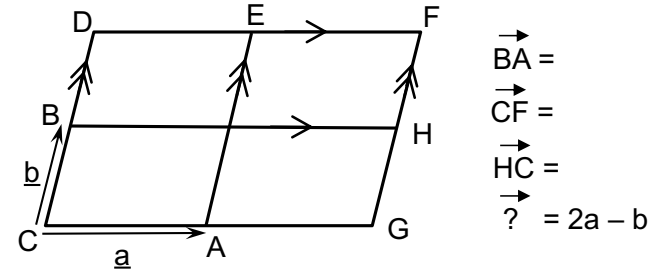


a triangle

M is the midpoint of PQ



a parallelogram



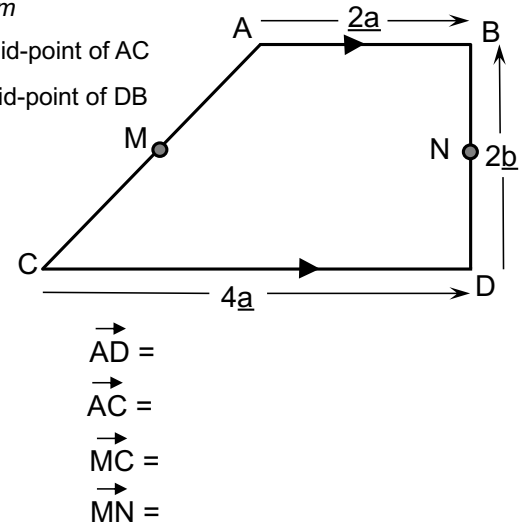
A is the mid-point of CG

B is the mid-point of CD

a trapezium

M is the mid-point of AC

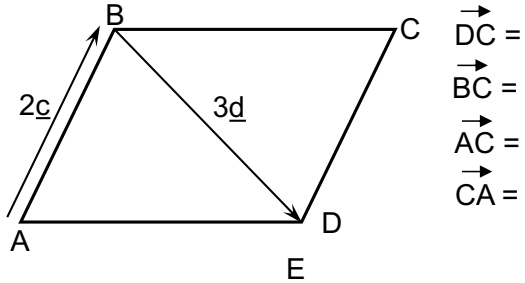
N is the mid-point of DB



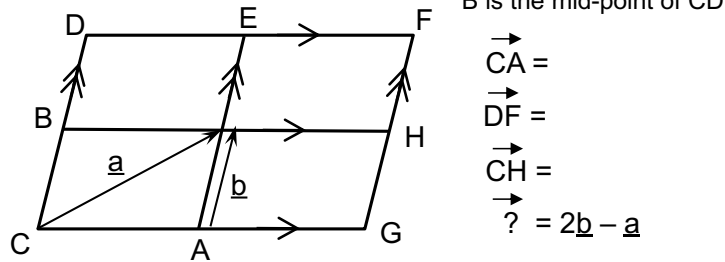
# Purposeful Practice

vector questions (ii)

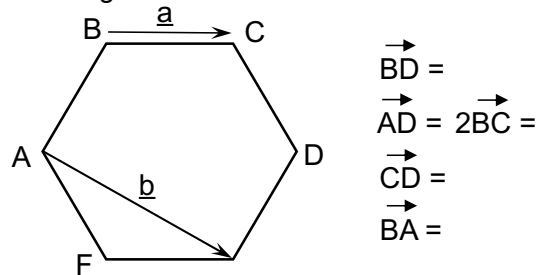
*a parallelogram*



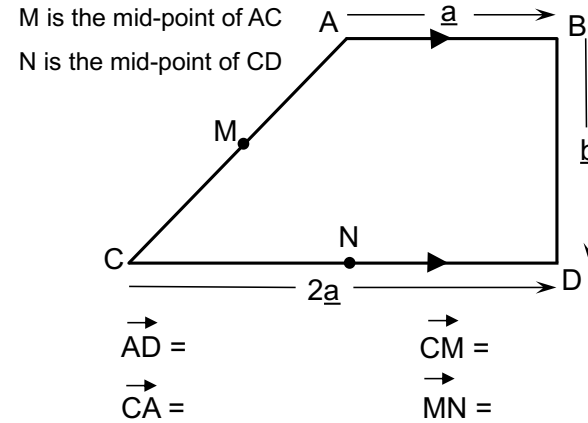
*a parallelogram*



*a regular hexagon*

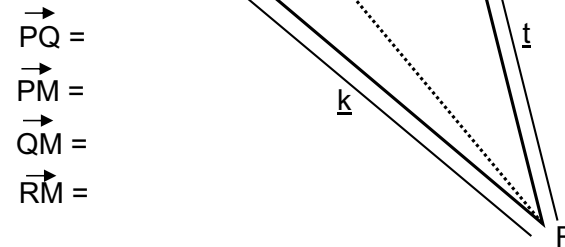


*a trapezium*



*a triangle*

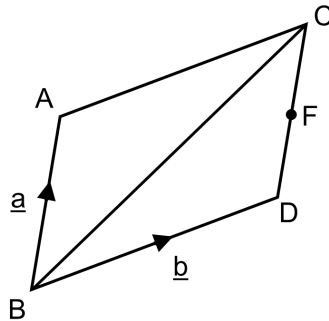
$M$  divides  $PQ$  in the ratio  $1 : 2$



# Purposeful Practice

## vector questions (iii)

ACDB is a parallelogram



F is the midpoint of CD.

work out the vectors:

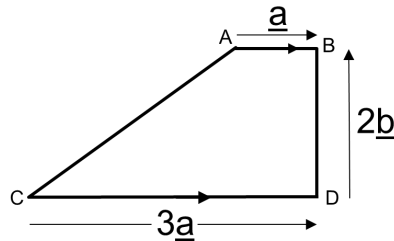
$$\vec{DC} = ?$$

$$\vec{DF} = ?$$

$$\vec{AD} = ?$$

$$\vec{CB} = ?$$

$$\vec{BF} = ?$$

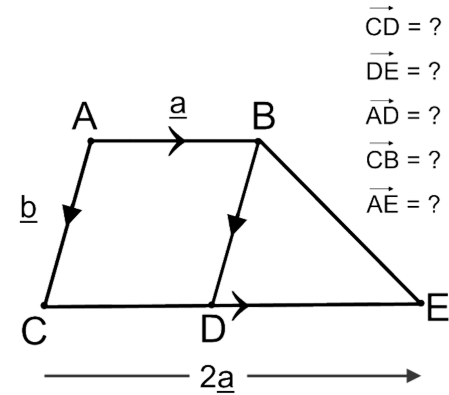


find, in terms of  $\mathbf{a}$  and  $\mathbf{b}$ , the vectors:

1)  $\vec{CA}$

2)  $\vec{CB}$

3)  $\vec{DA}$



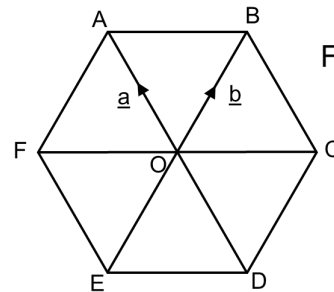
$$\vec{CD} = ?$$

$$\vec{DE} = ?$$

$$\vec{AD} = ?$$

$$\vec{CB} = ?$$

$$\vec{AE} = ?$$



FEDCBA is a regular hexagon

$$OA = \underline{\mathbf{a}}$$

$$OB = \underline{\mathbf{b}}$$

write down, in terms of  $\mathbf{a}$  and  $\mathbf{b}$ , the vectors

(i)  $\vec{AB}$

(ii)  $\vec{FC}$

what fact can you deduce about  $\vec{AB}$  and  $\vec{FC}$ ?

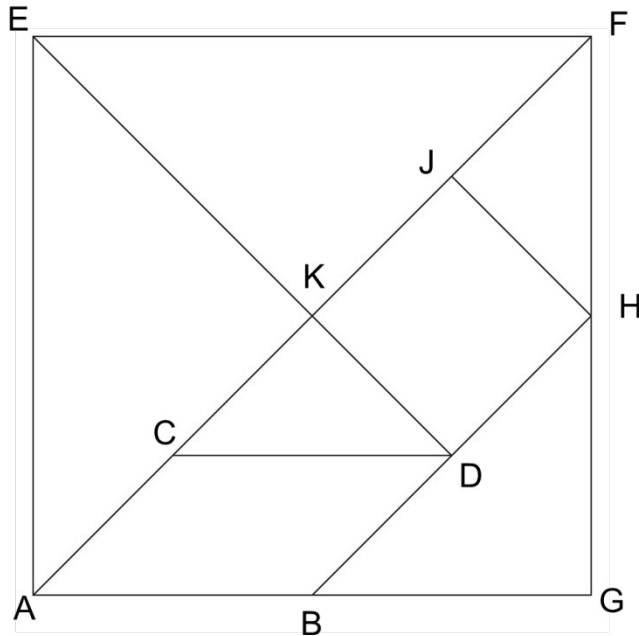
# Purposeful Practice

vector questions (iv)

a tangram puzzle

$$AB = \underline{b}$$

$$AC = \underline{c}$$



$$\vec{AF} =$$

$$\vec{BH} =$$

$$\vec{DK} =$$

$$\vec{GH} =$$

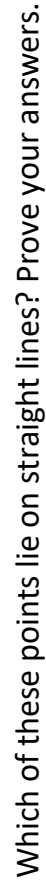
$$\vec{JC} =$$

$$\vec{BH} =$$

$$\vec{HC} =$$

$$\vec{DE} =$$

## Page 213

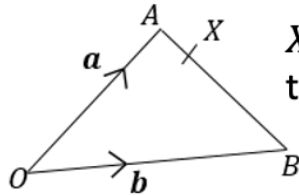


- Work out the ratio of the lengths of the line segments

The point X is between A and B so that XDN is a straight line. Find the ratio AX:XB

## Purposeful Practice

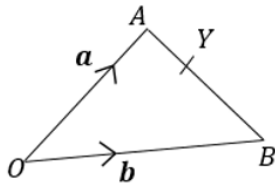
1



$X$  is a point such that  $AX:XB = 1:4$

- $\overrightarrow{AB} =$
- $\overrightarrow{AX} =$
- $\overrightarrow{OX} =$
- $\overrightarrow{BX} =$

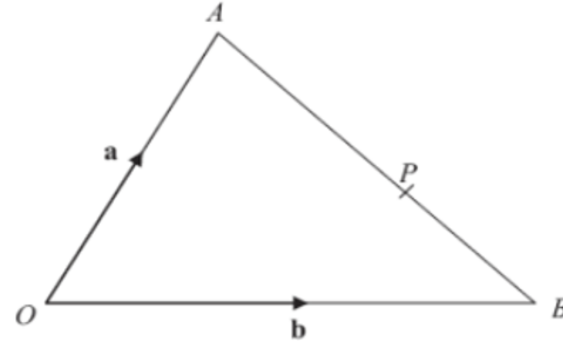
2



$Y$  is a point such that  $YB = 2AY$

- $\overrightarrow{AY} =$
- $\overrightarrow{OY} =$
- $\overrightarrow{YO} =$

3



[June 2009 2H Q23]

a) Find  $\overrightarrow{AB}$  in terms of  $\mathbf{a}$  and  $\mathbf{b}$ .

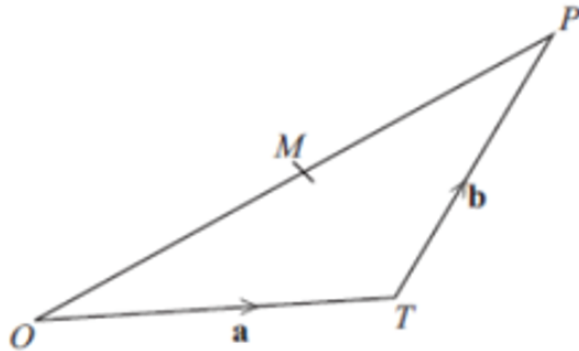
b)  $P$  is on  $AB$  such that  $AP:PB = 3:2$ .

Show that  $\overrightarrow{OP} = \frac{1}{5}(2\mathbf{a} + 3\mathbf{b})$



## Purposeful Practice

- 4 [Nov 2010 1H Q27]  $M$  is the midpoint of  $OP$ .



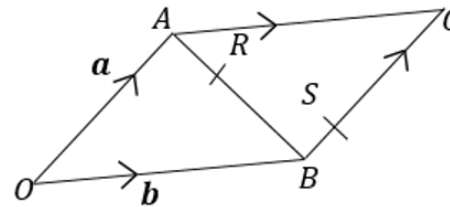
- a) Express  $\overrightarrow{OM}$  in terms of  $\mathbf{a}$  and  $\mathbf{b}$ .

?

- b) Express  $\overrightarrow{TM}$  in terms of  $\mathbf{a}$  and  $\mathbf{b}$  giving your answer in its simplest form.

?

5



$OACB$  is a parallelogram.  $R$  is a point such that  $AR:RB = 2:3$ .  $S$  is a point such that  $BS:SC = 1:3$ .

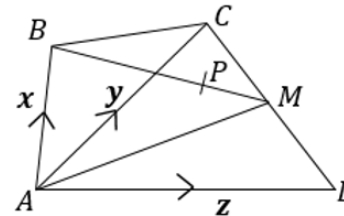
a.  $\overrightarrow{OR} =$  ?

b.  $\overrightarrow{BS} =$  ?

c.  $\overrightarrow{OS} =$  ?

d.  $\overrightarrow{RS} =$  ?

6



$M$  is the midpoint of  $CD$ ,  $BP:PM = 2:1$

a.  $\overrightarrow{DC} =$  ?

b.  $\overrightarrow{DM} =$  ?

c.  $\overrightarrow{AM} =$  ?

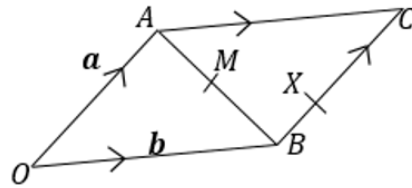
d.  $\overrightarrow{BM} =$  ?

e.  $\overrightarrow{BP} =$  ?

f.  $\overrightarrow{AP} =$  ?

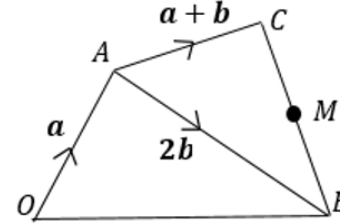
## Purposeful Practice

7



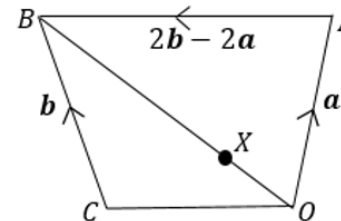
- a.  $\overrightarrow{AB} =$
- b.  $\overrightarrow{BC} =$
- c.  $\overrightarrow{MB} =$
- d.  $\overrightarrow{MX} =$
- e.  $\overrightarrow{XA} =$
- f.  $\overrightarrow{CM} =$
- g.  $\overrightarrow{XO} =$

8



- a.  $\overrightarrow{OB} =$
- b.  $\overrightarrow{BC} =$
- c.  $\overrightarrow{AM} =$
- d.  $\overrightarrow{OM} =$

9



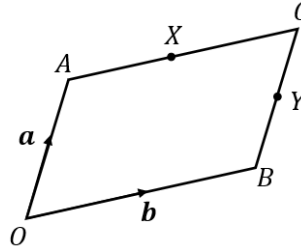
- a.  $\overrightarrow{OC} =$
- b.  $\overrightarrow{XC} =$
- c.  $\overrightarrow{AX} =$

## Purposeful Practice

### Vector Proof – Parallel Lines

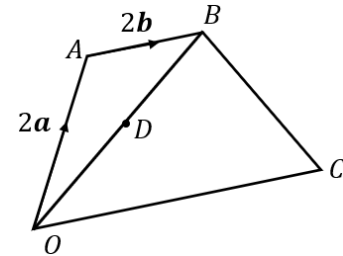
**(a)**

$OACB$  is a parallelogram.  $\vec{OA} = \mathbf{a}$  and  $\vec{OB} = \mathbf{b}$ .  $X$  is the midpoint of  $AC$  and  $Y$  is the midpoint of  $BC$ . Show that  $XY$  and  $AB$  are parallel.



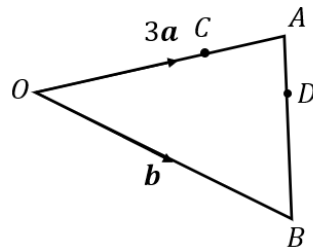
**(b)**

$OACB$  is a trapezium.  $\vec{OA} = 2\mathbf{a}$  and  $\vec{AB} = 2\mathbf{b}$ .  $\vec{OC} = 2\vec{AB}$  and  $D$  is the midpoint of  $OB$ . Show that  $AD$  is parallel to  $BC$ .



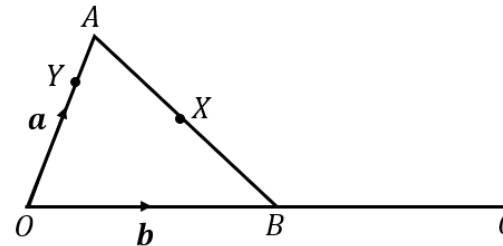
**(c)**

In the triangle  $OAB$ ,  $\vec{OB} = \mathbf{b}$  and  $\vec{OA} = 3\mathbf{a}$ . The point  $C$  divides the line  $OA$  in the ratio  $2 : 1$  and the point  $D$  divides the line  $AB$  in the ratio  $1 : 2$ . Show that  $CD$  is parallel to  $OB$ .



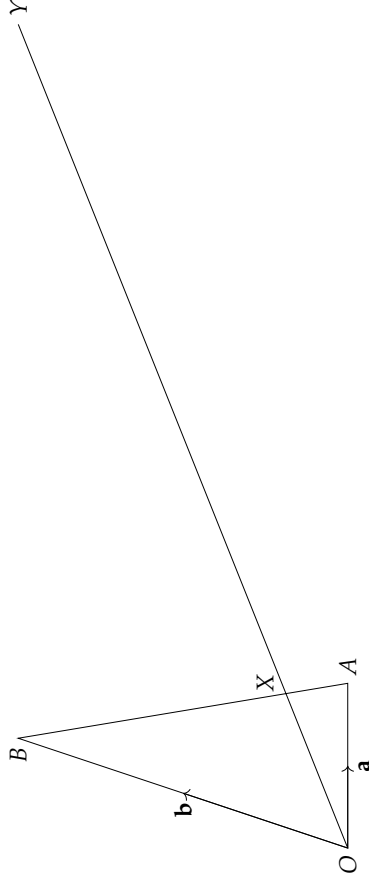
**(d)**

In the triangle  $OAB$ ,  $\vec{OB} = \mathbf{b}$  and  $\vec{OA} = \mathbf{a}$ . Point  $B$  is the midpoint of the line  $OC$  and  $X$  is the midpoint of  $AB$ . The point  $Y$  divides the line  $OA$  in the ratio  $3 : 1$ . Show that  $YX$  is parallel to  $AC$ .



## Purposeful Practice

1.

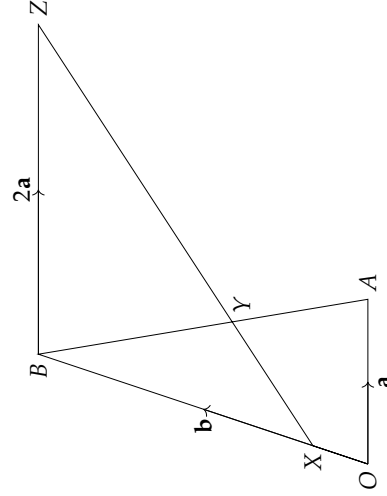


In the diagram,

- $\vec{OA} = \mathbf{a}$
- $\vec{OB} = \mathbf{b}$
- $AX : XB = 1 : 4$
- $\vec{OY} = 5 \times \vec{OX}$

Show that  $\vec{OA}$  and  $\vec{BY}$  are parallel.

2.



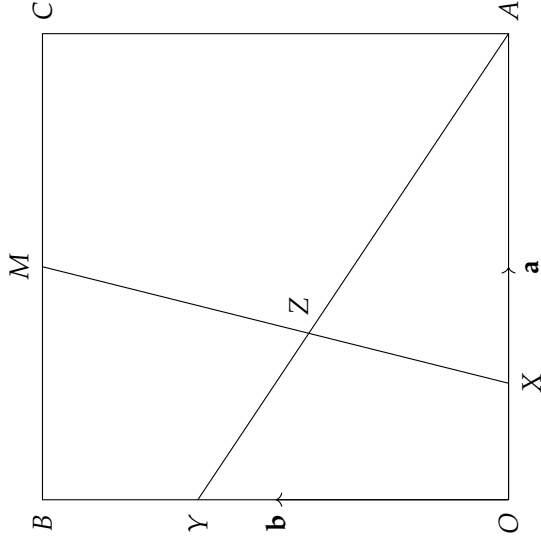
In the diagram,

- $\vec{OA} = \mathbf{a}$
- $\vec{OB} = \mathbf{b}$
- $\vec{BY} = 2\mathbf{b}$
- $OX : XB = 1 : 5$

Find the ratio  $BY : BA$ .

## Purposeful Practice

3.

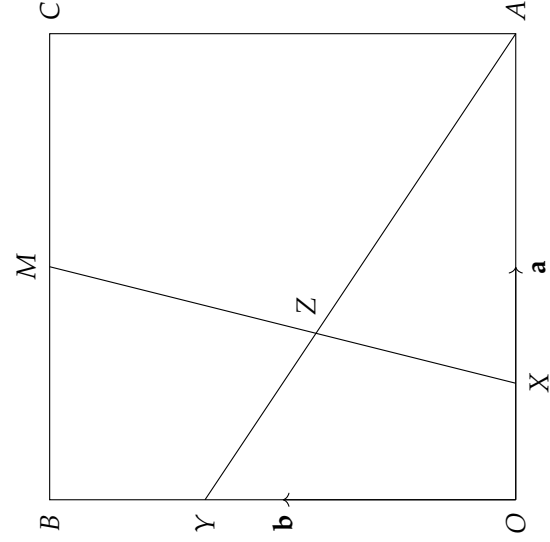


In the diagram,

- $ABCD$  is a square.
- $\vec{OA} = \mathbf{a}$
- $\vec{OB} = \mathbf{b}$
- $M$  is the midpoint of  $BC$
- $OY : YB = 2 : 1$
- $OX : XA = 1 : 3$

Find the vector  $\vec{OZ}$ .

4.



In the diagram,

- $ABCD$  is a square.

## Purposeful Practice

- $\vec{OA} = \mathbf{a}$
- $\vec{OB} = \mathbf{b}$
- $M$  is the midpoint of  $BC$
- $\vec{OX} = m \times \vec{OA}$
- $\vec{OY} = n \times \vec{OB}$

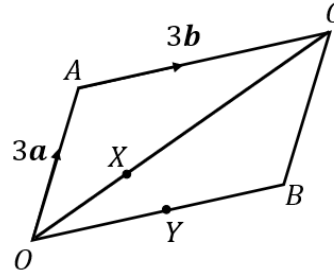
Find the vector  $\vec{OZ}$ , giving your answer in terms of  $m$  and  $n$ .

## Purposeful Practice

### Vector Proof – Collinear Points

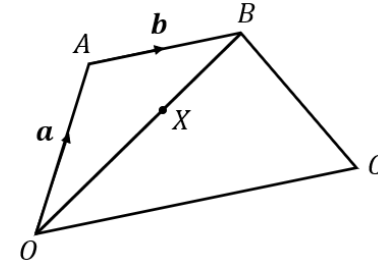
**(a)**

$OACB$  is a parallelogram.  $\overrightarrow{OA} = 3\mathbf{a}$  and  $\overrightarrow{AC} = 3\mathbf{b}$ .  $Y$  is the midpoint of  $OB$  and  $X$  divides the line  $OC$  in the ratio  $1 : 2$ . Show that the points  $A, X$  and  $Y$  are collinear.



**(b)**

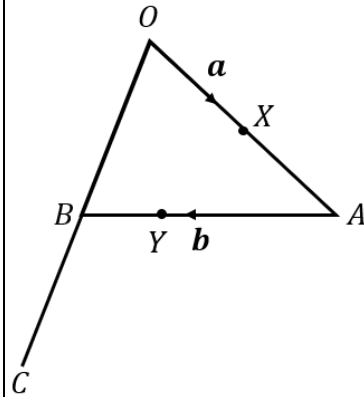
$OACB$  is a trapezium.  $\overrightarrow{OA} = \mathbf{a}$  and  $\overrightarrow{AB} = \mathbf{b}$ .  $\overrightarrow{OC} = 3\overrightarrow{AB}$  and  $X$  divides the line  $OB$  in the ratio  $3 : 1$ . Show that the points  $A, X$  and  $C$  are collinear.



**(c)**

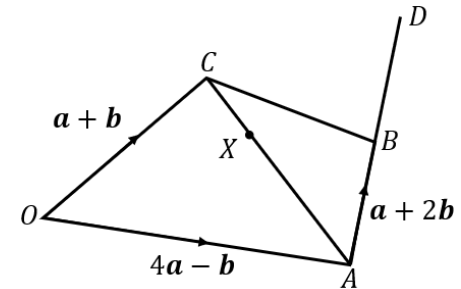
In the triangle  $OAB$ ,  $\overrightarrow{OX} = \mathbf{a}$  and  $\overrightarrow{AB} = \mathbf{b}$ .  $X$  is the midpoint of  $OA$  and the point  $Y$  divides the line  $AB$  in the ratio  $2 : 1$ .

$\overrightarrow{OB} = \overrightarrow{BC}$ . Show that the points  $X, Y$  and  $C$  are collinear.



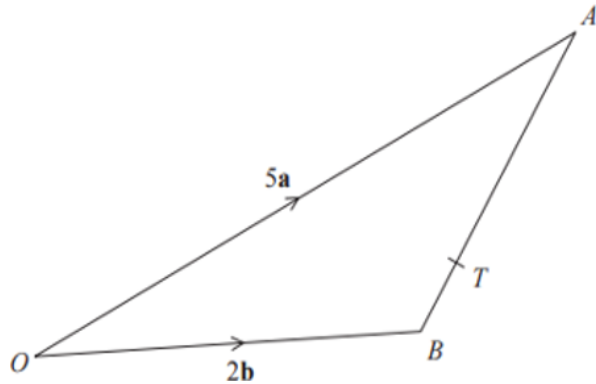
**(d)**

$\overrightarrow{OA} = 4\mathbf{a} - \mathbf{b}$ ,  $\overrightarrow{AB} = \mathbf{a} + 2\mathbf{b}$  and  $\overrightarrow{OC} = \mathbf{a} + \mathbf{b}$ .  $\overrightarrow{AB} = \overrightarrow{BD}$ . The point  $X$  divides the line  $AC$  in the ratio  $6 : 1$ . Show that  $O, X$  and  $D$  are collinear.



## Purposeful Practice

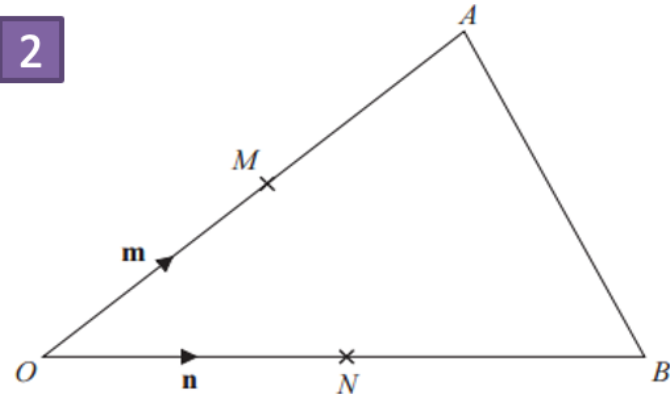
1



$T$  is the point on  $AB$  such that  $AT:TB = 5:1$ . Show that  $OT$  is parallel to the vector  $\mathbf{a} + 2\mathbf{b}$ .

?

2



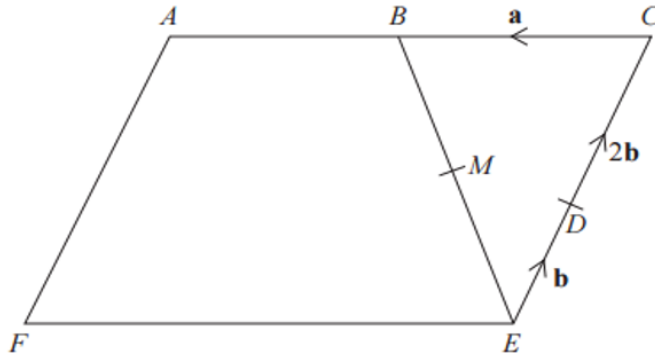
$M$  is the midpoint of  $OA$ .  $N$  is the midpoint of  $OB$ . Prove that  $AB$  is parallel to  $MN$ .

?



## Purposeful Practice

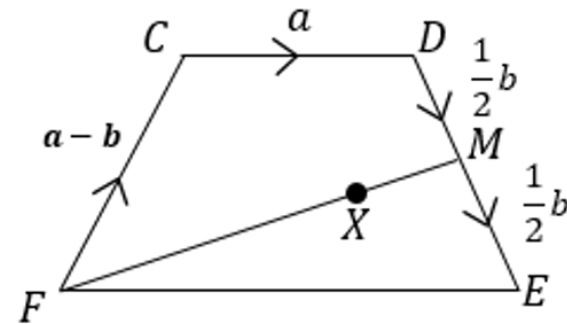
3



$ACEF$  is a parallelogram.  $B$  is the midpoint of  $AC$ .  $M$  is the midpoint of  $BE$ . Show that  $AMD$  is a straight line.

?

4



$\vec{CD} = \mathbf{a}$ ,  $\vec{DE} = \mathbf{b}$  and  $\vec{FC} = \mathbf{a} - \mathbf{b}$

i) Express  $\vec{CE}$  in terms of  $\mathbf{a}$  and  $\mathbf{b}$ .

?

ii) Prove that  $\vec{FE}$  is parallel to  $\vec{CD}$ .

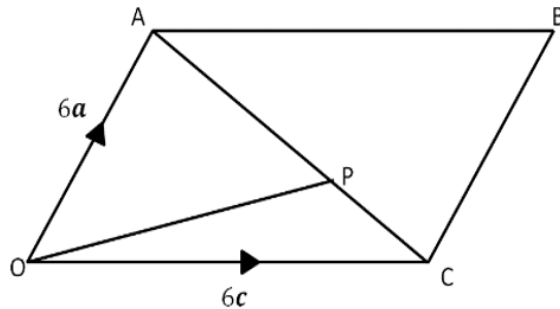
?

iii)  $X$  is the point on  $FM$  such that  $FX:XM = 4:1$ . Prove that  $C$ ,  $X$  and  $E$  lie on the same straight line.

?

## Purposeful Practice

5



$OABC$  is a parallelogram.  $P$  is the point on  $AC$  such that  $AP = \frac{2}{3}AC$ .

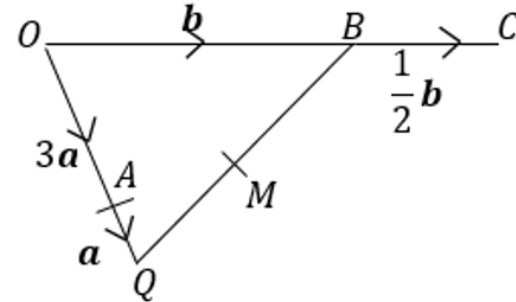
i) Find the vector  $\overrightarrow{OP}$ . Give your answer in terms of  $\mathbf{a}$  and  $\mathbf{c}$ .

?

ii) Given that the midpoint of  $CB$  is  $M$ , prove that  $OPM$  is a straight line.

?

6



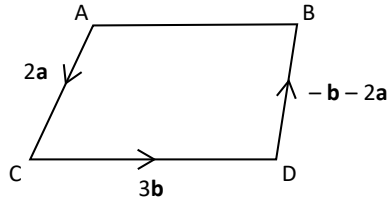
$\overrightarrow{OA} = 3\mathbf{a}$  and  $\overrightarrow{AQ} = \mathbf{a}$  and  $\overrightarrow{OB} = \mathbf{b}$  and  $\overrightarrow{BC} = \frac{1}{2}\mathbf{b}$ .  $M$  is the midpoint of  $QB$ . Prove that  $AMC$  is a straight line.

?

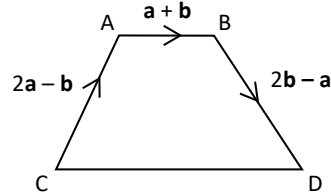
# Purposeful Practice

## Vectors: Proving Parallel & Straight Lines

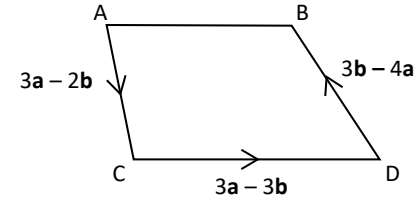
1) Prove  $\overrightarrow{AB}$  is parallel to  $\overrightarrow{CD}$



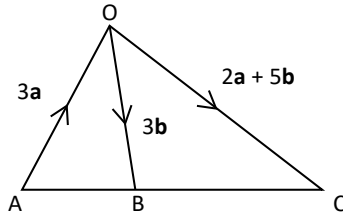
2) Prove  $\overrightarrow{CD}$  is parallel to  $\overrightarrow{AB}$



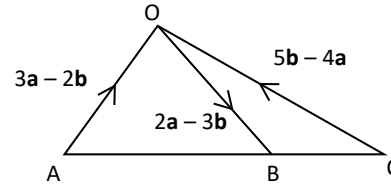
3) Prove  $\overrightarrow{AB}$  is parallel to  $\overrightarrow{CD}$



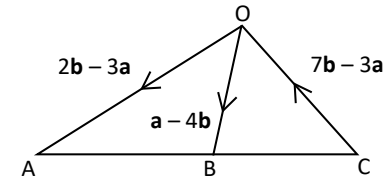
4) Prove  $\overrightarrow{ABC}$  is a straight line.



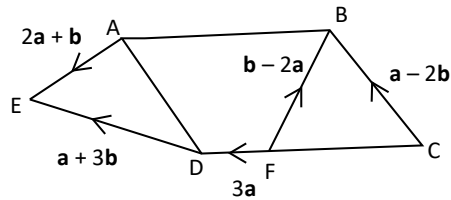
5) Prove  $\overrightarrow{ABC}$  is a straight line.



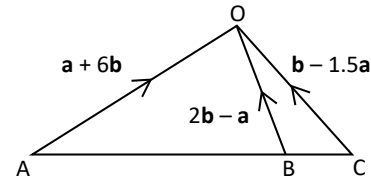
6) Prove  $\overrightarrow{ABC}$  is a straight line.



7) Prove ABCD is a parallelogram.



8) Prove  $\overrightarrow{ABC}$  is a straight line.



## Purposeful Practice

Find the values of  $\lambda$  and  $\mu$  by equating coefficients of  $\mathbf{a}$  and  $\mathbf{b}$ .

(a)  $4\mathbf{a} + \lambda\mathbf{b} = \mu\mathbf{a} + 6\mathbf{b}$

(b)  $\lambda\mathbf{a} - 5\mathbf{b} = -2\mathbf{a} + \mu\mathbf{b}$

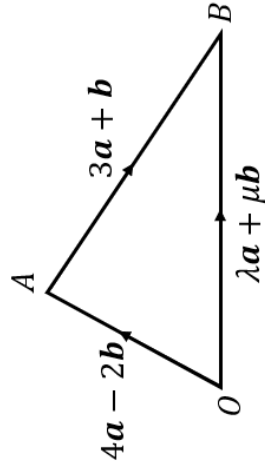
(c)  $\lambda\mathbf{a} + \mu\mathbf{b} = 2\mathbf{a} + 6\mathbf{b} + 3\mathbf{a} - 3\mathbf{b}$

(d)  $-3\mathbf{a} + \lambda\mathbf{b} = 2\mathbf{a} - \mathbf{b} + \mu\mathbf{a} + 5\mathbf{b}$

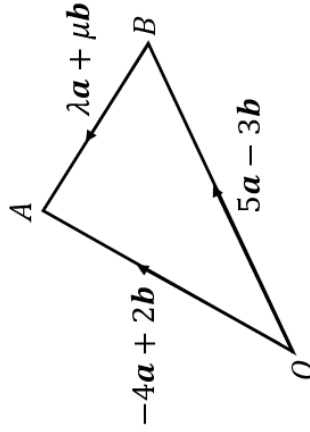
(e)  $2(2\mathbf{a} - \mu\mathbf{b}) + 3\mathbf{a} + 10\mathbf{b} = \lambda\mathbf{a} + 6\mathbf{b}$

In the vector diagrams shown, find the values of  $\lambda$  and  $\mu$ .

(a)

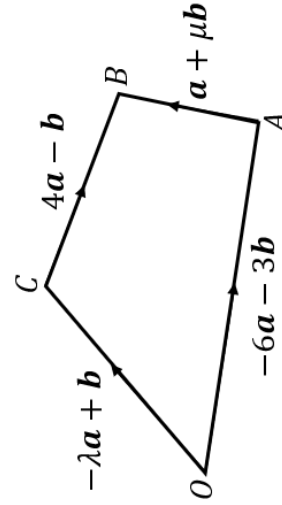


(b)

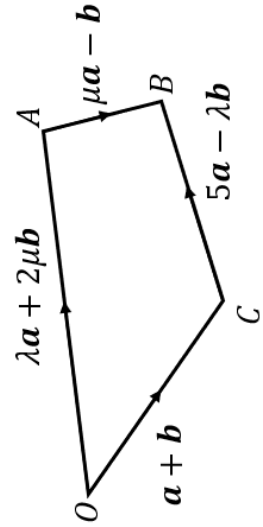


In the vector diagrams shown, find the values of  $\lambda$  and  $\mu$ .

(a)



(b)

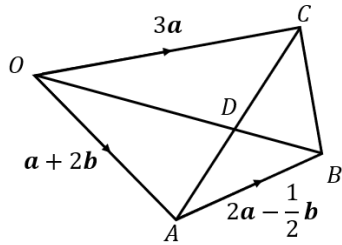


## Purposeful Practice

### Vector Proof – Equating Coefficients

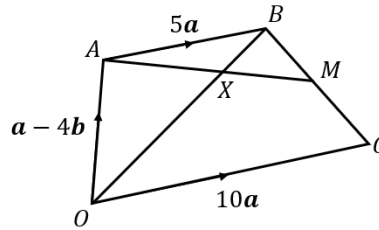
**(a)**

$OABC$  is a quadrilateral, where  $\overrightarrow{OC} = 3\mathbf{a}$ ,  $\overrightarrow{OA} = \mathbf{a} + 2\mathbf{b}$  and  $\overrightarrow{AB} = 2\mathbf{a} - \frac{1}{2}\mathbf{b}$ . The point  $D$  is on  $OB$  and  $AC$  such that  $OD : OB = \lambda : 1$  and  $AD : AC = \mu : 1$ . By finding two ways to express the vector  $\overrightarrow{OD}$ , find the values of  $\lambda$  and  $\mu$ .



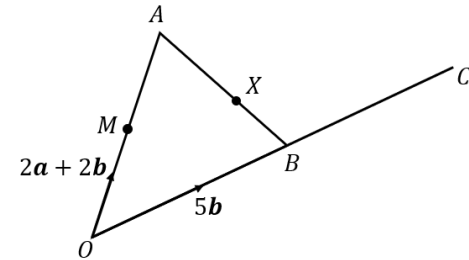
**(b)**

$OABC$  is a trapezium, where  $\overrightarrow{OC} = 10\mathbf{a}$ ,  $\overrightarrow{OA} = \mathbf{a} - 4\mathbf{b}$  and  $\overrightarrow{AB} = 5\mathbf{a}$ .  $M$  is the midpoint of the line  $BC$ . The point  $X$  is on  $OB$  and  $AM$  such that  $OX : OB = \lambda : 1$  and  $AX : AM = \mu : 1$ . Find the values of  $\lambda$  and  $\mu$  and the vector  $\overrightarrow{OX}$  in terms of  $\mathbf{a}$  and  $\mathbf{b}$ .



**(c)**

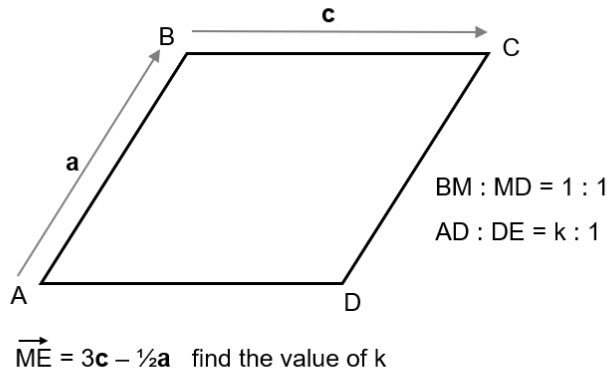
In the triangle  $OAB$ ,  $\overrightarrow{OB} = 5\mathbf{b}$  and  $\overrightarrow{OM} = 2\mathbf{a} + 2\mathbf{b}$ , where  $M$  is the midpoint of  $OA$ .  $OC$  is the line  $OB$  produced and  $\overrightarrow{OB} = \overrightarrow{BC}$ . The point  $X$  is on the line  $AB$  such that  $AX : AB = \lambda : 1$ . Given that  $MXC$  is a straight line, find the value of  $\lambda$  and the vector  $\overrightarrow{MX}$  in terms of  $\mathbf{a}$  and  $\mathbf{b}$ .



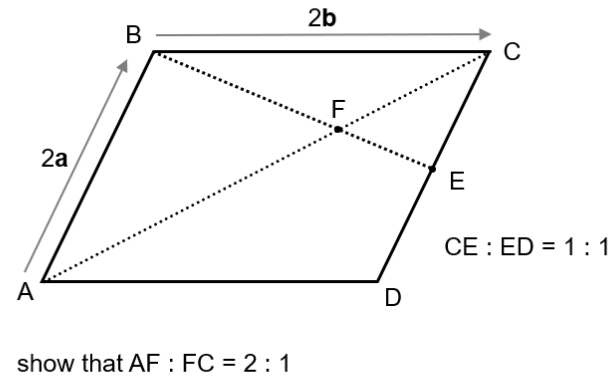
# Purposeful Practice

vector proofs

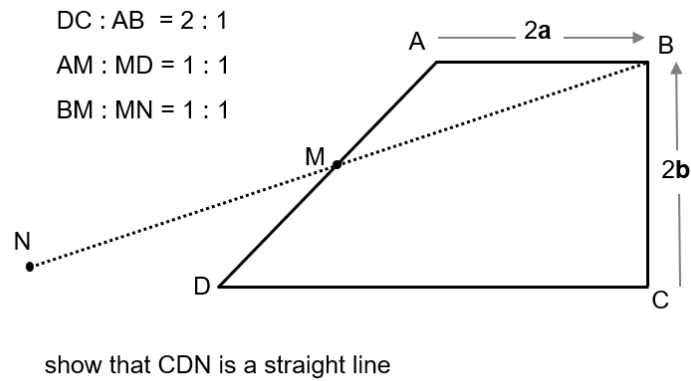
(1) a **parallelogram** (ABCD)



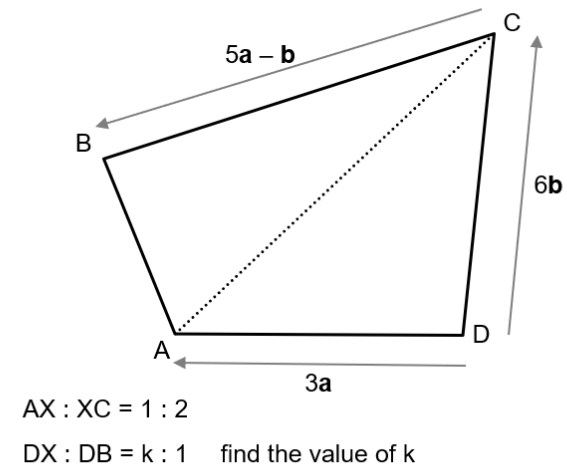
(2) a **parallelogram** (ABCD)



(3) a **trapezium** (ABCD)



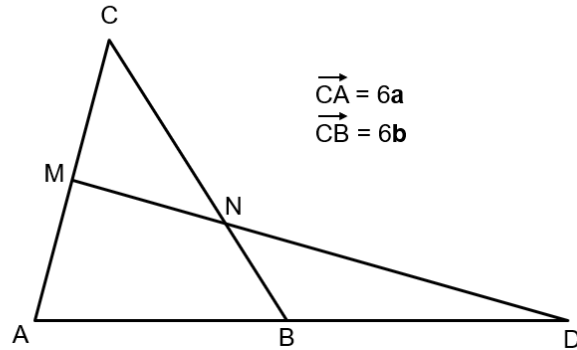
(4)



# Purposeful Practice

vector proofs

(1)

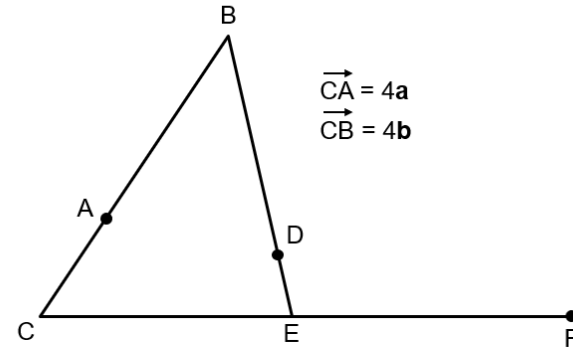


$$CM : MA = 1 : 1$$

$$AB : BD = 1 : 1$$

$$\vec{CN} = k\mathbf{b} \text{ find the value of } k$$

(2)



$$CA : AB = 1 : 2$$

$$ED : DB = 1 : 3$$

$$CE : EF = 1 : 2$$

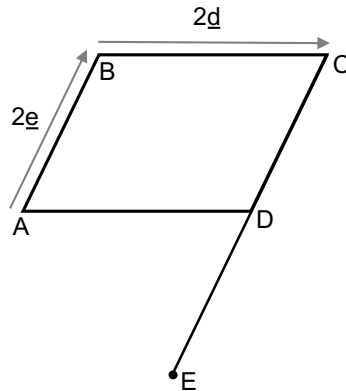
show that  $ADF$  is  
a straight line

(3) choose some other ratios to explore

# Purposeful Practice

vector proofs (i)

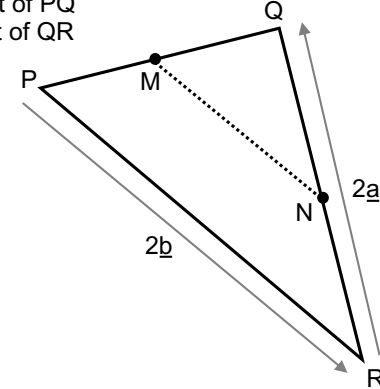
(1) a **parallelogram** (ABCD)



CE is twice CD  
establish that  
ABDE is a  
parallelogram

(2) a **triangle** (PQR)

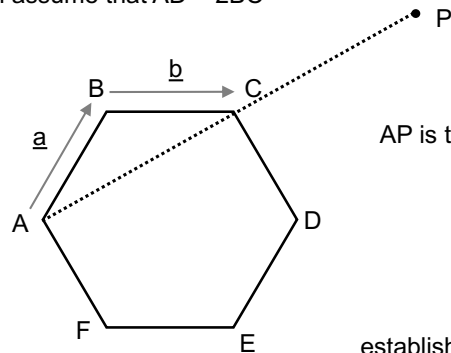
M is the midpoint of PQ  
N is the midpoint of QR



prove that MN is  
parallel to PR  
and half the  
length of it

(3) a **regular hexagon**

you can assume that  $AD = 2BC$

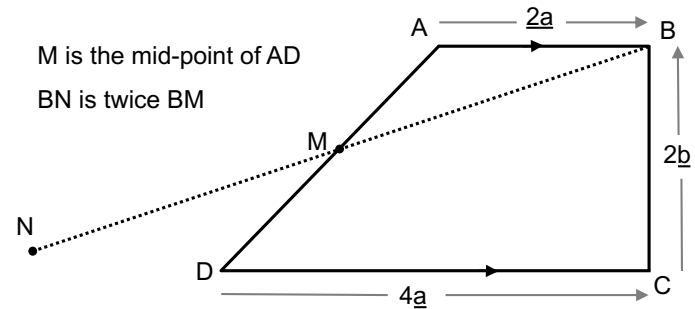


AP is twice AC

establish that EDP  
is a straight line

(4) a **trapezium** (ABCD)

M is the mid-point of AD  
BN is twice BM



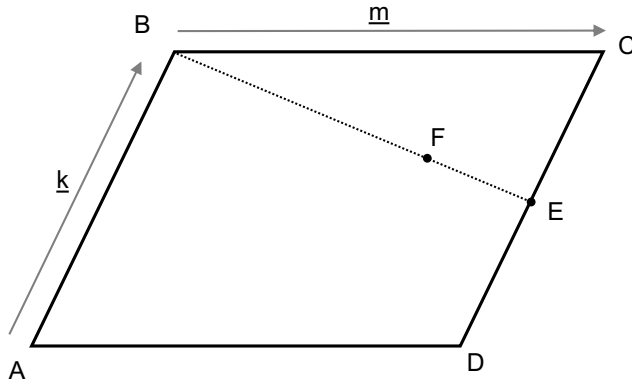
prove that CDN is a straight line



# Purposeful Practice

vector proofs (ii)

(4) a **parallelogram** (ABCD)



E is the mid-point of CD

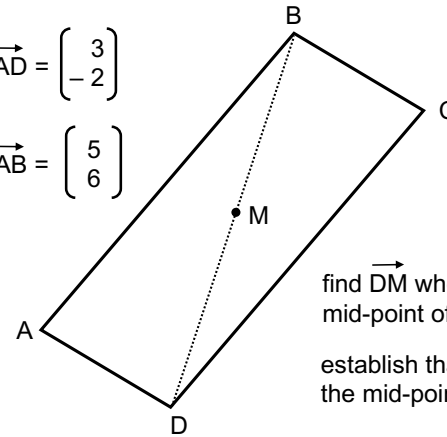
BF : FE = 2 : 1

establish that F is  $\frac{2}{3}$  of the way along AC

(5) a **parallelogram** (ABCD)

$$\vec{AD} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

$$\vec{AB} = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$$



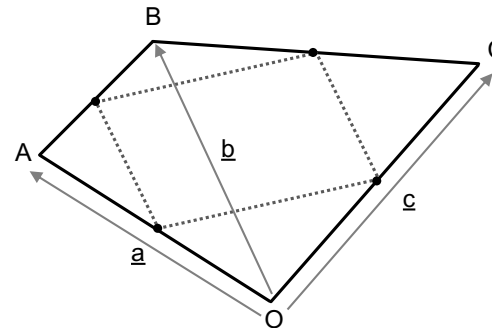
find  $\vec{DM}$  where M is the mid-point of DB

establish that M is also the mid-point of AC

(6) any **quadrilateral** (OABC)

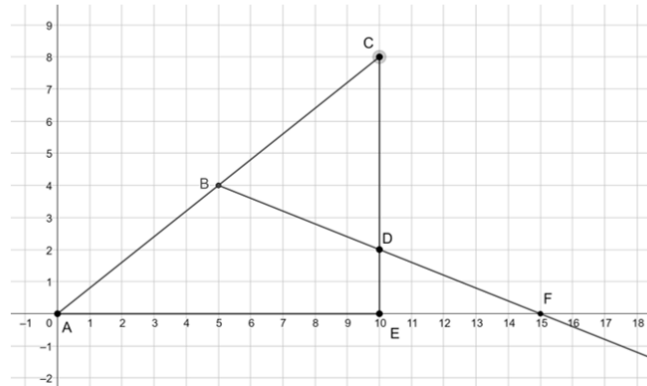
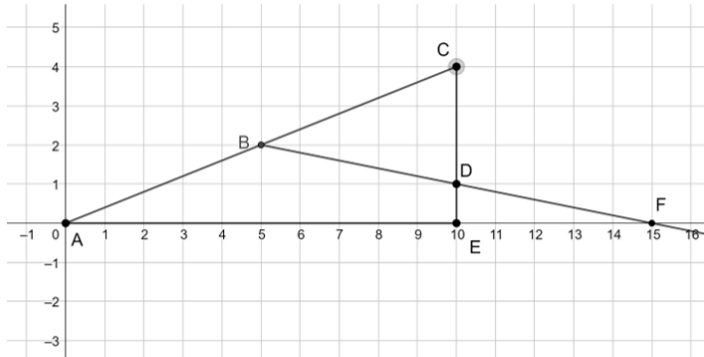
any quadrilateral (OABC) is drawn and the mid-points of the four sides are joined

establish that the resulting shape is a parallelogram



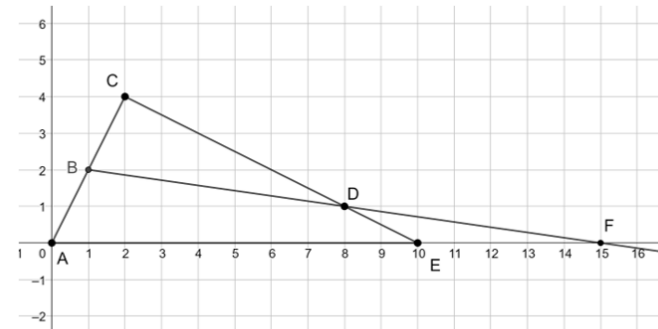
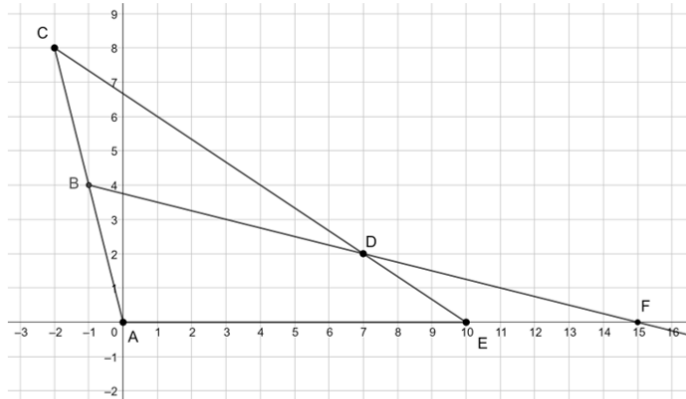
# Purposeful Practice

prove the general result



$$\begin{aligned} AB : BC &= 1 : 1 \\ ED : DC &= 1 : 3 \\ AE : EF &= 2 : 1 \end{aligned}$$

prove  
 $BD : DF = 1 : 1$



## Exam Questions

Q1.

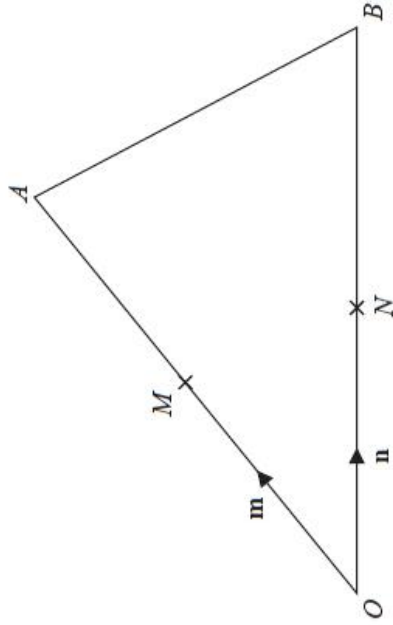


Diagram NOT  
accurately drawn

OAB is a triangle.  
M is the midpoint of OA.  
N is the midpoint of OB.

$$\vec{OM} = \mathbf{m}$$

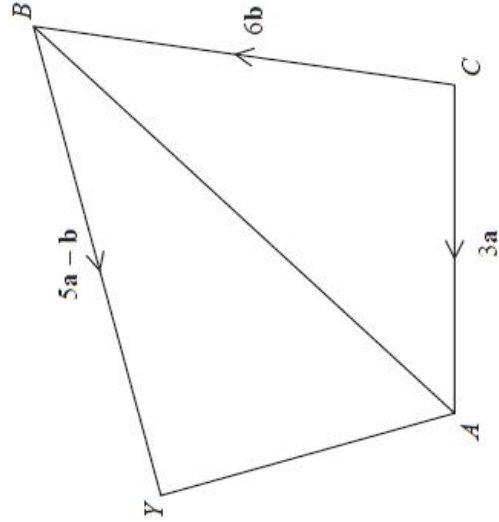
$$\vec{ON} = \mathbf{n}$$

Show that AB is parallel to MN.

(Total for Question is 3 marks)

## Exam Questions

Q2.



CAYB is a quadrilateral.

$$\vec{CA} = 3\mathbf{a}$$

$$\vec{CB} = 6\mathbf{b}$$

$$\vec{BY} = 5\mathbf{a} - \mathbf{b}$$

X is the point on AB such that  $AX : XB = 1 : 2$

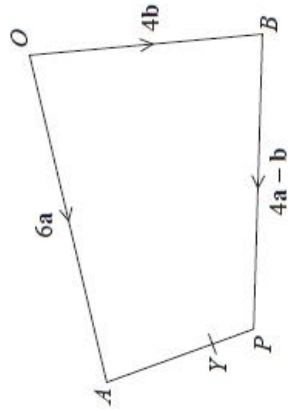
$$\vec{CX} = \frac{2}{5} \vec{CY}$$

Prove that

(Total for question is 5 marks)

## Exam Questions

Q3.



OBPA is a quadrilateral.

$$\vec{OA} = 6\mathbf{a}$$

$$\vec{OB} = 4\mathbf{b}$$

$$\vec{BP} = 4\mathbf{a} - \mathbf{b}$$

Y is the point on AP such that AY : YP = 2 : 1

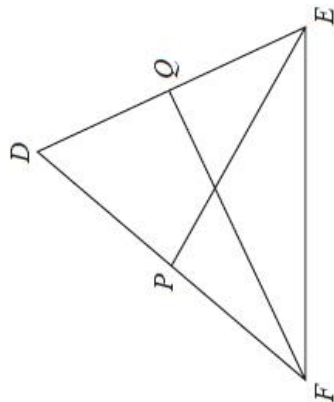
Show that  $\vec{OY}$  is parallel to the vector  $7\mathbf{a} + 3\mathbf{b}$

(Total for question = 4 marks)

## Exam Questions

**Q4.**

$DEF$  is a triangle.



$P$  is the midpoint of  $FD$ .

$Q$  is the midpoint of  $DE$ .

$$\vec{FD} = \mathbf{a} \quad \text{and} \quad \vec{FE} = \mathbf{b}$$

Use a vector method to prove that  $PQ$  is parallel to  $FE$ .

(Total for question = 4 marks)

## Exam Questions

1.



$OABC$  is a trapezium.

$OC$  is parallel to  $AB$ .

$$\vec{OA} = \mathbf{a}, \vec{OC} = \mathbf{c}$$

$$AB = 2OC.$$

$X$  is the point on  $AB$  such that  $AX:XB = 3:1$ .

Express  $\vec{XC}$  in terms of  $\mathbf{a}$  and  $\mathbf{c}$ .

2.

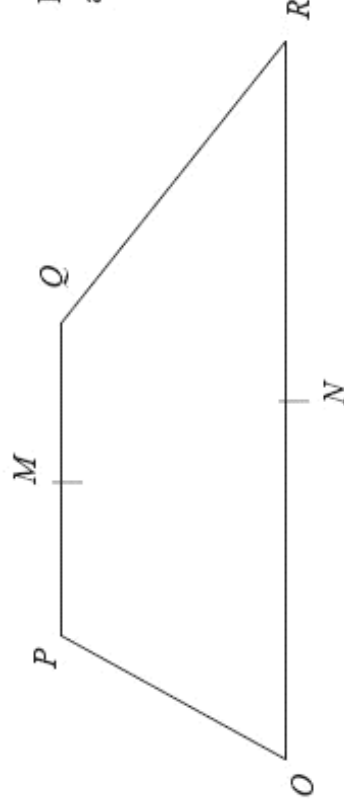


Diagram NOT  
accurately drawn

$OPQR$  is a trapezium with  $PQ$  parallel to  $OR$ .

$$\vec{OP} = 2\mathbf{b} \quad \vec{PQ} = 2\mathbf{a} \quad \vec{OR} = 6\mathbf{a}$$

$M$  is the midpoint of  $PQ$  and  $N$  is the midpoint of  $OR$ .

(a) Find the vector  $\vec{MN}$  in terms of  $\mathbf{a}$  and  $\mathbf{b}$ .

$X$  is the midpoint of  $MN$  and  $Y$  is the midpoint of  $QR$ .

(b) Prove that  $XY$  is parallel to  $OR$ .

## Exam Questions

3.

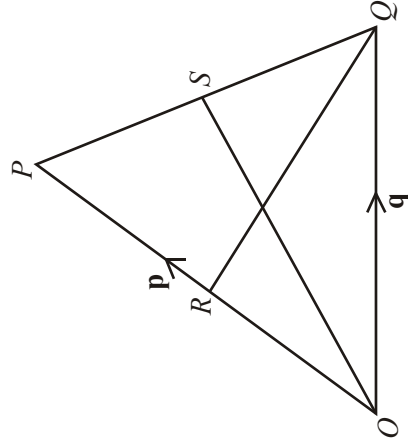


Diagram **NOT**  
accurately drawn

$OPQ$  is a triangle.  
 $R$  is the midpoint of  $OP$ .  
 $S$  is the midpoint of  $OQ$ .  
 $\vec{OP} = \mathbf{p}$  and  $\vec{OQ} = \mathbf{q}$

- (i) Find  $\vec{OS}$  in terms of  $\mathbf{p}$  and  $\mathbf{q}$ .      (ii) Show that  $RS$  is parallel to  $OQ$ .

4.

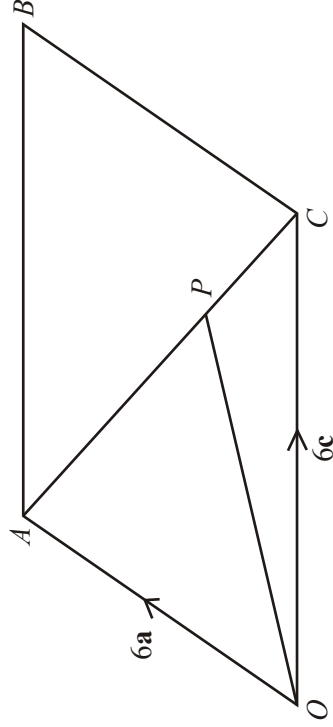


Diagram **NOT**  
accurately drawn

$OACB$  is a parallelogram.

$P$  is the point on  $AC$  such that  $AP = \frac{2}{3}AC$ .

$$\vec{OA} = 6\mathbf{a}, \vec{OC} = 6\mathbf{c}.$$

- (a) Find the vector  $\vec{OP}$ . Give your answer in terms of  $\mathbf{a}$  and  $\mathbf{c}$ .

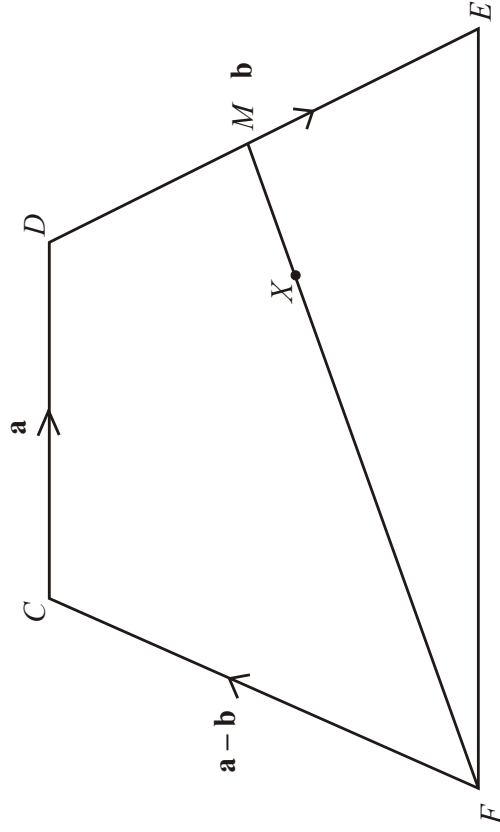
The midpoint of  $CB$  is  $M$ .

- (b) Prove that  $OPM$  is a straight line.



## Exam Questions

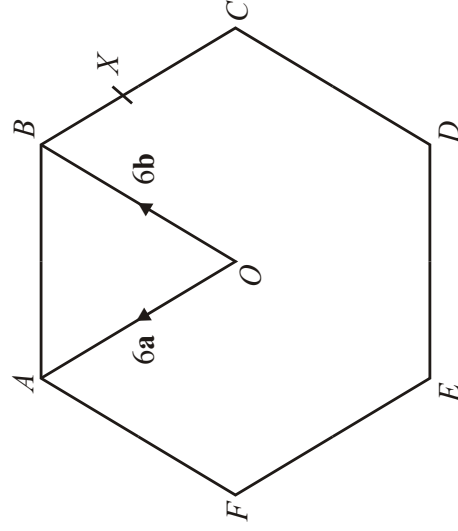
5.



$CDEF$  is a quadrilateral with  $\overrightarrow{CD} = \mathbf{a}$ ,  $\overrightarrow{DE} = \mathbf{b}$  and  $\overrightarrow{FC} = \mathbf{a} - \mathbf{b}$ .

- Express  $\overrightarrow{CE}$  in terms of  $\mathbf{a}$  and  $\mathbf{b}$ .
- Prove that  $FE$  is parallel to  $CD$ .  
 $M$  is the midpoint of  $DE$ .
- Express  $\overrightarrow{FM}$  in terms of  $\mathbf{a}$  and  $\mathbf{b}$ .
- $X$  is the point on  $FM$  such that  $FX : XM = 4 : 1$ .
- Prove that  $C$ ,  $X$  and  $E$  lie on the same straight line.

6.



The diagram shows a regular hexagon  $ABCDEF$  with centre  $O$ .

- $\overrightarrow{OA} = 6\mathbf{a}$        $\overrightarrow{OB} = 6\mathbf{b}$
- Express in terms of  $\mathbf{a}$  and/or  $\mathbf{b}$

## Exam Questions

- (i)  $\overrightarrow{AB}$ ,
- (ii)  $\overrightarrow{EF}$ .

$X$  is the midpoint of  $BC$ .

- (b) Express  $\overrightarrow{EX}$  in terms of **a** and/or **b**

$Y$  is the point on  $AB$  extended, such that  $AB : BY = 3 : 2$

- (c) Prove that  $E, X$  and  $Y$  lie on the same straight line.

7.

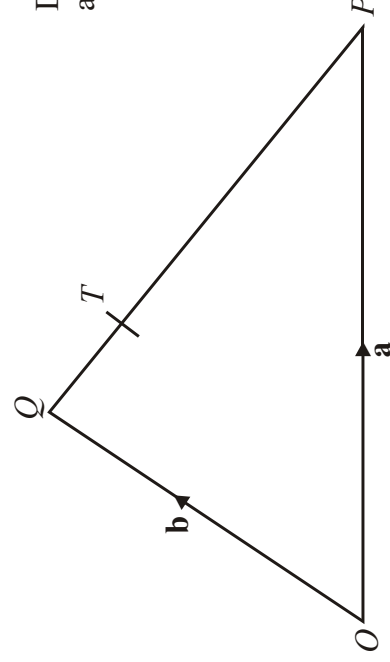


Diagram NOT  
accurately drawn

$OPQ$  is a triangle.

$T$  is the point on  $PQ$  for which  $PT : TQ = 2 : 1$ .

$\overrightarrow{OP} = \mathbf{a}$  and  $\overrightarrow{OQ} = \mathbf{b}$ .

- (a) Write down, in terms of **a** and **b**, an expression for  $\overrightarrow{PQ}$ .
- (b) Express  $\overrightarrow{OT}$  in terms of **a** and **b**.  
Give your answer in its simplest form.

# Exam Questions

1.  $\overrightarrow{OP} = -4\mathbf{a} + 5\mathbf{b}$  and  $\overrightarrow{OQ} = 5\mathbf{a} - \mathbf{b}$ .

(a) Express  $\overrightarrow{PQ}$  in terms of  $\mathbf{a}$  and  $\mathbf{b}$ .

$R$  is a point on  $\overrightarrow{PQ}$  such that  $PR : RQ = 1 : 2$ .

(b) Express  $\overrightarrow{OR}$  in terms of  $\mathbf{a}$  and  $\mathbf{b}$ .

(Total 3 marks)

2. In the diagram  $\overrightarrow{OP} = 4\mathbf{a}$ ,  $\overrightarrow{PA} = \mathbf{a}$ ,  
 $\overrightarrow{OB} = 5\mathbf{b}$ ,  $\overrightarrow{BR} = 3\mathbf{b}$  and  $\overrightarrow{AQ} = \frac{2}{5} \overrightarrow{AB}$

(a) Find, in terms of  $\mathbf{a}$  and  $\mathbf{b}$ , simplifying your answers,  
 (i)  $\overrightarrow{AB}$   
 (ii)  $\overrightarrow{PQ}$

(b) Show clearly that points  $P$ ,  $Q$  and  $R$  lie on a straight line.

(Total 6 marks)

3.  $ABCDEF$  is a regular hexagon with centre  $O$ .  
 $\overrightarrow{OA} = \mathbf{a}$  and  $\overrightarrow{AB} = \mathbf{b}$

(a) Find expressions, in terms of  $\mathbf{a}$  and  $\mathbf{b}$ , for  
 (i)  $\overrightarrow{OB}$   
 (ii)  $\overrightarrow{AC}$   
 (iii)  $\overrightarrow{EC}$

(b) The positions of points  $P$  and  $Q$  are given by the vectors  
 $\overrightarrow{OP} = \mathbf{a} - \mathbf{b}$   $\overrightarrow{OQ} = \mathbf{a} + 2\mathbf{b}$

(i) Draw and label the positions of points  $P$  and  $Q$  on the diagram.

(ii) Hence, or otherwise, deduce an expression for  $\overrightarrow{PQ}$ .

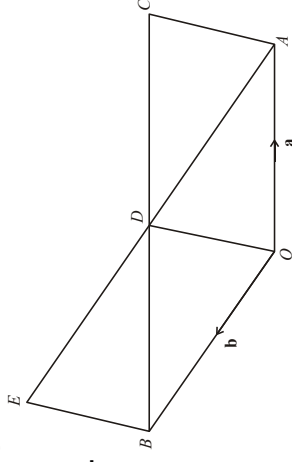
Diagram drawn accurately

(Total 3 marks)

(Total 6 marks)

# Exam Questions

4. In the diagram  $OACD$ ,  $OADB$  and  $ODEB$  are parallelograms.  
 $\overrightarrow{OA} = \mathbf{a}$  and  $\overrightarrow{OB} = \mathbf{b}$



- (a) Express, in terms of  $\mathbf{a}$  and  $\mathbf{b}$ , the following vectors.  
 Give your answers in their simplest form.

(i)  $\overrightarrow{OD}$

(ii)  $\overrightarrow{OC}$

(iii)  $\overrightarrow{AB}$

- (b) The point  $F$  is such that  $OCFE$  is a parallelogram.  
 Write the vector  $\overrightarrow{CF}$  in terms of  $\mathbf{a}$  and  $\mathbf{b}$ .

- (c) What geometrical relationship is there between the points  $O$ ,  $D$  and  $F$ ?  
 Justify your answer.

(Total 7 marks)

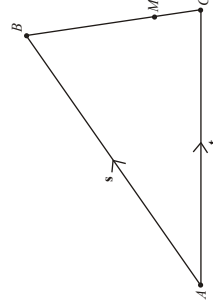
Not drawn accurately

5. In triangle  $ABC$ ,  $M$  lies on  $BC$  such that  $BM = \frac{3}{4} BC$ .

$\overrightarrow{AB} = \mathbf{s}$  and  $\overrightarrow{AC} = \mathbf{t}$

Find  $\overrightarrow{AB}$  in terms of  $\mathbf{s}$  and  $\mathbf{t}$ .

Give your answer in its simplest form.



(Total 3 marks)

6. In the diagram  $\overrightarrow{OP} = 4\mathbf{a}$ ,  $\overrightarrow{PA} = \mathbf{a}$ ,  
 $\overrightarrow{OB} = 5\mathbf{b}$ ,  $\overrightarrow{BR} = 3\mathbf{b}$  and  $\overrightarrow{AQ} = \frac{2}{5} \overrightarrow{AB}$

Not drawn accurately

- (a) Find, in terms of  $\mathbf{a}$  and  $\mathbf{b}$ , simplifying your answers,

(i)  $\overrightarrow{AB}$

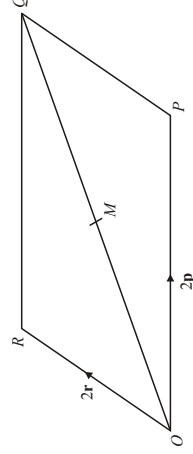
(ii)  $\overrightarrow{PQ}$

- (b) Show clearly that points  $P$ ,  $Q$  and  $R$  lie on a straight line.

(Total 6 marks)

7.  $OPQR$  is a parallelogram.  
 $M$  is the mid-point of the diagonal  $OQ$ .

$\overrightarrow{OP} = 2\mathbf{p}$  and  $\overrightarrow{OR} = 2\mathbf{r}$



- (a) Express  $\overrightarrow{OM}$  in terms of  $\mathbf{p}$  and  $\mathbf{r}$ .

- (b) Use vectors to prove that  $M$  is also the mid-point of  $PR$ .

(Total 4 marks)

# Exam Questions

8.  $PQRSTU$  is a regular hexagon and  $O$  is the centre of the hexagon.

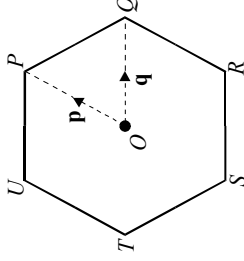
$$\vec{OP} = \mathbf{p} \text{ and } \vec{OQ} = \mathbf{q}$$

Express each of the following vectors in terms of  $\mathbf{p}$  and  $\mathbf{q}$

(a)  $\vec{PQ}$

(b)  $\vec{SP}$

(c)  $\vec{SQ}$



(Total 4 marks)

9. The diagram shows two sets of parallel lines.

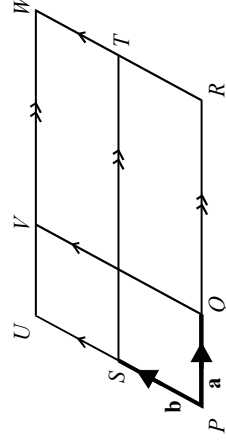
$$\text{Vector } \vec{PQ} = \mathbf{a} \text{ and vector } \vec{PS} = \mathbf{b}$$

$$\vec{PR} = 3\vec{PQ} \text{ and } \vec{PU} = 2\vec{PS}$$

(a) Write the vector  $\vec{PV}$  in terms of  $\mathbf{a}$  and  $\mathbf{b}$

(b) Write the vector  $\vec{RU}$  in terms of  $\mathbf{a}$  and  $\mathbf{b}$

(c) Find **two** vectors that can be written as  $3\mathbf{a} - \mathbf{b}$



(Total 4 marks)

10.  $OACB$  is a parallelogram and  $M$  is the mid-point of  $BC$ .

$$\vec{OA} = \mathbf{a} \text{ and } \vec{OB} = \mathbf{b}$$

(a) Express the following vectors in terms of  $\mathbf{a}$  and  $\mathbf{b}$

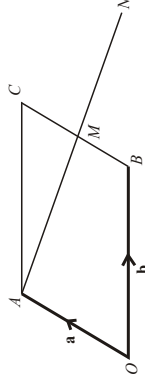
(i)  $\vec{BA}$

(iii)  $\vec{AM}$

(b)  $AM$  is extended to  $N$ , where  $\vec{AN} = 2\vec{AM}$ .

$$\text{Show that } \vec{BN} = \mathbf{b}$$

(c) What does this tell you about the position of  $N$ ?



(Total 5 marks)

11. In triangle  $ABC$ ,  $M$  is the mid-point of  $BC$ .

$$\vec{AB} = \mathbf{s} \text{ and } \vec{AC} = \mathbf{t}$$

(a) Find  $\vec{AM}$  in terms of  $\mathbf{s}$  and  $\mathbf{t}$ .

Give your answer in its simplest form.

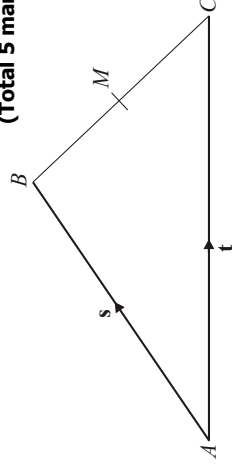
(b)  $\vec{AD} = \mathbf{s} + \mathbf{t}$

The length of  $AB$  is **not** equal to the length of  $AC$ .

(i) Write down the name of the shape  $ABDC$ .

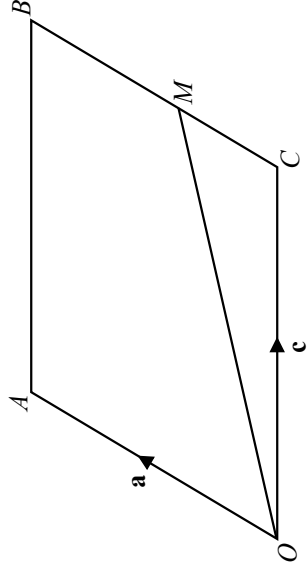
(ii) Write down one fact about the points  $A$ ,  $M$  and  $D$ .  
Explain your answer.

(Total 6 marks)



## Exam Questions

- 1**  $OABC$  is a parallelogram.



$$\vec{OA} = \mathbf{a} \quad \vec{OC} = \mathbf{c}$$

$M$  is the point on  $BC$  such that  $BM : MC = 3 : 2$

The line  $AB$  is extended to the point  $N$ .

$OMN$  is a straight line.

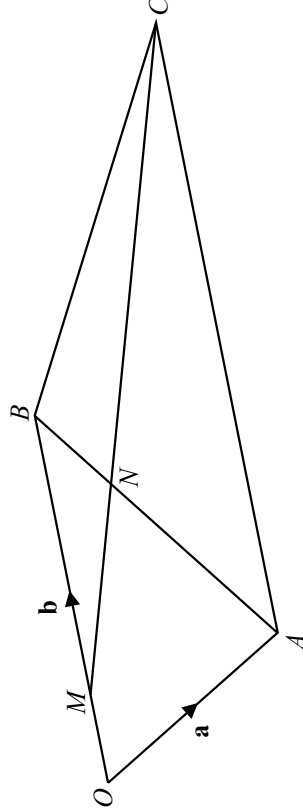
Find the ratio  $AB : BN$ .

Give your answer in the form  $k : 1$

$AB : BN = \dots\dots\dots$   
 (Total for Question 1 is 4 marks)

## Exam Questions

- 2  $OABC$  is a quadrilateral.



$$\vec{OA} = \mathbf{a} \quad \vec{OB} = \mathbf{b}$$

$MNC$  and  $ANB$  are straight lines.

$M$  is the point on the line  $OB$  such that  $OM : MB = 1 : 5$

$N$  is the point such that  $AN : NB = 3 : 2$

$\vec{OB}$  is parallel to  $\vec{AC}$ .

Work out the ratio  $OB : AC$ .

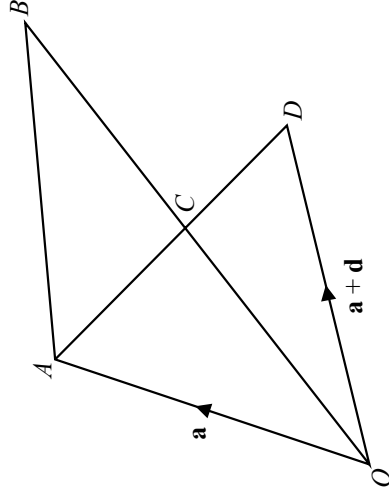
Give your answer in the form  $k : 1$

$$OB : AC = \dots\dots\dots$$

(Total for Question 2 is 5 marks)

## Exam Questions

3  $OAD$  is a triangle.



$$\vec{OA} = \mathbf{a} \quad \vec{OD} = \mathbf{a} + \mathbf{d}$$

$C$  is the point on the line  $AD$  such that  $AC : AD = 2 : 3$   
 $OCB$  is a straight line such that  $OC : CB = 3 : 5$

The vector  $\vec{AB} = \lambda \mathbf{a} + \mu \mathbf{d}$

Work out the ratio  $\lambda : \mu$

Give your answer as a ratio in its simplest form.

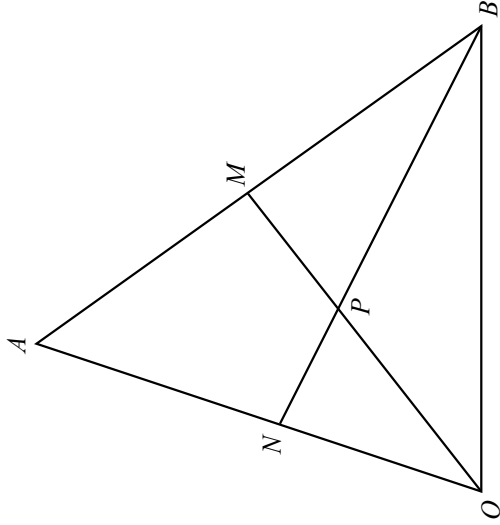
$\lambda : \mu = \dots\dots\dots$

(Total for Question 3 is 4 marks)



## Exam Questions

- 4  $OAB$  is a triangle.



$$\vec{OA} = \mathbf{a} \quad \vec{OB} = \mathbf{b}$$

$M$  is the midpoint of  $AB$ .

$OPM$  and  $BPN$  are straight lines.

$P$  is the point such that  $OP : PM = 4 : 3$

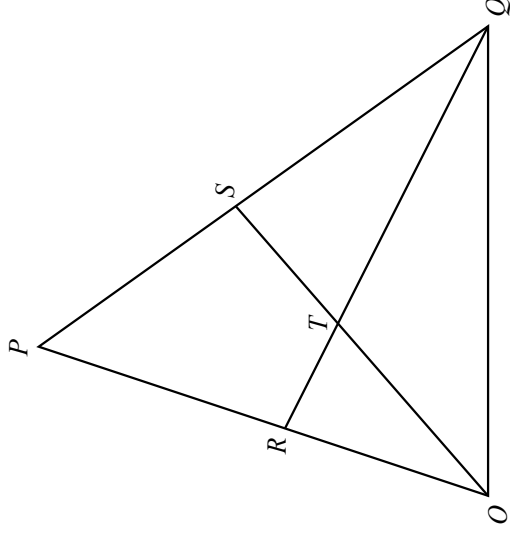
Find the ratio  $ON : NA$

$$ON : NA = \dots\dots\dots$$

(Total for Question 4 is 5 marks)

## Exam Questions

- 5  $OPQ$  is a triangle.



$$\vec{OP} = \mathbf{p} \quad \vec{OQ} = \mathbf{q}$$

$OTS$  and  $RTQ$  are straight lines.

$R$  is the point on  $OP$  such that  $OR : RP = 3 : 4$

$T$  is the point such that  $TQ : RQ = 3 : 4$

Find the ratio  $PS : SQ$

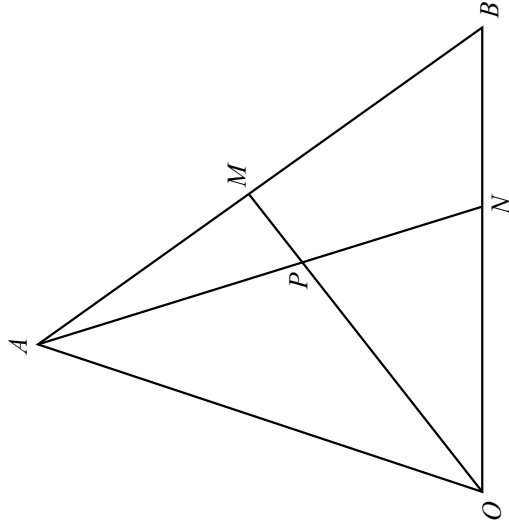
Give your answer in the form  $a : b$  where  $a$  and  $b$  are integers.

$$PS : SQ = \dots\dots\dots$$

(Total for Question 5 is 5 marks)

## Exam Questions

6  $OAB$  is a triangle.



$OPM$  and  $APN$  are straight lines.  
 $M$  is the midpoint of  $AB$ .

$$\vec{OA} = \mathbf{a} \quad \vec{OB} = \mathbf{b}$$

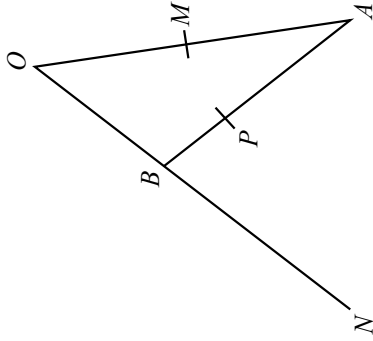
$$OP : PM = 5 : 2$$

Work out the ratio  $ON : NB$

.....  
 (Total for Question 6 is 5 marks)

## Exam Questions

7



$OBN$ ,  $OMA$  and  $APB$  are straight lines.

$BN = 2OB$ .

$M$  is the midpoint of  $OA$ .

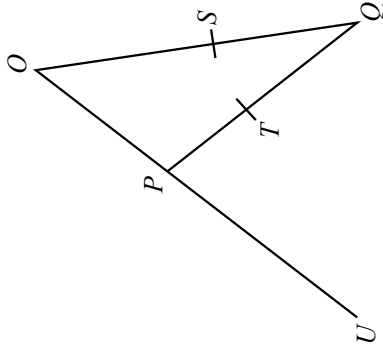
$$\vec{OA} = \mathbf{a} \quad \vec{OB} = \mathbf{b}$$

Given that  $MPN$  is a straight line, work out the ratio  $AP : PB$

.....  
(Total for Question 7 is 5 marks)

## Exam Questions

8



$OPU$ ,  $OSQ$  and  $PTQ$  are straight lines.

$$OP : PU = 3 : 4$$

$$OS : OQ = 5 : 8$$

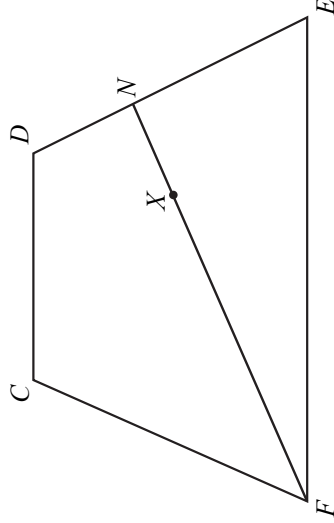
$$\vec{OP} = \mathbf{p} \quad \vec{OQ} = \mathbf{q}$$

Given that  $STU$  is a straight line, work out the ratio  $PT : TQ$

.....  
(Total for Question 8 is 5 marks)

## Exam Questions

- 9  $CDEF$  is a quadrilateral.



$\vec{CD} = \mathbf{a}$ ,  $\vec{DE} = \mathbf{b}$  and  $\vec{FC} = \mathbf{a} - \mathbf{b}$ .

$N$  is the point on  $DE$  such that  $DN : NE = 2 : 3$

$X$  is the point on  $FN$  such that  $FX : XN = k : 1$

$CXE$  is a straight line.

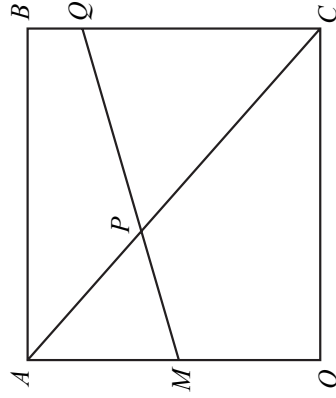
Work out the value of  $k$ .

$k = \dots\dots\dots$

(Total for Question 9 is 5 marks)

## Exam Questions

**10**  $OABC$  is a square.



$$\vec{OA} = \mathbf{a}, \vec{OC} = \mathbf{c}.$$

$M$  is the midpoint of  $OA$ .

$Q$  divides  $BC$  in the ratio  $BQ : QC = 1 : 3$

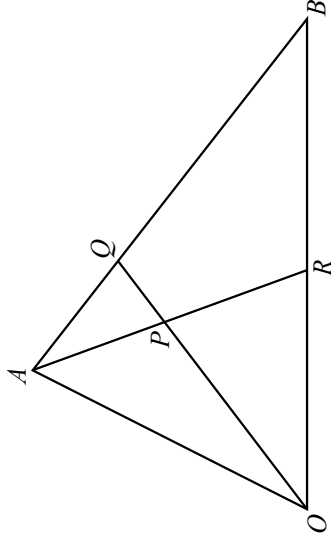
$APC$  and  $MPQ$  are straight lines.

Show that  $P$  divides  $AC$  in the ratio  $2 : 3$

(Total for Question 10 is 5 marks)

## Exam Questions

- 11**  $OAB$  is a triangle.



$$\vec{OA} = \mathbf{a} \quad \vec{OB} = \mathbf{b}$$

$OPQ$  and  $APR$  are straight lines.

The point  $P$  divides  $APR$  in the ratio  $AP : PR = 3 : 4$

The point  $Q$  divides  $AQB$  in the ratio  $AQ : QB = 2 : 5$

Find the ratio  $OR : RB$

Give your answer in its simplest form.

.....  
(Total for Question 11 is 6 marks)