



KING EDWARD VI
HANDSWORTH GRAMMAR
SCHOOL FOR BOYS



KING EDWARD VI
ACADEMY TRUST
BIRMINGHAM

Year 11

2025

Level 2 Further Mathematics

2026

Unit 29 Tasks

DO NOT WRITE INSIDE

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1 Differentiation

Find the gradient function $\frac{dy}{dx}$ when:

- (a) $y = x^4$ (b) $y = x^9$
 (c) $y = x^7$ (d) $y = x^6$
 (e) $y = x$ (f) $y = x^{10}$

Find the gradient function $\frac{dy}{dx}$ when:

- (a) $y = 7x^2$
 (b) $y = 3x^5$
 (c) $y = 10x^6$
 (d) $y = 2x^9$
 (e) $y = \frac{1}{2}x^8$
 (f) $y = \frac{1}{5}x^4$
 (g) $y = 0.3x^5$
 (h) $y = -6x^3$

Find the gradient function $\frac{dy}{dx}$ when:

- (a) $y = x^2 + x^5$
 (b) $y = 3x^2 + 7x^5$
 (c) $y = 5x^4 - x^3$
 (d) $y = 2x^3 - x^2 + 5x$
 (e) $y = 3x + 6x^4$
 (f) $y = 0.5x^7 + 3$
 (g) $y = \frac{1}{4}x^5 - x^3 + 7x$
 (h) $y = x^3 + 2x^2 - 7x + 10$

(a) Expand and simplify $(x + 3)(x^2 - 5)$

(b) Hence find the gradient function $\frac{dy}{dx}$

when $y = (x + 3)(x^2 - 5)$

Find $\frac{dy}{dx}$ when:

- (a) $y = x^3(x + 2)$
 (b) $y = 2x(x^5 - 4x^3)$
 (c) $y = (x + 7)(x - 3)$
 (d) $y = (3x - 5)(2x + 1)$
 (e) $y = (x^2 + 3)(x - 5)$
 (f) $y = x(x + 4)(x - 4)$

Find $\frac{dy}{dx}$ when:

- (a) $y = \frac{8x^5 + 6x^2}{2}$
 (b) $y = \frac{x^4 - 2x^3}{x}$
 (c) $y = \frac{10x^4 - 5x^3}{2x}$
 (d) $y = \frac{9x^7 + 2x^3}{3x^2}$
 (e) $y = \frac{4x^2(x-7)}{2x}$

Find $\frac{dy}{dx}$ when:

- (a) $y = \frac{7}{x}$ (b) $y = -\frac{3}{x^2}$
 (c) $y = \frac{5}{2x}$ (d) $y = \frac{4}{5x^3}$
 (e) $y = 2x^5 + x^3 - \frac{3}{x}$
 (f) $y = 7x^2 + 4x + \frac{5}{2x}$
 (g) $y = 6x^3 + \frac{1}{x} - \frac{5}{x^2}$
 (h) $y = (x + 3)\left(x + \frac{1}{x}\right)$
 (i) $y = \frac{10x^4 + 4x^2 + 2}{2x}$

Fluency Practice

1 For each of the following, find the gradient function $\frac{dy}{dx}$, and hence find the gradient of the tangent to the curve when $x = 2$

a $y = 2x^5$ →

?

b $y = 7x^3$ →

?

c $y = 3x + 4$ →

?

d $y = 2x + x^2$ →

?

e $y = 2x^3 + x^{-1}$ →

?

f $y = \frac{1}{2}x^{-2}$ →

?

g $y = 2\pi x^2 - 1$ →

?

2 The tangent to the curve $y = x^3 + 2x^2 - x$ has gradient 6. Determine the possible values of x .

?

3 For the curve $y = x^2 + x - 2$, determine:

(a) The gradient of the tangent to the curve at the point $(1,0)$

?

(b) The point on the curve where the gradient is 5.

?

4 Find the points on the curve $y = \frac{1}{3}x^3 + x^2 - 4x$ where the gradient is 11.

?

Purposeful Practice

① Differentiate the following functions.

(i) $y = x^4$

(ii) $y = 2x^3$

(iii) $y = 5x^2$

(iv) $y = 7x^9$

(v) $y = -3x^6$

(vi) $y = 5$

(vii) $y = 10x$

(viii) $y = \frac{1}{4}x^3$

(ix) $y = 2\pi x$

(x) $y = \pi x^2$

② Differentiate the following functions.

(i) $y = 2x^5 + 4x^2$

(ii) $y = 3x^4 + 8x$

(iii) $y = x^3 + 4$

(iv) $y = x - 5x^3$

(v) $y = 4x^3 + 2x$

(vi) $y = 2x + 6$

(vii) $y = 3x^5 + 2$

③ Differentiate the following functions.

(i) $y = 3x^5 + 4x^4 - 3x^2 + 2$

(ii) $y = x^5 + 12x^3 + 3x$

(iii) $y = x^3 + 42x^2 - 5x + 24$

④ Write down the rate of change of the following functions with respect to y .

(i) $y = x^{-4}$

(ii) $y = 3x^{-2}$

(iii) $y = 3x^2 + 4x^{-1}$

(iv) $y = 2x^{-3} - 4$

(v) $y = x^2 + x^{-2}$

(vi) $y = 3x^{-2} + 2x^{-3}$

⑤ Differentiate the following functions.

(i) $y = 3x^2 + \frac{2}{x^3}$

(ii) $y = x^2 + \frac{1}{x^2}$

(iii) $y = 3x^3 + \frac{3}{x^3}$

(iv) $y = \frac{2}{x} - \frac{3}{x^2}$

(v) $y = \frac{1}{2x} - \frac{1}{3x^2}$

(vi) $y = \frac{2}{3x} - \frac{3}{4x^2}$

⑥ A rectangle has length $6x$ and width $3x$.

The area of the rectangle is y .

(i) Write down y in terms of x .

(ii) Work out $\frac{dy}{dx}$.

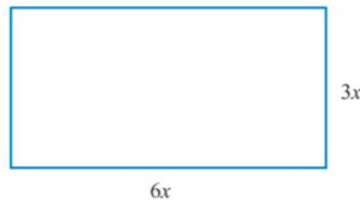


Figure 8.6

⑦ When a stone is thrown into a lake circular ripples appear centred on the point at which the stone entered the water and spreading outwards. After a time, t seconds, the radius of the circle is r cm where $r = 10t^2$.

(i) Work out the rate at which the radius is increasing (include the units).

With time, the definition of the ripples becomes negligible so that after 8 seconds they cannot be seen by the human eye.

(ii) What is the area of the largest ripple that you can see? Give your answer to the nearest 10 square metres.

⑧ An expanding sphere has radius $2x$.

(i) Show that the volume, y , of the sphere is given by the formula

$$y = \frac{32}{3}\pi x^3.$$

(ii) Work out the rate of change of y with respect to x when $x = 2$

Fluency Practice

- ① Work out the gradient function for each of the following functions.
- (i) $y = x(x^2 + 2)$ (ii) $y = 2x^2(3x - 4)$
(iii) $y = (x + 3)(x + 2)$ (iv) $y = (x + 5)(x + 2)$
(v) $y = x^3(4 + x - x^2)$ (vi) $y = (x + 2)(x - 5)$
- ② Work out an expression for the rate of change of y with respect to x for each of the following.
- (i) $y = \frac{x^5 + x^3}{4}$ (ii) $y = \frac{x^7 + x^3}{x^2}$
(iii) $y = \frac{4x^6 - 2x^2}{x^2}$ (iv) $y = (3x + 1)(x - 2)$
(v) $y = x^{\frac{1}{2}}\left(x^{\frac{3}{2}} + x^{\frac{1}{2}}\right)$ (vi) $y = x^{\frac{1}{2}}\left(x^{\frac{7}{2}} + x^{-\frac{1}{2}}\right)$
- ③ (i) Simplify $\frac{3x^3 - 2x^2}{x}$.
(ii) Use your answer to (i) to differentiate $y = \frac{3x^3 - 2x^2}{x}$.
- ④ Work out the gradient of the curve $y = x^3(x - 2)$ at the point $(3, 27)$.
- ⑤ Work out the rate of change of y with respect to x for $\frac{6x^4 + 2x^5}{2x^3}$ when $x = -1$
- ⑥ Work out the rate of change of y with respect to x for $y = x^{\frac{1}{3}}\left(x^{\frac{5}{3}} - x^{\frac{2}{3}}\right)$ when $x = -3$
- ⑦ Work out the gradient of the curve $y = \frac{3x^4 + x^2 - 5x}{x}$ at the point $(1, -1)$.
- ⑧ Work out the gradient of the curve $y = 3\sqrt{x} - \frac{3}{\sqrt{x}}$ at the point $(4, 4.5)$.

Note

The rule

$$y = x^n \Rightarrow \frac{dy}{dx} = nx^{n-1}$$

is valid for all values of n but will only be examined when n is an integer.

(a) Find the equation of the tangent to the curve $y = x^2 + 2x - 3$ at the point $(1, 0)$.

(b) Find the equation of the tangent to the curve $y = x^2 + 4x - 5$ at the point $(-1, -7)$.

(c) Find the equation of the tangent to the curve $y = x^3 + x$ at the point $(2, 10)$.

(a) Find the equation of the normal to the curve $y = x^2 - 4$ at the point $(1, -3)$.

(b) Find the equation of the normal to the curve $y = x^2 - 5x - 6$ at the point $(3, -12)$.

(c) Find the equation of the normal to the curve $y = 2x^3 - 3x + 1$ at the point $(1, 0)$.

(a) Find the equation of the tangent to the curve $y = x^2 + \frac{1}{x}$ at the point where $x = 1$.

(b) Find the equation of the normal to the curve $y = x(x + 2)(x - 1)$ at the point where $x = -2$.

(a) Find the equation of the tangent to the curve $y = 3x - x^2$ at the point $x = 2$.

(b) The tangent crosses the x -axis and y -axis at A and B respectively. Find the area of the triangle AOB.

Fluency Practice

1

[IGCSEFM June 2012 Paper 1 Q8] A curve has equation $y = x^3 + 5x^2 + 1$

(a) When $x = -1$, show that the value of $\frac{dy}{dx}$ is -7.

?

(b) Work out the equation of the tangent to the curve $y = x^3 + 5x^2 + 1$ at the point where $x = -1$.

?

2

[IGCSEFM June 2013 Paper Q8] A curve has equation $y = x^4 - 5x^2 + 9$

(a) Work out $\frac{dy}{dx}$. $\frac{dy}{dx} =$?

(b) Work out the equation of the tangent to the curve at the point where $x = 2$
Give your answer in the form $y = mx + c$

?

3

[IGCSEFM Set Paper 1 Q11] Show that the tangents to the curve $y = x^3 + 3x^2 + 3x + 1$ at $x = 1$ and $x = -3$ are parallel.

?

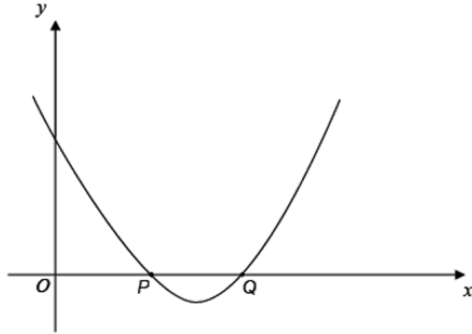
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[IGCSEFM Set 1 Paper 2 Q17] Work out the equation of the normal to the curve $y = 2x^3 - x^2 + 1$ at the point $(1, 2)$. Give your answer in the form $y = mx + c$.

?

Fluency Practice

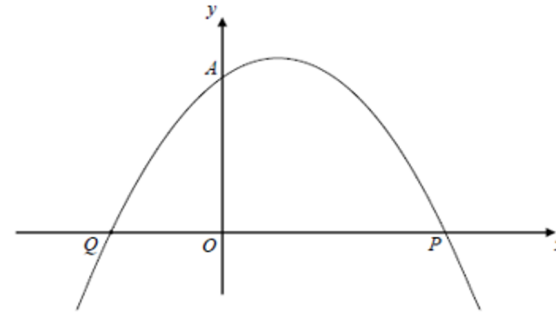
- 5 [IGCSEFM Set 2 Paper 1 Q15] The graph shows a sketch of $y = (x - 2)(x - 3)$. The curve intersects the x -axis at P and Q .



Show that the tangents at P and Q are perpendicular.



- 6 [IGCSEFM Set 4 Paper 2 Q20] A sketch of the curve $y = (x + 1)(2 - x)$ is shown. $A(0,2)$, $P(2,0)$ and Q are points on the curve.

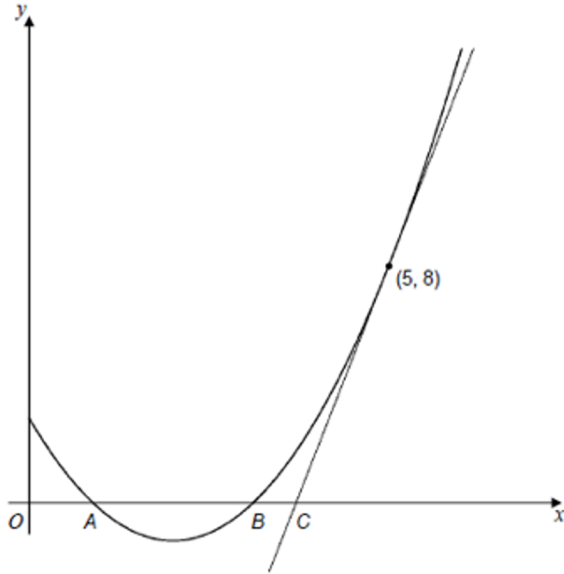


- (a) Write down the coordinates of point Q .
(b) Show that the normal to the curve at A intersects the curve again at P .



Fluency Practice

7



[IGCSEFM Specimen Paper 2 Q22] The diagram shows the graph of $y = x^2 - 4x + 3$. The curve cuts the x -axis at the points A and B . The tangent to the curve at the point $(5, 8)$ cuts the x -axis at the point C . Show that $AB = 3BC$.

?

Purposeful Practice

- ① The sketch shows the graph of $y = 5x - x^2$.
The marked point, P, has coordinates (3, 6). Work out

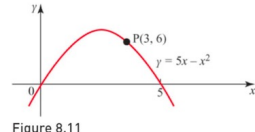


Figure 8.11

- the gradient function $\frac{dy}{dx}$
 - the gradient of the curve at P
 - the equation of the tangent at P
 - the equation of the normal at P.
- ② The sketch shows the graph of $y = 3x^2 - x^3$.

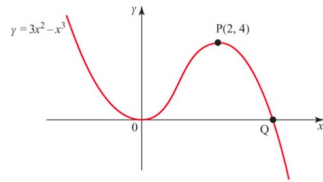


Figure 8.12

- The marked point, P, has coordinates (2, 4). Work out
 - the equation of the tangent at P
 - the equation of the normal at P.
 - The graph touches the x -axis at the origin O and crosses it at the point Q. Work out the equation of the tangent at Q.
 - Without further calculation, state the equation of the tangent to the curve at O.
- ③ The sketch shows the graph of $y = x^5 - x^3$.

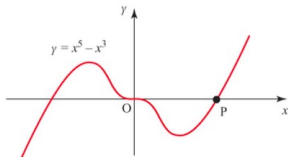


Figure 8.13

- Work out the coordinates of the point P where the curve crosses the positive x -axis.
- Work out the equation of the tangent at P.
- Work out the equation of the normal at P.
The tangent at P meets the y -axis at Q and the normal meets the y -axis at R.
- Work out the coordinates of Q and R and hence calculate the area of triangle PQR.

- Given that $y = x^3 - 3x^2 + 4x + 1$, work out the gradient function $\frac{dy}{dx}$.
 - The point P is on the curve $y = x^3 - 3x^2 + 4x + 1$ and its x -coordinate is 2.
 - Work out the equation of the tangent at P.
 - Work out the equation of the normal at P.
 - Work out the values of x for which the curve has a gradient of 13.
- ⑤ The sketch shows the graph of $y = x^3 - 9x^2 + 23x - 15$

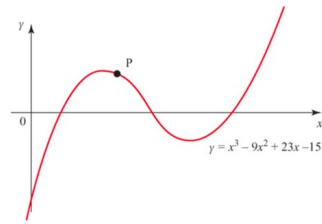


Figure 8.14

- The point P marked on the curve has its x -coordinate equal to 2
- Work out the equation of the tangent at P.
- Q is a point on the curve where the tangent is parallel to the tangent at P.
- Work out the equation of the tangent at Q.
- ⑥ The point (2, -8) is on the curve $y = x^3 - px + q$.
- Identify a relationship between p and q .
The tangent to this curve at the point (2, -8) is parallel to the x -axis.
 - Work out the value of p .
 - Work out the coordinates of the other point where the tangent is parallel to the x -axis.
 - Work out the equation of the normal to the curve at the point where it crosses the y -axis.
- ⑦ The sketch shows the graph of $y = x^2 - x - 1$

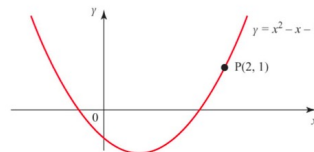


Figure 8.15

- Work out the equation of the tangent at the point P.
- The normal at a point Q on the curve is parallel to the tangent at P.
- Work out the coordinates of the point Q.

- A curve has the equation $y = (x - 3)(7 - x)$.
 - Work out the equation of the tangent at the point (6, 3).
 - Work out the equation of the normal at the point (6, 3).
 - Which one of these lines passes through the origin?
- A curve has the equation $y = 1.5x^3 - 3.5x^2 + 2x$.
 - Show that the curve passes through the points (0, 0) and (1, 0).
 - Work out the equations of the tangents and normals at each of these points.
 - What shape is formed by the four lines in part (ii)?
- Figure 8.16 shows the curve with the equation $y = x^2 + \frac{2}{x}$ for $x > 0$
 - Work out the gradient function $\frac{dy}{dx}$ and calculate the coordinates of the minimum point.
 - State the equations of the tangent and the normal at that minimum point.
 - Work out the equations of the tangent and normal at the point where $x = 2$

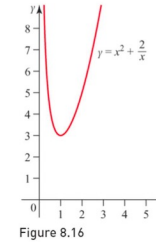


Figure 8.16

Fluency Practice

1

$$y = \frac{1}{3}x^3 + 3x^2 + 10x + 1$$

- a) Find $\frac{dy}{dx}$
- b) By completing the square, hence show that this function is increasing for all x .

2

$$f(x) = x^2 + 3x + 5$$

- a) Find $f'(x)$
- b) Hence or otherwise, determine the values of x for which $f(x)$ is increasing.

3

$$y = x^2 - x + 2$$

- Determine the values of x for which y is decreasing.

4

$$y = x^3 - 7x^2 - 5x$$

- Determine the values of x for which y is increasing.

5

- Show that $f(x) = x^3 - 9x^2 + 39x - 4$ is an increasing function for all values of x .

6

- [AQA IGCSEFM June 2012 Paper 2 Q20] For what values of x is $y = 150x - 2x^3$ an increasing function?

7

- [Set 4 Paper 1 Q10] $y = 10 - 8x - x^3$ for all values of x . Show that y is a decreasing function for all values x .

Purposeful Practice

- ① Work out the values of x for which the following functions are increasing.
- | | |
|--------------------------------|-------------------------------|
| (i) $y = x^2 + 4$ | (ii) $y = 2x - 3$ |
| (iii) $y = x^2 + 2x - 5$ | (iv) $y = x^2 - 3x$ |
| (v) $y = 3x^2 + 4x + 7$ | (vi) $y = (x + 6)(x - 2)$ |
| (vii) $y = x^3 - 2x^2$ | (viii) $y = x^3 + 6x^2 - 15x$ |
| (ix) $y = x^3 - 3x^2 - 9x + 1$ | |
- ② Work out the values of x for which the following functions are decreasing.
- | | |
|----------------------------------|--------------------------------|
| (i) $y = 4x^2$ | (ii) $y = x^2 - 6x + 2$ |
| (iii) $y = x(x + 2)$ | (iv) $y = 3 + 4x - x^2$ |
| (v) $y = 12 - x$ | (vi) $y = (2x + 1)^2$ |
| (vii) $y = \frac{1}{3}x^3 + x^2$ | (viii) $y = 2x^3 - 3x^2 - 72x$ |
| (ix) $y = 27x - x^3$ | |
- ③ Prove that $y = \frac{1}{3}x^3 + 2x^2 + 7x + 1$ is an increasing function for all values of x .
- ④ Prove that $y = x^3 - 6x^2 + 27x - 4$ is an increasing function for all values of x .
- ⑤ Work out the values of x for which $y = x^2 + \frac{2}{x}$ is an increasing function.
- ⑥ Prove that $y = 12 - 2x - x^3$ is a decreasing function for all values of x .
- ⑦ Prove that $y = \frac{1}{x}$ is a decreasing function for all $x \neq 0$.
- ⑧ Work out the values of x for which the following functions are
- | | |
|---------------------------------|--------------------------------|
| (a) increasing | (b) decreasing. |
| (i) $y = x + \frac{1}{x}$ | (ii) $y = x - \frac{1}{x}$ |
| (iii) $y = x^2 + \frac{1}{x^2}$ | (iv) $y = x^2 - \frac{1}{x^2}$ |
- ⑨ Air is being pumped into a spherical balloon at the rate of $1000 \text{ cm}^3 \text{ s}^{-1}$. Initially the balloon contains no air. (The formula for the volume of a sphere is $V = \frac{4}{3}\pi r^3$).
- | |
|---|
| (i) Calculate the volume V of the balloon after 10 seconds. |
| (ii) Calculate the volume of the balloon after t seconds. |
| (iii) State the value of $\frac{dV}{dt}$. |
| (iv) Calculate the radius of the balloon after t seconds. |

Purposeful Practice

- ① Work out $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for each of the following expressions.
- (i) $y = 3x^3 + 3x$ (ii) $y = x^5 - 25$
(iii) $y = 3x - 5x^4$
- ② Work out $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for each of the following expressions.
- (i) $y = x^4 - 2x^2 + 5x - 4$ (ii) $y = 2x^3 + 3x - 4$
(iii) $y = x^3 - 2x^2 + 1$
- ③ Work out $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for each of the following expressions. Remember that when an expression involves brackets you need to multiply out before differentiating.
- (i) $y = (2x - 1)(x + 2)$ (ii) $y = (2x - 1)^2$
(iii) $y = (1 - 3x)(2x - 3)$
- ④ Work out $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for each of the following expressions.
- (i) $y = 3(x - 2)(x^2 - 2x + 3)$ (ii) $y = 2x^2(x - 1)^2$
(iii) $y = x^3(3x + 1)^2$
- ⑤ The sum of two numbers x and y is 13 and their product P is 40.
- (i) Write down an expression for y in terms of x .
(ii) Write down an expression for P in terms of x .
(iii) Write down expressions for $\frac{dy}{dx}$ and $\frac{dP}{dx}$.
(iv) Write down the rate of change of $\frac{dP}{dx}$.
- ⑥ For the curve $y = 3x^3 - 2x^2 - 6x - 4$
- (i) write down expressions for $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.
(ii) work out the gradient of the curve at the points $(-1, -7)$, $(1, -9)$ and $(2, 0)$.
(iii) work out the rate of change of the gradient at each of these points.
- ⑦ A formula which you will meet in Mechanics or Physics is $s = ut + \frac{1}{2}at^2$, where the letters in this case are $t =$ time, u is the initial velocity (which will be a constant, or zero if starting from rest), a is the acceleration (which must also be constant, for this formula) and s is the distance travelled. The only variables in the formula are s and t . Using this formula $\frac{ds}{dt}$ will give the velocity after a time t has elapsed.
- (i) Work out $\frac{ds}{dt}$ and hence the velocity after 12 seconds when the distance is measured in metres and time in seconds.
(ii) Work out $\frac{d^2s}{dt^2}$.

Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ when:

(a) $y = x^2 + 4x - 3$

(b) $y = 5x^3 + x^2 + 8x - 3$

(c) $y = x^4 - 7x^2$

(d) $y = x^2 - \frac{2}{x}$

Find the coordinates of the stationary points on each of these curves. By differentiating for a second time, establish whether these points are maximums or minimums.

(a) $y = 4x^2 - 8x$

(b) $y = 5 + 2x - x^2$

(c) $y = (8 + x)(2 - x)$

(d) $y = x^4 - 8x^2$

(e) $y = 2x^3 - 3x^2 - 12x + 5$

(f) $y = x + \frac{1}{x}$

(a) Find the gradient of the curve
 $y = x^2 - 3x + 7$ at the point $(3, 7)$

(b) Find the gradient of the curve
 $y = x^3 + 4x^2 - 9x$ at the point $(2, 6)$

(c) Find the gradient of the curve
 $y = x + \frac{9}{x}$ at the point $(3, 6)$

(a) Find the coordinates of the minimum point on the curve $y = x^2 - 4$

(b) Find the coordinates of the minimum point on the curve $y = x^2 + 8x + 15$

(c) Find the coordinates of the maximum point on the curve $y = 7 - 6x - x^2$

(d) Find the coordinates of the maximum point on the curve $y = 2 + 5x - x^2$

(a) Find the coordinates of the stationary points on the curve $y = x^3 - 3x^2 + 4$. By sketching the graph, determine whether each point is a minimum point or a maximum point.

(b) Find the coordinates of the stationary point on the curve $y = 3x + \frac{12}{x^2}$. Is this point a minimum point or a maximum point?

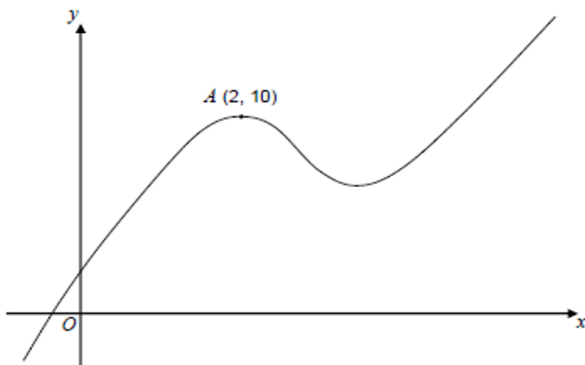
(a) The curve with equation
 $y = x^2 + ax + b$ has a stationary point at $(-4, -11)$. Find the values of a and b .

(b) The curve with equation
 $y = c + dx - x^2$ has a stationary point at $(3, 10)$. Find the values of c and d .

Fluency Practice

1

[Set 4 Paper 2 Q22] A sketch of $y = f(x)$, where $f(x)$ is a cubic function, is shown.



There is a maximum point at $A(2, 10)$.

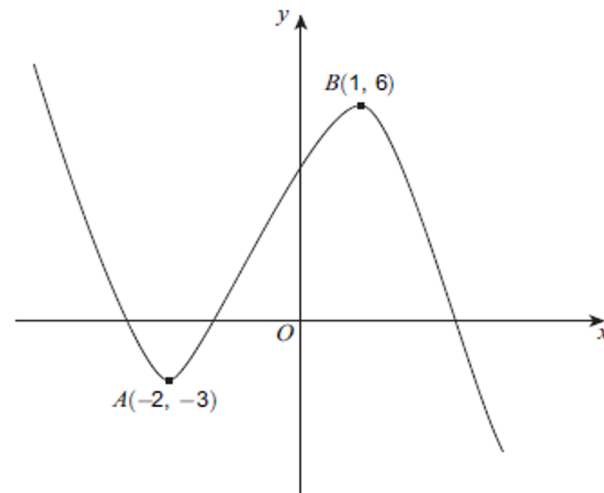
- (a) Write down the equation of the tangent to the curve at A . ?
- (b) Write down the equation of the normal to the curve at A . ?
- (c) Circle the word that describes the cubic function when $x < 2$.

positive negative increasing decreasing

?

2

[June 2013 Paper 2 Q8] A sketch of $y = f(x)$ is shown. There are stationary points at A and B .



- (a) Write down the equation of the tangent to the curve at A . ?
- (b) Write down the equation of the normal to the curve at B . ?
- (c) Circle the range of values of x for which $f(x)$ is an increasing function.

$x < -2,$ $-2 < x < 1,$ $-3 < x < 6$
 $x > 1$

?

Fluency Practice

3

[Set 2 Paper 2 Q12] A curve has equation
 $y = x^3 - 9x^2 + 24x - 16$

(a) Work out $\frac{dy}{dx}$

?

(b) Work out the coordinates of the two stationary points on the curve.

?

4

[Set 1 Paper 2 Q14] (a) Work out the stationary points on the curve $y = x^3 - 12x$.
(b) Sketch the curve $y = x^3 - 12x$

?

6

[Specimen Paper 1 Q13] (a) Work out the coordinates of the stationary point for the curve $y = x^2 + 3x + 4$.
(b) Explain why $x^2 + 3x + 4 = 0$ has no real solutions.

?

(a) A rectangle has a width x cm and a length $(30 - 2x)$ cm. Using calculus, find the maximum area of the rectangle.

(b) A car sales company sells x cars per week. Its revenue R per week is given by the equation $R = 0.2x^2 - 10x + 1750$. Using differentiation, find the number of cars which generates the maximum revenue, and the value of this revenue.

(a) The cost C of a car journey when driving at a speed of x mph is given by

$$C = \frac{720}{x} + 0.2x + 6.$$

Using differentiation, find the value of x that minimises the cost, and the minimum value of C .

(b) The volume of a box is given by $V = x(5 - x)^2$. Use calculus to find the maximum volume of the box, and the value of x for which this occurs.

(a) A picture frame has a perimeter of 120 cm. If the width of the frame is x cm, then show that the height of the frame is $(60 - x)$ cm. Hence use calculus to find the value of x that gives a maximum area for the frame. Calculate this maximum area.

(b) A farmer has enough stone for 80 m of dry stone walling. He wants to create a field with the largest area possible. Find the dimensions of the field that gives this maximum area.

Purposeful Practice

If you have access to a graphic calculator you will find it helpful to use it to check your answers.

- ① For each of the curves given below
- (a) work out $\frac{dy}{dx}$ and the value(s) of x for which $\frac{dy}{dx} = 0$
 - (b) work out the value(s) of $\frac{d^2y}{dx^2}$ at those points
 - (c) classify the point(s) on the curve with these x -values
 - (d) work out the corresponding y -value(s)
 - (e) sketch the curve.
 - (i) $y = 1 + x - 2x^2$
 - (ii) $y = 12x + 3x^2 - 2x^3$
 - (iii) $y = x^3 - 4x^2 + 9$
 - (iv) $y = x(x - 1)^2$
 - (v) $y = x^2(x - 1)^2$
 - (vi) $y = x^3 - 48x$
 - (vii) $y = x^3 + 6x^2 - 36x + 25$
 - (viii) $y = 2x^3 - 15x^2 + 24x + 8$
- ② The graph of $y = px + qx^2$ passes through the point $(3, -15)$ and its gradient at that point is -14
- (i) Work out the values of p and q .
 - (ii) Calculate the maximum value of y and state the value of x at which it occurs.
- ③ (i) Identify the stationary points of the function $f(x) = x^2(3x^2 - 2x - 3)$ and distinguish between them.
- (ii) Sketch the curve $y = f(x)$.
- ④ The curve $y = ax^2 + bx + c$ crosses the y -axis at the point $(0, 2)$ and has a minimum point at $(3, 1)$.
- (i) Work out the equation of the curve.
 - (ii) Check that the stationary point is a minimum.
- ⑤ The sum of two positive numbers p and q is 12
- (i) Write q in terms of p .
 - (ii) S is the sum of the squares of the two numbers. Write down an expression for S in terms of p .
 - (iii) Work out the least value of S , checking that it is a minimum.
- ⑥ The sum of two positive numbers a and b is 40
- (i) Write $2ab$ in terms of a .
 - (ii) Work out values of a and b when $2ab$ is a maximum, checking that it is a maximum.
 - (iii) Work out the maximum value of $2ab$.
- ⑦ x and y are two positive numbers whose sum is 10
- (i) Express $P = xy^2$ in terms of x .
 - (ii) Work out the values of x for which $\frac{dP}{dx} = 0$
 - (iii) Use the second derivative test to identify which one gives the maximum value of P .
 - (iv) Comment on the implication of the other value of x that you calculated.
- ⑧ Netty and Mackenzie are going to climb a mountain and the equation of their path is given by $10y = x + 4x^2 - x^3$ for $x \geq 0$. Distances horizontally and vertically are measured in units of 1000 metres. Give all answers to 3 significant figures.
- (i) How far away, horizontally, is the summit?
 - (ii) How much higher is the summit than where they are now?
- ⑨ The base of a cuboid is x cm by x cm and its height is y cm. Its volume is 216 cm^3 .
- (i) Write down an expression for the surface area in terms of x .
 - (ii) Work out the dimensions that give the minimum surface area, proving that this is a minimum.

Optimisation

One of the most useful applications of differentiation is optimisation.

It allows us to exactly calculate the ideal size for a tin or a box to minimise surface area, and therefore minimise the cost of production.



Dimensions

Baked Beans:
Radius = 3.8cm
Height = 11cm

Tuna:
Radius = 4.3cm
Height = 3.5cm



1. Calculate the volume and surface area of the baked beans tin, to 2 significant figures.
Hint: consider the net of a cylinder.
2. Find an expression for the surface area of a cylinder of radius r and volume 500cm^3 .
Hint: start by writing the height in terms of r .
3. Use differentiation to find the optimal radius and the resulting minimum surface area.
Hint: stationary points occur when the derivative equals 0.
4. Calculate the potential saving in steel if tuna were to be sold in optimal cylindrical tins (of the same volume) rather than using current dimensions.
Hint: use scale factors to generalise the result you've already found.

Extension:

The majority of tins are cylindrical in shape – an optimal circular prism (cylinder with equal diameter and height) has a surface area almost 8% smaller than that of an optimal rectangular-based prism (cube). However, even among cylindrical packaging, some are far from optimal (eg Pringles tubes, spice jars, olive oil bottles). Why is this?

2 Domain and Range

Fluency Practice

Work out a suitable domain and the range of each function.
A sketch may help with each one.

1

Function	$f(x) = 2x$
Domain	?
Range	?

2

Function	$f(x) = 2^x$
Domain	?
Range	?

3

Function	$f(x) = \frac{1}{x}$
Domain	?
Range	?

4

Function	$f(x) = \sin x$
Domain	?
Range	?

5

Function	$f(x) = 2 \cos x$
Domain	?
Range	?

6

Function	$f(x) = x^3 + 1$
Domain	?
Range	?

7

Function	$f(x) = \frac{1}{x-2} + 1$
Domain	?
Range	?

8

Function	$f(x) = 2\cos(\sqrt{x+1})$
Domain	?
Range	?

Find the ranges for each of these functions and their domains:

(a) $f(x) = 5x + 1$ $x = \{1, 2, 3\}$

(b) $g(x) = x^2 - 3$ $x = \{3, 4, 5\}$

(c) $h(x) = \sqrt{2x + 1}$ $x = \{2, 4, 12\}$

The domain is $\{1, 2, 3, 4\}$. Find the ranges of these functions:

(d) $f: x \rightarrow x + 9$

(e) $g: x \rightarrow 2x^2$

(f) $h: x \rightarrow \frac{x}{x+1}$

What value of x must be excluded from the domains for the following functions?

(g) $f(x) = \frac{3}{x}$

(h) $g(x) = \frac{x}{x-2}$

(i) $h(x) = \frac{x+1}{x+2}$

What values of x must be excluded from the domains for the following functions?

(j) $f: x \rightarrow \sqrt{x}$

(k) $g: x \rightarrow \sqrt{x - 3}$

(l) $h: x \rightarrow \sqrt{x + 2}$

What values of x must be excluded from the domains for the following functions?

(m) $f(x) = \frac{2}{x-1} + \frac{3}{x+5}$

(n) $g(x) = \sqrt{2x - 1}$

1: Give the domain.

a) $g(x) = \frac{1 - 7x}{4x}$

b) $f(x) = \frac{-7x - 5}{x - 1}$

c) $h(x) = \frac{-5x - 9}{5x - 2}$

d) $f(x) = \frac{-8}{x} - 5$

2: Give the domain.

a) $g(x) = 7\sqrt{x} + 6$

b) $h(x) = \sqrt{\left(\frac{-x + 10}{4}\right)}$

c) $g(x) = \sqrt{-3x - 9}$

d) $h(x) = \sqrt{\left(\frac{-x}{2}\right)} - 5$

3: Give the domain.

a) $f(x) = \frac{2x - 5}{3x - 9}$

b) $g(x) = \sqrt{\left(\frac{x - 9}{6}\right)}$

c) $f(x) = 9\sqrt{x} - 3$

d) $h(x) = \frac{x + 7}{x - 4}$

4: What values must be excluded from the domain?

a) $h(x) = \frac{1}{-5 - 4x}$

b) $g(x) = \frac{10}{x + 6}$

c) $f(x) = \frac{1 - 4x}{6x}$

d) $g(x) = \frac{2x - 3}{6x - 5}$

5: What values must be excluded from the domain?

a) $f(x) = \sqrt{-4x + 5}$

b) $h(x) = \sqrt{\left(\frac{-x}{2}\right) + 3}$

c) $h(x) = 8\sqrt{x - 4}$

d) $g(x) = \sqrt{-5x - 3}$

6: What values must be excluded from the domain?

a) $f(x) = \frac{x + 4}{x - 3}$

b) $g(x) = \sqrt{\left(\frac{x - 6}{8}\right)}$

c) $h(x) = \sqrt{\left(\frac{x}{2}\right) + 7}$

d) $f(x) = \frac{5x + 5}{x - 1}$

Fluency Practice

<p>A1</p> $f(x) = 3x - 5$ <p>Find $f(6)$</p>	<p>A2</p> $f(x) = x^2 - \frac{10}{x}$ <p>Find $f(-2)$</p>	<p>A3</p> $f(x) = \frac{3x+2}{x}$ <p>Find $f(0.5)$</p>	<p>A4</p> $f(x) = \frac{9}{x+2} + \frac{3}{x-1}$ <p>Find $f(0)$</p>
<p>B1</p> $f(x) = \sqrt{8-x}$ <p>State the values of x which must be excluded from the domain of f.</p>	<p>B2</p> $f(x) = \frac{7}{3x+1}$ <p>State the value of x which must be excluded from the domain of f.</p>	<p>B3</p> $f(x) = \frac{5}{x+1} + \frac{2}{x-3}$ <p>State the values of x which cannot be included in any domain of f.</p>	<p>B4</p> $f(x) = \sqrt{x-4}$ <p>State the values of x which cannot be included in any domain of f.</p>
<p>C1</p> $f(x) = 4x - 9$ <p>Express the inverse function f^{-1} in the form $f^{-1}(x) = \dots$</p>	<p>C2</p> $f(x) = \frac{2x}{x-1}$ <p>Express the inverse function f^{-1} in the form $f^{-1}(x) = \dots$</p>	<p>C3</p> $f(x) = \frac{x}{3x+1}$ <p>Find $f^{-1}(x)$</p>	<p>C4</p> $f(x) = \sqrt{2x-1}$ <p>Express the inverse function f^{-1} in the form $f^{-1}(x) = \dots$</p>
<p>D1</p> $f(x) = 2x - 7$ <p>Given that $f(a) = 3$, work out the value of a</p>	<p>D2</p> $f(x) = \frac{1}{2}x + 4$ <p>$f(a) = -2$ Work out the value of a.</p>	<p>D3</p> $f(x) = \frac{x}{x-1}$ <p>Solve the equation $f(x) = 1.2$ Show your working clearly.</p>	<p>D4</p> $f(x) = \frac{3}{x+1} + \frac{1}{x-2}$ <p>Find the value of x for which $f(x) = 0$ Show your working clearly.</p>

Fluency Practice

① Write down the range of $f(x)$ in each of the following.

- (i) $f(x) = 3x$ $x < 2$
- (ii) $f(x) = x + 4$ $x \geq 1$
- (iii) $f(x) = 2x + 4$ $x \geq -1$
- (iv) $f(x) = 10 - x$ $x \leq 4$

② Write down the range of $f(x)$ in each of the following.

- (i) $f(x) = 2x$ $1 \leq x \leq 5$
- (ii) $f(x) = x - 3$ $0 < x < 10$
- (iii) $f(x) = 5 - 2x$ $x \geq -3$
- (iv) $f(x) = 3 - 4x$ $-2 \leq x < 3$

③ Write down the range of $f(x)$ in each of the following.

- (i) $f(x) = \frac{x+5}{2}$ $0 \leq x \leq 5$
- (ii) $f(x) = \frac{2x-3}{4}$ $-2 \leq x \leq 2$
- (iii) $f(x) = \frac{3-2x}{3}$ $-3 \leq x \leq 5$
- (iv) $f(x) = \frac{1-3x}{2}$ $-3 \leq x \leq 5$

④ Write down the range of $f(x)$ in each of the following.

- (i) $f(x) = x^2$ $-2 \leq x < 2$
- (ii) $f(x) = x^2$ $0 < x < 4$
- (iii) $f(x) = x^3$ $x \geq 0$
- (iv) $f(x) = x^3$ $-1 \leq x \leq 3$

⑤ Write down the range of $f(x)$ in each of the following.

- (i) $f(x) = 2x^2 - 3$ $0 \leq x \leq 4$
- (ii) $f(x) = 3x^2 - 2$ $0 \leq x \leq 4$
- (iii) $f(x) = 3 - 2x^2$ $-1 \leq x \leq 2$
- (iv) $f(x) = 2 - 3x^2$ $-1 \leq x \leq 2$

⑥ Write down the range of $f(x)$ in each of the following.

- (i) $f(x) = x^2 + 2x$ $0 \leq x \leq 3$
- (ii) $f(x) = x^2 + 2x$ $-2 \leq x \leq 3$
- (iii) $f(x) = x^2 - 2x$ $0 \leq x \leq 3$
- (iv) $f(x) = x^2 - 2x$ $2 \leq x \leq 3$

⑦ In each of the following, a sketch of a function, $f(x)$, is shown. Write down the domain and the range for $f(x)$.

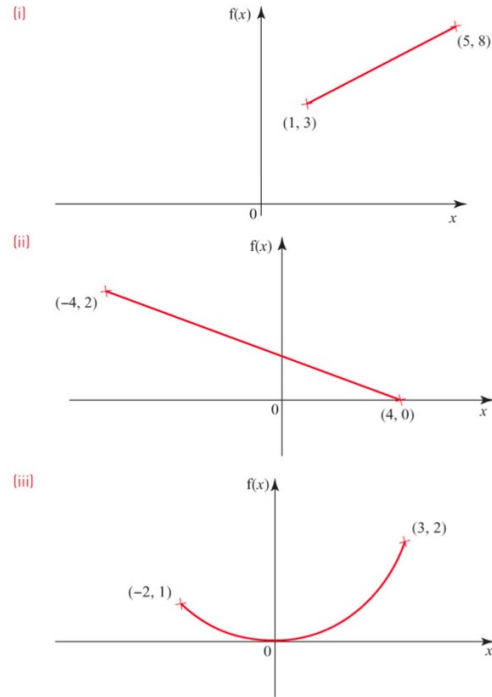


Figure 3.4

⑧ Sketch the graph of each of the following functions and write down the corresponding range.

- (i) $f(x) = 3x - 2$ for $-1 \leq x \leq 3$
- (ii) $f(x) = 2 - 3x$ for $-1 \leq x \leq 2$
- (iii) $f(x) = x^2 + 3$ for $-2 \leq x \leq 2$
- (iv) $f(x) = 4 - x^2$ for $-2 \leq x \leq 3$

Fluency Practice

- 1** Work out the range for each of these functions.

(a) $f(x) = x^2 + 6$ for all x

?

(b) $f(x) = 3x - 5, -2 \leq x \leq 6$

?

(c) $f(x) = 3x^4, x < -2$

?

- 2** (a) $f(x) = \frac{x+2}{x-3}$

Give a reason why $x > 0$ is not a suitable domain for $f(x)$.

?

(b) Give a possible domain for

$f(x) = \sqrt{x-5}$

?

3 $f(x) = 3 - 2x, a < x < b$

The range of $f(x)$ is $-5 < f(x) < 5$

Work out a and b .

?

- 4** [Set 1 Paper 2] (a) The function $f(x)$ is defined as:

$$f(x) = 22 - 7x, \quad -2 \leq x \leq p$$

The range of $f(x)$ is $-13 \leq f(x) \leq 36$

Work out the value of p .

?

(b) The function $g(x)$ is defined as

$$g(x) = x^2 - 4x + 5 \text{ for all } x.$$

(i) Express $g(x)$ in the form $(x - a)^2 + b$

?

(ii) Hence write down the range of $g(x)$.

?

- 5** [June 2012 Paper 1] $f(x) = 2x^2 + 7$ for all values of x .

(a) What is the value of $f(-1)$?

?

(b) What is the range of $f(x)$?

?

Fluency Practice

6 [Jan 2013 Paper 2]

$$f(x) = \sin x \quad 180^\circ \leq x \leq 360^\circ$$

$$g(x) = \cos x \quad 0^\circ \leq x \leq \theta$$

(a) What is the range of $f(x)$?

?

(b) You are given that $0 \leq g(x) \leq 1$.

Work out the value of θ .

?

7 By completing the square or otherwise, determine the range of the following functions:

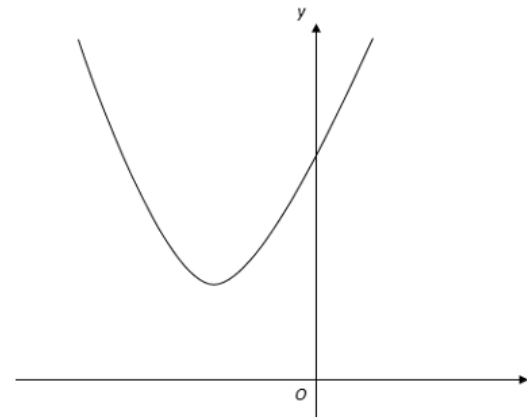
(a) $f(x) = x^2 - 2x + 5$, for all x

?

(b) $f(x) = x^2 + 6x - 2$, for all x

?

8

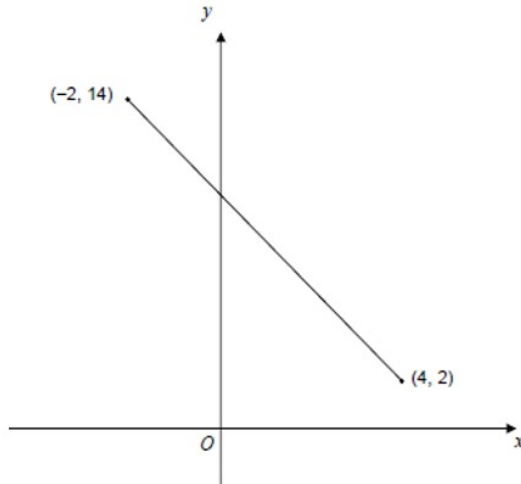


Here is a sketch of $f(x) = x^2 + 6x + a$ for all x , where a is a constant. The range of $f(x)$ is $f(x) \geq 11$. Work out the value of a .

?

Fluency Practice

9



The straight line shows a sketch of $y = f(x)$ for the full domain of the function.

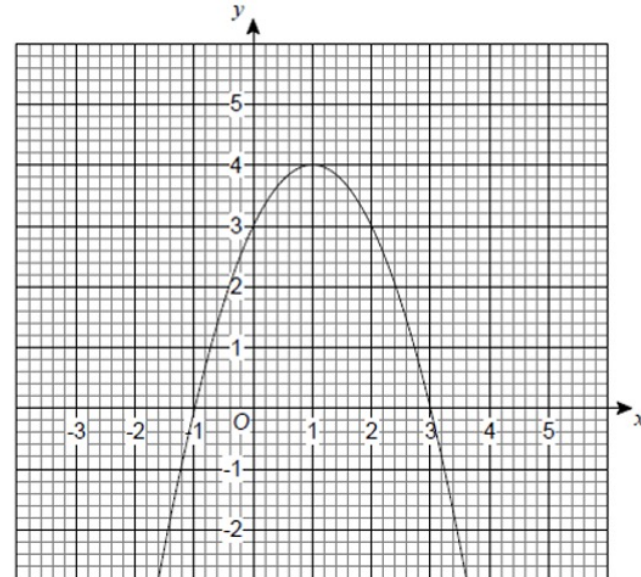
(a) State the domain of the function.

$$2 < f(x) < 14$$

(b) Work out the line.

$$f(x) = -2x + 10$$

10



$f(x)$ is a quadratic function with domain all real values of x . Part of the graph of $y = f(x)$ is shown.

(a) Write down the range of $f(x)$.

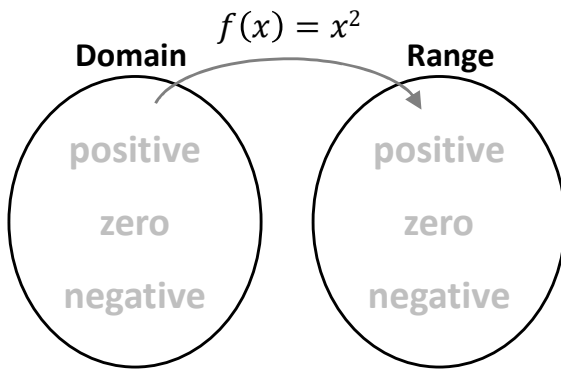
(b) Use the graph to find solutions of the equation $f(x) = 1$.

(c) Use the graph to solve $f(x) < 0$.

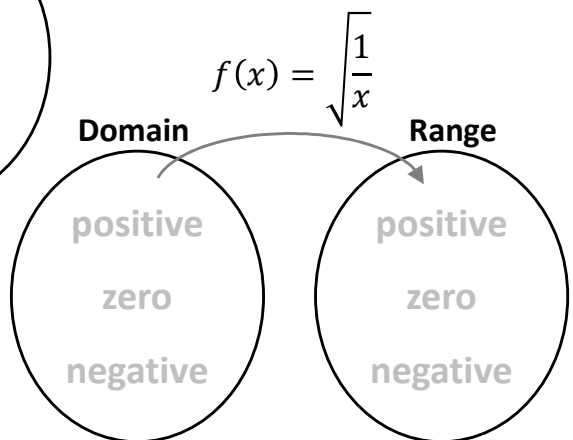
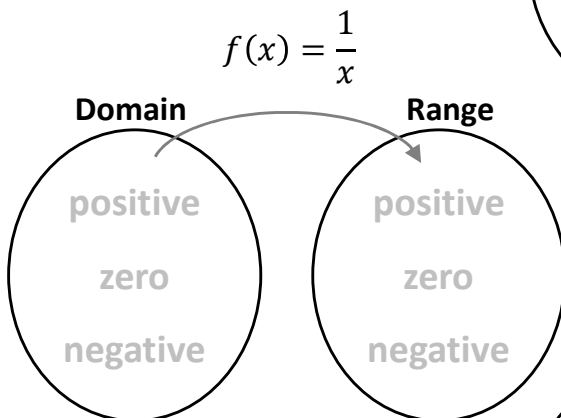
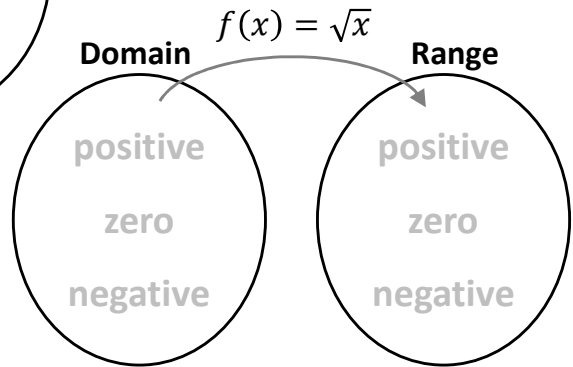
Special Functions

①

Think about these 4 functions.
How do positive & negative numbers (& zero) in
the Domain map onto the Range?



Which numbers
cannot map between
the domain & range?



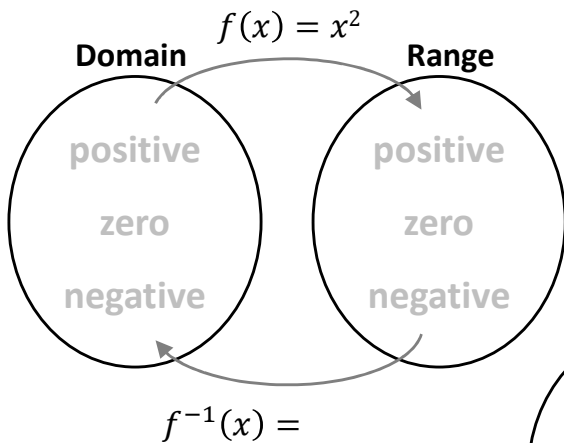
Special Functions

②

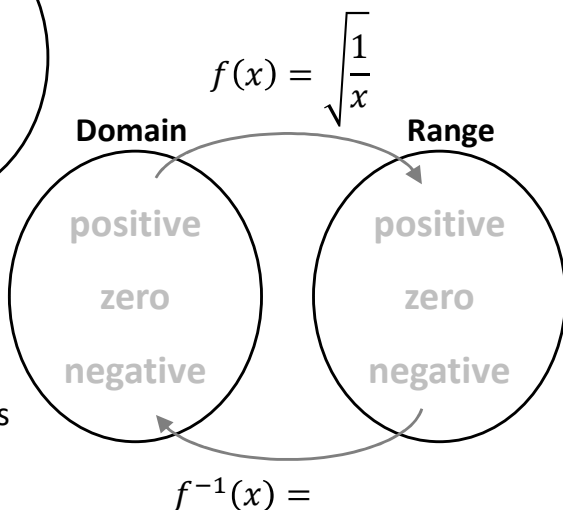
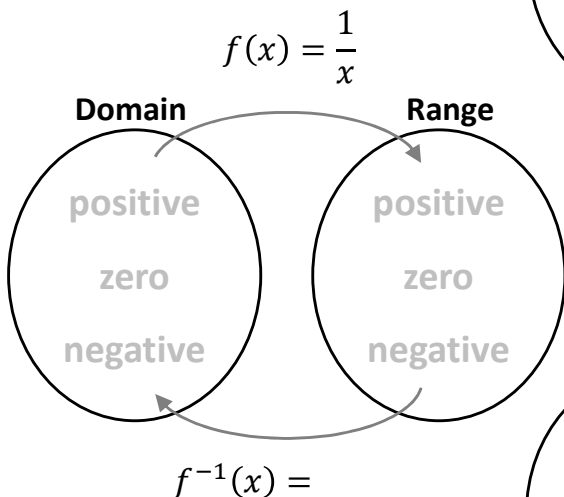
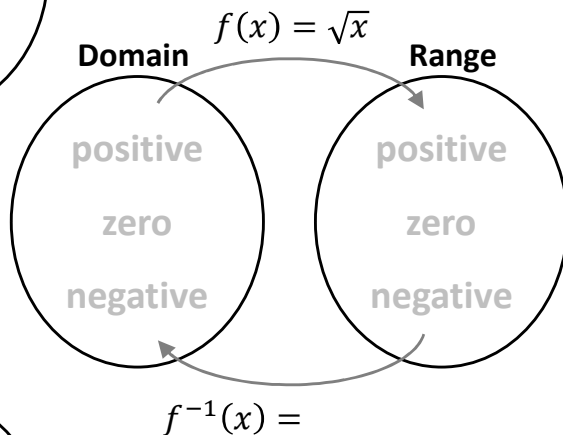
Think about these 4 functions.

How do positive & negative numbers (& zero) in the Domain map onto the Range?

Express the inverse function, how do numbers map in the reverse direction?



Which numbers **cannot** map between the domain & range?



For specific values of x , how are the domain & range values connected for the normal & inverse operations?

Fluency Practice

1 Domain is $1 \leq x < 3$. Range $1 \leq f(x) \leq 3$. $f(x)$ is an increasing function.

?

2 Domain is $1 \leq x \leq 3$. Range $1 \leq f(x) \leq 3$. $f(x)$ is a decreasing function.

?

3 Domain is $5 \leq x \leq 7$. Range $7 \leq f(x) \leq 11$. $f(x)$ is an increasing function.

?

4 Domain is $5 \leq x \leq 7$. Range $7 \leq f(x) \leq 11$. $f(x)$ is a decreasing function.

?

5 Domain is $-4 \leq x \leq 7$. Range $4 \leq f(x) \leq 8$. $f(x)$ is a decreasing function.

?

3 Piecewise Functions

Sketch the following functions:

(a) $f(x) = 2$ for $-5 \leq x < 0$

$f(x) = 2 - x$ for $0 \leq x \leq 3$

$f(x) = -1$ for $3 < x \leq 5$

(b) $f(x) = x$ for $-4 \leq x < 0$

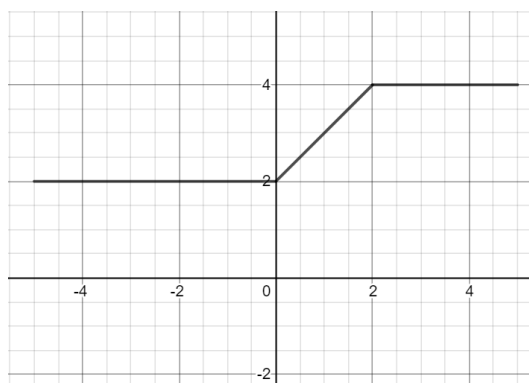
$f(x) = x^2$ for $0 \leq x \leq 4$

(c) $f(x) = x^2 + 1$ for $-4 \leq x < 0$

$f(x) = 1$ for $0 \leq x \leq 2$

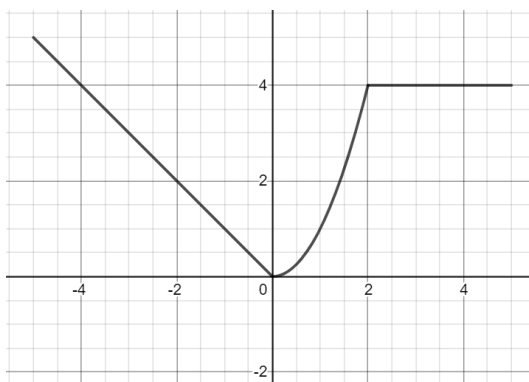
$f(x) = x - 1$ for $2 < x \leq 4$

(a) Given the graph of $y = f(x)$, define the function, stating the domain of each part clearly.



(b) Evaluate $f(1)$

(a) Given the graph of $y = f(x)$, define the function, stating the domain of each part clearly.



(b) Solve $f(x) = 1$

Fluency Practice

- ① Draw the graph of $y = f(x)$ where
 $f(x) = 2$ $-2 \leq x < 1$
 $= 2x$ $1 \leq x \leq 3$
- ② Draw the graph of $y = f(x)$ where
 $f(x) = x^2$ $0 \leq x < 3$
 $= 9$ $3 \leq x \leq 5$
- ③ Draw the graph of $y = g(x)$ where
 $g(x) = x + 3$ $-3 \leq x < 0$
 $= 3 - x$ $0 \leq x \leq 3$
- ④ Draw the graph of $y = f(x)$ where
 $f(x) = 3x - 1$ $-2 \leq x < 1$
 $= 3 - x$ $1 \leq x < 4$
 $= -1$ $4 \leq x \leq 6$
- ⑤ Draw the graph of $y = f(x)$ where
 $f(x) = 2x$ $-2 \leq x < 0$
 $= \frac{1}{2}x$ $0 \leq x < 2$
 $= 5 - 2x$ $2 \leq x \leq 4$
- ⑥ Draw the graph of $y = g(x)$ where
 $g(x) = -4$ $-3 \leq x < -2$
 $= -x^2$ $-2 \leq x < 2$
 $= 3x - 10$ $2 \leq x \leq 4$

- ⑦ Figure 3.39 shows the graph of $y = f(x)$.
- (i) Define $f(x)$, stating clearly the domain for each part.
 - (ii) State the range of $f(x)$.
 - (iii) Solve $f(x) = 5$.



Figure 3.39

- ⑧ Figure 3.40 shows the graph of $y = f(x)$.
- (i) Define $f(x)$, stating clearly the domain for each part.
 - (ii) State the range of $f(x)$.
 - (iii) Solve $f(x) = 3$.

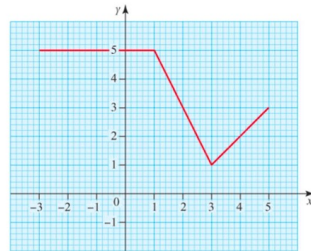


Figure 3.40

- ⑨ The graph of $y = f(x)$ is shown in Figure 3.41.
- (i) Define $f(x)$, stating clearly the domain for each part.
 - (ii) Work out the area between the graph and the x -axis.

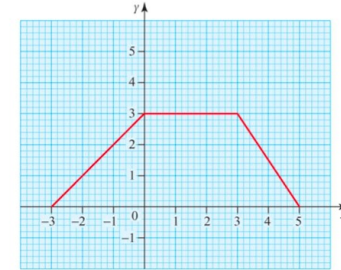


Figure 3.41

- ⑩ The graph of $y = g(x)$ is shown in Figure 3.42.
- (i) Define $g(x)$, stating clearly the domain for each part.
 - (ii) Work out the area between the graph and the x -axis.

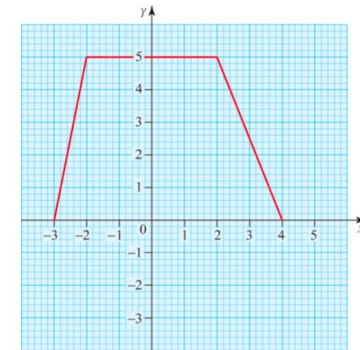


Figure 3.42

- ⑪ The distance–time graph in Figure 3.43 shows the relationship between distance travelled and time for a student who leaves home at 8:15 a.m., walks to the bus stop and then catches the bus to school.
- (i) Describe what is happening between O and A.
 - (ii) Why do you think the distance doesn't change between A and B?
 - (iii) Think about the part of the graph from B to C. What does this tell you about the bus journey? How realistic is that?

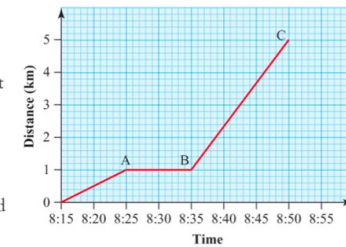


Figure 3.43

Fluency Practice

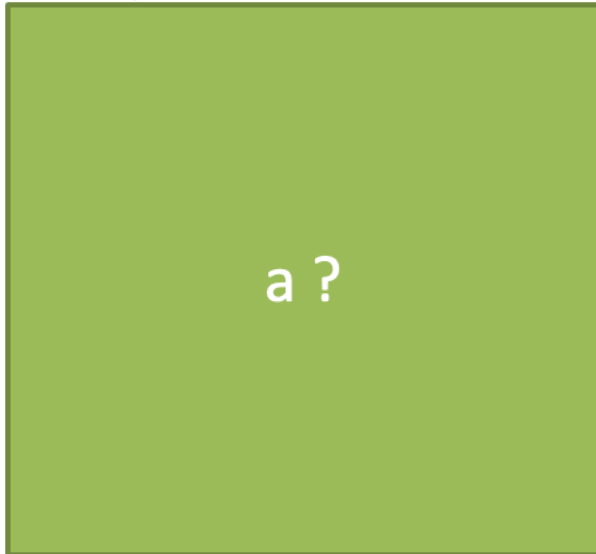
1 [Jan 2013 Paper 2] A function $f(x)$ is defined as:

$$f(x) = \begin{cases} 4 & x < -2 \\ x^2 & -2 \leq x \leq 2 \\ 12 - 4x & x > 2 \end{cases}$$

(a) Draw the graph of $y = f(x)$ for $-4 \leq x \leq 4$

(b) Use your graph to write down **how** b? solutions there are to $f(x) = 3$

(c) Solve $f(x) = -10$ c?



2 [June 2013 Paper 2] A function $f(x)$ is defined as:

$$f(x) = \begin{cases} x + 3 & -3 \leq x < 0 \\ 3 & 0 \leq x < 1 \\ 5 - 2x & 1 \leq x \leq 2 \end{cases}$$

Draw the graph of $y = f(x)$ for $-3 \leq x < 2$



Fluency Practice

3 [Set 1 Paper 1] A function $f(x)$ is defined as:

$$f(x) = \begin{cases} 3 & 0 \leq x < 2 \\ x + 1 & 2 \leq x < 4 \\ 9 - x & 4 \leq x \leq 9 \end{cases}$$

Draw the graph of $y = f(x)$ for $0 \leq x \leq 9$.



4 [Specimen 1 Q4] A function $f(x)$ is defined as:

$$f(x) = \begin{cases} 3x & 0 \leq x < 1 \\ 3 & 1 \leq x < 3 \\ 12 - 3x & 3 \leq x \leq 4 \end{cases}$$

Calculate the area enclosed by the graph of $y = f(x)$ and the x -axis.



Area =

Fluency Practice

5 [AQA Worksheet Q9]

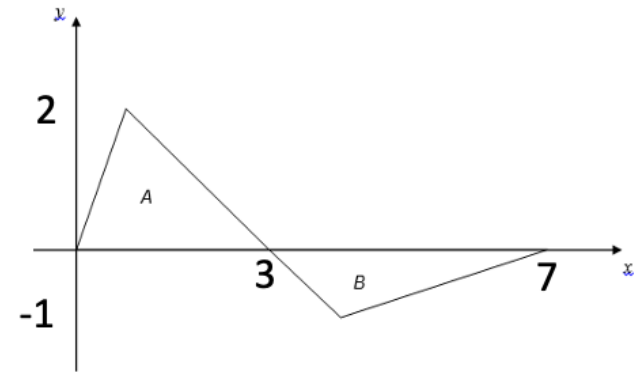
$$f(x) = \begin{cases} -x^2 & 0 \leq x < 2 \\ -4 & 2 \leq x < 3 \\ 2x - 10 & 3 \leq x \leq 5 \end{cases}$$

Draw the graph of $f(x)$ from $0 \leq x \leq 5$.

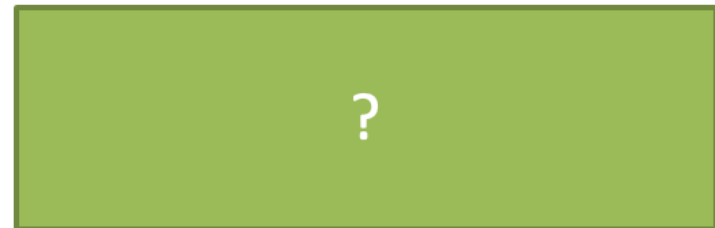


6 [AQA Worksheet Q10]

$$f(x) = \begin{cases} 2x & 0 \leq x < 1 \\ 3 - x & 1 \leq x < 4 \\ \frac{x - 7}{3} & 4 \leq x \leq 7 \end{cases}$$



Show that *area of A*: *area of B* = 3:2



Fluency Practice

11 The function $f(x)$ is defined as:

$$f(x) = \begin{cases} x^2 - 4 & 0 \leq x < 3 \\ 14 - 3x & 3 \leq x \leq 5 \end{cases}$$

Work out the range of $f(x)$.

?

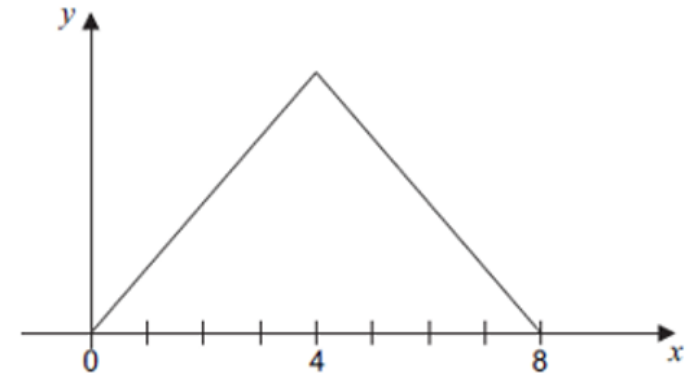
12 The function $f(x)$ has the domain $-3 \leq x \leq 3$ and is defined as:

$$f(x) = \begin{cases} x^2 + 3x + 2 & -3 \leq x < 0 \\ 2 + x & 0 \leq x \leq 3 \end{cases}$$

Work out the range of $f(x)$.

?

13



[June 2012 Paper 2] A sketch of $y = g(x)$ for domain $0 \leq x \leq 8$ is shown. The graph is symmetrical about $x = 4$. The range of $g(x)$ is $0 \leq g(x) \leq 12$. Work out the function $g(x)$.

$$g(x) = \begin{cases} ? & 0 \leq x \leq 4 \\ ? & 4 < x \leq 8 \end{cases}$$

?