1 Equations of Circles & Tangents

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THE EQUATION OF A CIRCLE

$$x^2 + y^2 = r^2$$

- This is a circle with centre **(0, 0)** — the origin.
- The number on the right is r^2, **not** the radius.
- So $x^2 + y^2 = 25$ has radius **5**, not 25.
- If r^2 is not a square number, leave the radius as a **surd**.

WORKED EXAMPLE — IS A POINT ON THE CIRCLE?

Does (−5, 7) lie on $x^2 + y^2 = 85$?

$(-5)^2 + 7^2$

$= 25 + 49 = 74$

$74 \neq 85$, so **it does not lie on the circle**

74 < 85, so the point is **inside** the circle.

WORKED EXAMPLE — RADIUS AND EQUATION

a) Find the radius of $x^2 + y^2 = 196$.

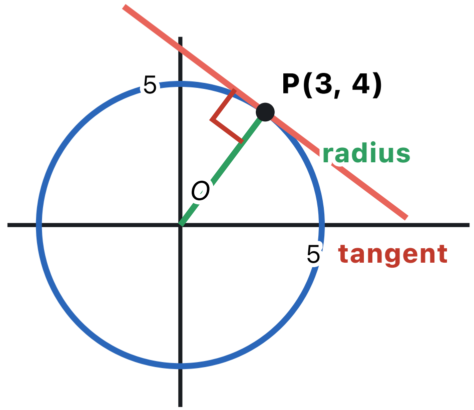
$r = \sqrt{196} = \mathbf{14}$

b) The point (−5, 3) is on a circle centred on the origin. Find its equation.

$r^2 = (-5)^2 + 3^2 = 34$

$\mathbf{x^2 + y^2 = 34}$

THE TANGENT IS PERPENDICULAR TO THE RADIUS



FINDING A TANGENT — THE METHOD

1. Gradient of the **radius** = $\frac{y}{x}$ at that point.
2. Gradient of the **tangent** = the **negative reciprocal**.
3. Put it into $y - y_1 = m(x - x_1)$.
4. **Rearrange** into the form asked for.

Perpendicular gradients **multiply to −1**.

WORKED EXAMPLE — EQUATION OF A TANGENT

Find the equation of the tangent to $x^2 + y^2 = 25$ at the point P(3, 4).

Gradient of OP $= \frac{4}{3}$

Gradient of tangent $= -\frac{3}{4}$

$y - 4 = -\frac{3}{4}(x - 3)$

$y - 4 = -\frac{3}{4}x + \frac{9}{4}$

$\mathbf{y = -\frac{3}{4}x + \frac{25}{4}}$

Check: at $x = 3$, $y = -\frac{9}{4} + \frac{25}{4} = 4$ ✓

WORKED EXAMPLE — WHERE IT CROSSES AN AXIS

The tangent above crosses the x-axis at Q. Find the coordinates of Q.

On the x-axis, $y = 0$

$0 = -\frac{3}{4}x + \frac{25}{4}$

$\frac{3}{4}x = \frac{25}{4}$

$x =$ **Q($\frac{25}{3}$, 0)**

NEGATIVE RECIPROCALS

Radius	Tangent
$\frac{4}{3}$	$-\frac{3}{4}$
$-\frac{1}{3}$	3
4	$-\frac{1}{4}$
$-\frac{3}{4}$	$\frac{4}{3}$

Flip it over and **change the sign**.

⚠ COMMON EXAM MISTAKES (recurring errors flagged in examiners' reports)

- ✕ Treating the number on the right as the **radius** instead of r^2.
- ✕ Only **flipping** the gradient and forgetting to change the sign.
- ✕ Using the gradient of the **tangent** where the radius gradient is needed.
- ✕ Sign slips with a **negative** coordinate when finding $\frac{y}{x}$.
- ✕ Rounding a surd radius when the question wants it **exact**.
- ✕ Leaving the answer as $y - 4 = \ldots$ when $y = mx + c$ was asked for.
- ✕ Putting $y = 0$ in to find the **y**-intercept, or $x = 0$ for the x-intercept.
- ✕ Not **checking** that the given point satisfies the circle equation.

Revise it: find the tangent to $x^2 + y^2 = 100$ at the point (6, −8).

2 Advanced Sequences

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GEOMETRIC SEQUENCES

nth term = ar^{n-1}

- Each term is the one before **multiplied** by a fixed number.
- That number is the **common ratio**, r.
- a is the **first** term.
- Find r by **dividing** a term by the one before it.

WORKED EXAMPLE — WRITING THE NTH TERM

Find the nth term of 3, 12, 48, 192, …

r = 12 ÷ 3 = 4

Check: 48 ÷ 12 = 4 ✓

a = 3

nth term = **$3 \times 4^{n-1}$**

WORKED EXAMPLE — FINDING R FROM TWO TERMS

The 2nd term of a geometric sequence is 78 and the 6th term is 101 088. Find the common ratio.

From the 2nd to the 6th is **4** steps

$r^4 = \frac{101\,088}{78} = 1296$

$r = \sqrt[4]{1296} = $ **6**

First term a = 78 ÷ 6 = 13

Count the **steps between** the terms, not the term numbers.

QUADRATIC SEQUENCES

nth term = $an^2 + bn + c$

- The **second differences** are constant.
- **2a** = the **second difference**
- **3a + b** = the **first** first difference
- **a + b + c** = the **first term**
- Solve in that **order** — each gives you the next.

THE DIFFERENCE LADDER

term	**0**		**11**		**28**		**51**		**80**
1st		+11		+17		+23		+29	
2nd			+6		+6		+6		

Second difference **6**, so a = 6 ÷ 2 = **3**.

WORKED EXAMPLE — NTH TERM OF A QUADRATIC

Find the nth term of 0, 11, 28, 51, 80.

Second difference = 6, so a = 3

$3n^2$: 3, 12, 27, 48, 75

Sequence − $3n^2$:

−3, −1, 1, 3, 5

Difference 2, zeroth term −5

So that part is 2n − 5

nth term = **$3n^2 + 2n - 5$**

Check n = 4: 48 + 8 − 5 = **51** ✓

WHICH KIND OF SEQUENCE?

Pattern	Type
Same difference	Linear
Same 2nd difference	Quadratic
Same ratio	Geometric
Add the two before	Fibonacci-style

Work out the differences **first** — they tell you which method to use.

⚠ COMMON EXAM MISTAKES (recurring errors flagged in examiners' reports)

- ✕ Writing ar^n instead of **ar^{n-1}** for a geometric sequence.
- ✕ Using the **first** difference for a as if the sequence were linear.
- ✕ Forgetting to **halve** the second difference to get a.
- ✕ Counting term numbers rather than the **steps between** them when finding r.
- ✕ Taking away n^2 rather than **an^2**.
- ✕ Using the **first** term instead of the **zeroth** term for the linear part.
- ✕ Stopping at an^2 and never finding b and c.
- ✕ Not **checking** the nth term against a term further down the sequence.

Revise it: find the nth term of 6, 13, 26, 45, 70.

3 Iterations

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SHOWING A ROOT LIES BETWEEN TWO NUMBERS

1. Substitute the **lower** number into f(x).
2. Substitute the **upper** number into f(x).
3. Show one answer is **positive** and the other **negative**.
4. Say there is a **change of sign**, so a root lies between them.

You must write the **sign change** sentence — the numbers alone are not enough.

WORKED EXAMPLE — SIGN CHANGE

Show that $x^3 - 5x + 1 = 0$ has a solution between 0 and 1.

$f(0) = 0 - 0 + 1 = \mathbf{1}$

$f(1) = 1 - 5 + 1 = \mathbf{-3}$

There is a **change of sign**, so a root lies **between 0 and 1**

The function must be **continuous** for this to work.

MAKING AN ITERATIVE FORMULA

Rearrange $x^3 - 5x + 1 = 0$ to make an iterative formula.

$x^3 + 1 = 5x$

$x = \frac{x^3 + 1}{5}$

$x_{n+1} = \frac{x_n^{\,3} + 1}{5}$

Get a **single** x on the left; every x on the right becomes x_n.

USING THE FORMULA

- Start with the given $\mathbf{x_0}$.
- Put it in to get $\mathbf{x_1}$, then put x_1 in to get x_2, and so on.
- Use the **ANS** key so nothing is rounded on the way.
- Stop when the digits you need have **settled down**.

WORKED EXAMPLE — THREE ITERATIONS

Use $x_{n+1} = \frac{x_n^{\,3} + 1}{5}$ with $x_0 = 0$ to find x_1, x_2 and x_3.

$x_1 = \frac{0^3 + 1}{5} = \mathbf{0.2}$

$x_2 = \frac{0.2^3 + 1}{5} = \mathbf{0.2016}$

$x_3 = \frac{0.2016^3 + 1}{5} = \mathbf{0.20164}$

Root = **0.20** to 2 d.p.

Give **every** iteration asked for, to the accuracy stated.

SHOWING A ROOT IS CORRECT TO 2 D.P.

1. Take the value **0.005 below** your root.
2. Take the value **0.005 above** it.
3. Substitute **both** into f(x).
4. Show the **signs are different**, then say so.

WORKED EXAMPLE — CONFIRMING 0.20

Show that the root is 0.20 correct to 2 d.p.

f(0.195) = **0.0324...**

f(0.205) = **−0.0164...**

There is a change of sign between 0.195 and 0.205

So the root is **0.20 to 2 d.p.**

Anything that rounds to 0.20 lies **between** these two values.

CHECKING A GIVEN REARRANGEMENT

Show that $x^3 - 5x + 1 = 0$ can be written as $x = \frac{x^3 + 1}{5}$.

Start from the **iterative** form:

$5x = x^3 + 1$

$0 = x^3 - 5x + 1$ ✓

Working **backwards** to the original equation is a full proof.

⚠ COMMON EXAM MISTAKES (recurring errors flagged in examiners' reports)

- ✕ Finding f(0) and f(1) but never writing "**change of sign**".
- ✕ Rounding part-way through, so later iterations drift.
- ✕ Using $\mathbf{x_0}$ where x_1 was wanted — count the iterations carefully.
- ✕ Testing the root itself instead of **0.005 either side** of it.
- ✕ Using 0.01 either side rather than **0.005** for 2 d.p.
- ✕ Leaving x on **both** sides when rearranging.
- ✕ Giving the answer to the wrong number of **decimal places**.
- ✕ Saying "it gets closer" instead of showing the sign change.

Revise it: show $x^3 + 4x^2 - x - 5 = 0$ has a root between 1 and 2.

4 Algebraic Proof

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WRITING NUMBERS ALGEBRAICALLY

In words	In algebra
Any number	n
An even number	$2n$
An odd number	$2n + 1$
Two consecutive numbers	$n, n + 1$
Two consecutive evens	$2n, 2n + 2$
Two consecutive odds	$2n + 1, 2n + 3$

n must stand for **any integer** — say so at the start.

MULTIPLES

- A multiple of 3 is **3n**; a multiple of 5 is **5n**.
- To **show** something is a multiple of 4, factorise it as **4(...)**.
- The bracket must contain only **whole numbers**.

SQUARES

In words	In algebra
A number squared	n^2
An even number squared	$(2n)^2 = 4n^2$
An odd number squared	$(2n+1)^2 = 4n^2+4n+1$

HOW TO SET OUT A PROOF

1. Say what your letter **stands for**.
2. Write the numbers **algebraically**.
3. **Expand** and simplify.
4. **Factorise** to show the required form.
5. Write a **concluding sentence** in words.

WORKED EXAMPLE — SUM OF TWO ODD NUMBERS

Prove that the sum of two consecutive odd numbers is a multiple of 4.

Let n be any integer.

The two odd numbers are $2n + 1$ and $2n + 3$

Sum $= (2n + 1) + (2n + 3)$

$= 4n + 4$

$= \mathbf{4(n + 1)}$

$4(n + 1)$ is **4 times an integer**, so the sum is a multiple of 4.

WORKED EXAMPLE — DIFFERENCE OF SQUARES

Prove that the difference between the squares of two consecutive numbers equals the sum of those numbers.

Let n be any integer.

The numbers are n and $n + 1$

$(n + 1)^2 - n^2$

$= n^2 + 2n + 1 - n^2$

$= \mathbf{2n + 1}$

And $n + (n + 1) = 2n + 1$

The two are **equal**, as required.

WORKED EXAMPLE — ALWAYS EVEN

Prove that $(n + 3)^2 - (n + 1)^2$ is always even.

$= (n^2 + 6n + 9) - (n^2 + 2n + 1)$

$= 4n + 8$

$= \mathbf{2(2n + 4)}$

This is **2 times an integer**, so it is always even.

Watch the **minus** in front of the second bracket — it changes both signs.

PROVE, SHOW OR VERIFY?

- **Prove** — must work for **every** number, so use algebra.
- Testing a few numbers is **never** a proof.
- One counter-example **is** enough to **disprove** a statement.

⚠ COMMON EXAM MISTAKES (recurring errors flagged in examiners' reports)

- ✕ Testing 3, 5 and 7 and calling it a proof — that shows nothing about other numbers.
- ✕ Using the **same** letter for two different numbers that need not be equal.
- ✕ Writing consecutive odds as $2n + 1$ and **$2n + 2$**, which is even.
- ✕ Expanding but never **factorising**, so the multiple is never shown.
- ✕ Dropping the **brackets** when subtracting, so signs go wrong.
- ✕ Writing $(2n + 1)^2$ as $4n^2 + 1$.
- ✕ Leaving out the final sentence **in words**.
- ✕ Never saying that **n is an integer**.

Revise it: prove that the sum of three consecutive numbers is always a multiple of 3.

5 Advanced Vectors

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THE BASICS

- A vector has **size** and **direction**.
- $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ means **3 right** and **2 up**.
- Bold **a** and $\overline{AB}$ both name vectors.
- $\overline{BA} = -\overline{AB}$ — same length, **opposite** direction.

MIDPOINTS AND RATIOS

- M is the **midpoint** of AB → $\overline{AM} = \frac{1}{2}\overline{AB}$.
- X divides AB in the ratio **2 : 3** → there are **5** parts in total.
- So $\overline{AX} = \frac{2}{5}\overline{AB}$ and $\overline{XB} = \frac{3}{5}\overline{AB}$.
- Always **add** the ratio parts to get the denominator.

WORKED EXAMPLE — STRAIGHT LINE

$\overline{PQ}$ = 2a + 4b and $\overline{QR}$ = 3a + 6b. Show that P, Q and R lie on a straight line.

2a + 4b = 2(a + 2b)

3a + 6b = 3(a + 2b)

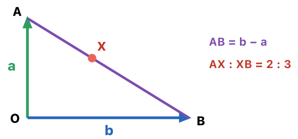
$\overline{QR} = \frac{3}{2}\overline{PQ}$

So $\overline{PQ}$ and $\overline{QR}$ are **parallel**

They share the point **Q**

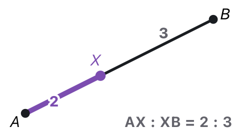
So **P, Q and R are in a straight line**

TRAVELLING ROUND A SHAPE



To get from A to B, go **back** along a then **forward** along b.

WORKED EXAMPLE — DIVIDING IN A RATIO



X divides AB in the ratio 2 : 3. Find $\overline{OX}$ in terms of a and b.

2 + 3 = 5 parts

$\overline{AX} = \frac{2}{5}(b - a)$

$\overline{OX} = \overline{OA} + \overline{AX}$

$= a + \frac{2}{5}(b - a)$

$= a + \frac{2}{5}b - \frac{2}{5}a$

$= \mathbf{\frac{3}{5}a + \frac{2}{5}b}$

The coefficients add to **1** because X is **on** the line AB.

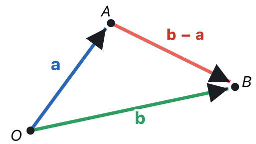
COLUMN VECTORS

$\begin{pmatrix} 3 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$

$3\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 6 \\ -3 \end{pmatrix}$

Add or multiply the **top** and **bottom** separately.

WORKED EXAMPLE — A PATH ROUND THE SHAPE



In the diagram, $\overline{OA}$ = a and $\overline{OB}$ = b. Find $\overline{AB}$.

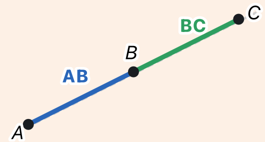
$\overline{AB} = \overline{AO} + \overline{OB}$

= −a + b

= **b − a**

Always route your journey through a point you **know**.

PROVING POINTS ARE IN A STRAIGHT LINE



1. Find **both** vectors in terms of a and b.
2. Show one is a **multiple** of the other.
3. Say they are therefore **parallel**.
4. Say they share a **common point**, so the three points are in a straight line.

Parallel alone is **not** enough — you must name the shared point.

⚠ COMMON EXAM MISTAKES (recurring errors flagged in examiners' reports)

- ✕ Writing $\overline{AB}$ = a − b when it should be **b − a**.
- ✕ Using the ratio 2 : 3 as $\frac{2}{3}$ instead of $\frac{2}{5}$.
- ✕ Forgetting to **add** the ratio parts to get the total.
- ✕ Not multiplying **both** terms inside the bracket by the fraction.
- ✕ Showing two vectors are parallel but never naming the **common point**.
- ✕ Treating a vector as a **coordinate**.
- ✕ Losing the **minus** when reversing a vector.
- ✕ Leaving the answer unsimplified — collect the a terms and the b terms.

Revise it: if X divides AB in the ratio 1 : 4, write $\overline{OX}$ in terms of a and b.