1 Completing the Square

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THE METHOD — HALVE AND SQUARE

$$x^2 + bx + c = (x + \frac{b}{2})^2 - (\frac{b}{2})^2 + c$$

1. **Halve** the number in front of x.
2. Put it inside the bracket: $(x + \frac{b}{2})^2$.
3. **Subtract** the square of that number.
4. **Add on** the constant c and tidy up.

WORKED EXAMPLE — INTEGERS

Write $x^2 + 4x + 6$ in the form $(x + a)^2 + b$.

Half of 4 is **2**

$x^2 + 4x = (x + 2)^2 - 4$

So $x^2 + 4x + 6 = (x + 2)^2 - 4 + 6$

$= \mathbf{(x + 2)^2 + 2}$

WORKED EXAMPLE — ODD COEFFICIENT

Write $x^2 + 5x - 7$ in the form $(x + a)^2 + b$.

Half of 5 is $\frac{5}{2}$, and $(\frac{5}{2})^2 = \frac{25}{4}$

$= (x + \frac{5}{2})^2 - \frac{25}{4} - 7$

$= \mathbf{(x + \frac{5}{2})^2 - \frac{53}{4}}$

Leave it as a **fraction** — do not round.

WHEN THE x^2 COEFFICIENT IS NOT 1

- **Factorise** the a out of the first **two** terms only.
- Complete the square **inside** the bracket.
- **Multiply back** through by a, then collect the numbers.

WORKED EXAMPLE — A = 4

Write $4x^2 - 8x - 9$ in the form $p(x + q)^2 + r$.

$= 4(x^2 - 2x) - 9$

$= 4[(x - 1)^2 - 1] - 9$

$= 4(x - 1)^2 - 4 - 9$

$= \mathbf{4(x - 1)^2 - 13}$

The −1 inside gets multiplied by 4, giving **−4**.

WORKED EXAMPLE — NEGATIVE A

Write $-3x^2 + 2x + 6$ in the form $a(x + b)^2 + c$.

$= -3(x^2 - \frac{2}{3}x) + 6$

$= -3[(x - \frac{1}{3})^2 - \frac{1}{9}] + 6$

$= -3(x - \frac{1}{3})^2 + \frac{1}{3} + 6$

$= \mathbf{-3(x - \frac{1}{3})^2 + \frac{19}{3}}$

WHY COMPLETE THE SQUARE?

- It **solves** any quadratic, even one that will not factorise.
- It gives the **exact** answer using a surd.
- It reads off the **turning point** straight away.

$$(x + a)^2 + b \rightarrow \text{turning point } (-a, b)$$

WORKED EXAMPLE — SOLVING IN EXACT FORM

Solve $x^2 - 6x + 2 = 0$, leaving your answer in exact form.

$(x - 3)^2 - 9 + 2 = 0$

$(x - 3)^2 - 7 = 0$

$(x - 3)^2 = 7$

$x - 3 = \pm\sqrt{7}$

$x = \mathbf{3 + \sqrt{7} \text{ or } 3 - \sqrt{7}}$

Take **both** square roots — the ± gives two solutions.

CHOOSING HOW TO SOLVE

The quadratic...	Use
factorises easily	Factorising
asks for exact form	Completing the square
asks for the turning point	Completing the square
asks to 2 d.p.	The formula

⚠ COMMON EXAM MISTAKES (recurring errors flagged in examiners' reports)

- ✕ Forgetting to **subtract** the square of the halved number.
- ✕ Subtracting **b** rather than $(\frac{b}{2})^2$.
- ✕ Factorising a out of the **constant** as well as the x terms.
- ✕ Not **multiplying back** through by a at the end.
- ✕ Giving only the **positive** square root when solving.
- ✕ Rounding when the question says **exact form** — keep the surd.
- ✕ Reading the turning point of $(x + 3)^2 - 1$ as (3, −1) instead of **(−3, −1)**.
- ✕ Sign slips with a **negative** a — check by expanding back.

Revise it: write $x^2 + 8x - 3$ in completed square form and state the turning point.

2 Advanced Functions

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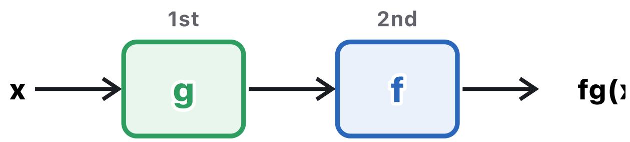
COMPOSITE FUNCTIONS — READ RIGHT TO LEFT

fg(x) → do g first, then f

- The function **nearest the x** goes first.
- **fg is not the same as gf** — the order matters.
- ff(x) can be written as $f^2(x)$.

THE MACHINE PICTURE



WORKED EXAMPLE — WITH A NUMBER

f(x) = 3x + 4 and g(x) = 2x − 5. Find fg(6) and gf(6).

fg(6): g(6) = 2(6) − 5 = 7

f(7) = 3(7) + 4 = **25**

gf(6): f(6) = 3(6) + 4 = 22

g(22) = 2(22) − 5 = **39**

Different answers — proof that **fg ≠ gf**.

COMPOSITE FUNCTIONS IN TERMS OF X

1. Write out the **outer** function.
2. Replace every **x** in it with the **whole** inner function.
3. Put the inner function in **brackets**.
4. **Expand** and simplify.

WORKED EXAMPLE — FG(X) AND GF(X)

f(x) = 3x + 4 and g(x) = 2x − 5. Find fg(x) and gf(x).

fg(x) = f(2x − 5)

= 3(2x − 5) + 4

= 6x − 15 + 4 = **6x − 11**

gf(x) = g(3x + 4)

= 2(3x + 4) − 5

= 6x + 8 − 5 = **6x + 3**

WORKED EXAMPLE — FF(X)

f(x) = 2x − 3. Find ff(x).

ff(x) = f(2x − 3)

= 2(2x − 3) − 3

= 4x − 6 − 3 = **4x − 9**

Substitute the function **into itself**.

WORKED EXAMPLE — WITH A SQUARE

$f(x) = 5x^2$ and g(x) = 2x + 3. Find fg(x) and gf(x).

fg(x) = f(2x + 3) = $5(2x + 3)^2$

= $5(4x^2 + 12x + 9)$

= $\mathbf{20x^2 + 60x + 45}$

gf(x) = $g(5x^2) = 2(5x^2) + 3$

= $\mathbf{10x^2 + 3}$

$(2x + 3)^2$ means (2x + 3)(2x + 3) — **not** $4x^2 + 9$.

CHECK YOURSELF

f(x)	g(x)	fg(x)	gf(x)
5x	x + 2	5(x + 2)	5x + 2
x − 1	x^2	$x^2 - 1$	$(x - 1)^2$
x + 3	$\sqrt{x}$	$\sqrt{x} + 3$	$\sqrt{(x + 3)}$

Swapping the order changes the answer **every time**.

WORKED EXAMPLE — SOLVING

f(x) = 3x + 4, g(x) = 2x − 5. Solve fg(x) = gf(x).

6x − 11 = 6x + 3

−11 = 3

No solutions — the lines are parallel.

If the x terms cancel, check whether the statement can ever be true.

⚠ COMMON EXAM MISTAKES (recurring errors flagged in examiners' reports)

- ✗ Doing **f first** in fg(x) — the function next to the x always goes first.
- ✗ Multiplying f(x) by g(x) instead of substituting one into the other.
- ✗ Leaving out the **brackets** round the inner function before expanding.
- ✗ Writing $(2x + 3)^2$ as $4x^2 + 9$ — the middle term is missing.
- ✗ Substituting the number into the **outer** function first.
- ✗ Assuming fg(x) = gf(x) because they agree for one value of x.
- ✗ Stopping before **simplifying** the final expression.
- ✗ Muddling $f^2(x)$, which means ff(x), with $[f(x)]^2$.

Revise it: with f(x) = 4x − 3 and g(x) = 5x + 2, find fg(x), gf(x) and ff(x).

3 Quadratic Graphs

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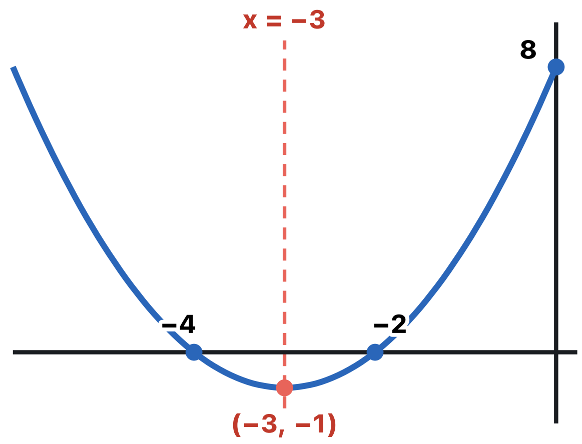
KEY FEATURES OF A QUADRATIC GRAPH

- **y-intercept** — put x = 0 into the equation.
- **x-intercepts (roots)** — put y = 0 and factorise.
- **Line of symmetry** — halfway between the roots.
- **Turning point** — on the line of symmetry.
- $a > 0$ gives a **U** shape; $a < 0$ gives an **∩** shape.

SKETCHING Y = X^2 + 6X + 8



WORKED EXAMPLE — INTERCEPTS

y = (8 − 2x)(3x + 7). Find the y-intercept and the x-intercepts.

y-intercept: x = 0 → y = 8 × 7 = **56**

x-intercepts: y = 0

8 − 2x = 0 → x = 4

3x + 7 = 0 → x = $-\frac{7}{3}$ **and 4**

FINDING THE LINE OF SYMMETRY

$$x = \frac{\text{root}_1 + \text{root}_2}{2}$$

- The graph is symmetrical, so the line sits **halfway** between the roots.
- From completed square form $(x + a)^2 + b$, the line is **x = −a**.

WORKED EXAMPLE — LINE OF SYMMETRY

Find the line of symmetry of y = −2(x − 1)(x + 5).

Roots: x = 1 and x = −5

Midpoint = $\frac{1 + (-5)}{2} = \frac{-4}{2} = -2$

Line of symmetry: **x = −2**

WORKED EXAMPLE — TURNING POINT

y = 2(x + 2)(x − 4). Find the turning point.

Roots: x = −2 and x = 4

$x = \frac{-2 + 4}{2} = 1$

y = 2(1 + 2)(1 − 4) = 2(3)(−3) = −18

Turning point: **(1, −18)**

Find x first, then **substitute back** for y.

HOW TO SKETCH

1. Decide the **shape** from the sign of a.
2. Find the **y-intercept** (put x = 0).
3. **Factorise** and find the roots.
4. Find the **turning point** from the midpoint.
5. Draw a **smooth** curve and label everything.

WORKED EXAMPLE — FULL SKETCH

Sketch y = x^2 + 6x + 8.

a = 1 > 0, so a **U** shape

y-intercept: x = 0 → y = 8

$x^2 + 6x + 8 = (x + 2)(x + 4)$

Roots: x = −2 and x = −4

$x = \frac{-2 + (-4)}{2} = -3$

y = 9 − 18 + 8 = −1

Turning point: **(−3, −1)**

WORKING BACKWARDS

A quadratic passes through (−2, 0) with turning point (−0.5, −4.5). Find the other root.

The turning point is **halfway** between the roots.

−2 is 1.5 to the **left** of −0.5

So the other root is 1.5 to the **right**

x = −0.5 + 1.5 = **(1, 0)**

⚠ COMMON EXAM MISTAKES (recurring errors flagged in examiners' reports)

- ✕ Getting the **sign** of a root wrong — (x + 4) gives the root x = **−4**.
- ✕ Giving the line of symmetry as a **number** rather than an equation like x = −3.
- ✕ Finding the turning point's x but forgetting to **substitute back** for y.
- ✕ Sketching a **U** when a is negative.
- ✕ Drawing the curve with a **flat bottom** or a sharp corner at the vertex.
- ✕ Leaving the sketch **unlabelled** — roots, intercept and vertex all earn marks.
- ✕ Reading the y-intercept off the x term instead of the constant.
- ✕ Averaging the roots wrongly when one is negative.

Revise it: sketch y = x^2 + 6x + 8, labelling both roots, the intercept and the vertex.

4 Quadratic Inequalities

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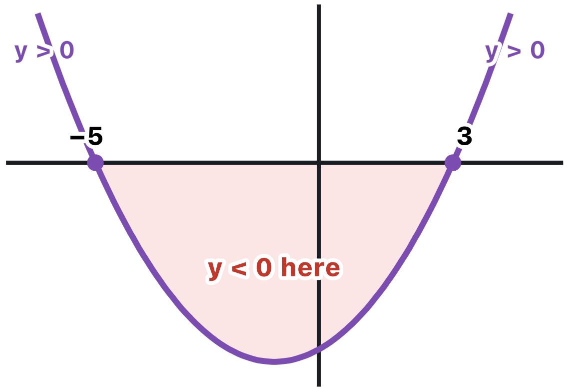
THE METHOD

1. **Rearrange** so one side is **zero**.
2. **Factorise** the quadratic.
3. Find the **critical values** (the roots).
4. **Sketch** the curve.
5. Read off where the curve is **above** or **below** the axis.

READ IT OFF THE SKETCH



Below the axis → between the roots. **Above** → outside them.

THE TWO SHAPES OF ANSWER

Inequality	Answer looks like
$(x+5)(x-3) < 0$	$-5 < x < 3$
$(x+5)(x-3) > 0$	$x < -5$ **or** $x > 3$

Less than gives one region. **Greater than** gives two.

WORKED EXAMPLE — REARRANGE FIRST

Solve $3x^2 \le 8x$.

$3x^2 - 8x \le 0$

$x(3x - 8) \le 0$

Critical values: $x = 0$ and $x = \frac{8}{3}$

U shape, so **below** the axis is between them

$0 \le x \le \frac{8}{3}$

Never **divide** both sides by x — you would lose the root x = 0.

WORKED EXAMPLE — TWO REGIONS

Solve $9x^2 \ge 4$.

$9x^2 - 4 \ge 0$

$(3x - 2)(3x + 2) \ge 0$

Critical values: $x = \frac{2}{3}$ and $x = -\frac{2}{3}$

Above the axis is outside the roots

$x \le -\frac{2}{3}$ or $x \ge \frac{2}{3}$

Do **not** just square root to get $x \ge \frac{2}{3}$ — that loses half the answer.

WORKED EXAMPLE — EXPAND FIRST

Solve $(5x + 4)(x + 4) \le 21$.

$5x^2 + 24x + 16 \le 21$

$5x^2 + 24x - 5 \le 0$

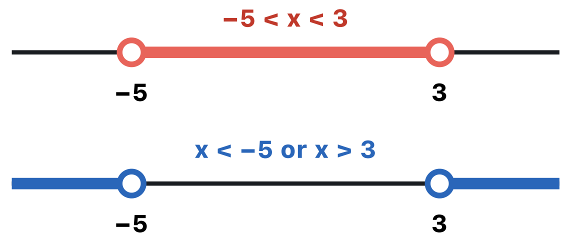
$(5x - 1)(x + 5) \le 0$

Critical values: $x = \frac{1}{5}$ and $x = -5$

$-5 \le x \le \frac{1}{5}$

Get **zero** on one side before factorising — never factorise across the 21.

SHOWING IT ON A NUMBER LINE



Open circle for < and >; **filled** circle for ≤ and ≥.

⚠ COMMON EXAM MISTAKES (recurring errors flagged in examiners' reports)

- ✕ Dividing by **x** and losing the root x = 0.
- ✕ Factorising before getting **zero** on one side.
- ✕ Giving only the **critical values** instead of a range.
- ✕ Writing $x > -5$ **and** $x < 3$ as one chain when the answer needs **or**.
- ✕ Square rooting $9x^2 \ge 4$ and losing the **negative** region.
- ✕ Not **sketching**, then picking the wrong side of the axis.
- ✕ Using < when the question says ≤ — the endpoints are included.
- ✕ Reversing the inequality after multiplying by a **negative**.

Revise it: solve $2x^2 > 3x$, then show the answer on a number line.

5 Real-Life Graphs

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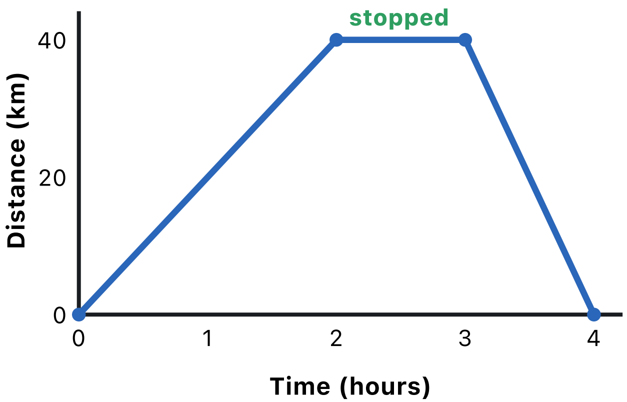
GRADIENT IS A RATE OF CHANGE

$$\text{gradient} = \frac{\text{change in y}}{\text{change in x}}$$

- On a **distance–time** graph the gradient is the **speed**.
- On a **speed–time** graph the gradient is the **acceleration**.
- Always give the **units** the axes imply.

DISTANCE–TIME GRAPHS



READING THE JOURNEY

0 to 2 h: 40 ÷ 2 = **20 km/h** away from home

2 to 3 h: flat, so **stopped**

3 to 4 h: 40 ÷ 1 = **40 km/h** back home

A line sloping **down** means returning, not a negative speed.

SPEED–TIME GRAPHS

WORKED EXAMPLE — ACCELERATION

Find the acceleration in the first 10 seconds.

Speed rises from 0 to 15 m/s

$a = \frac{15 - 0}{10 - 0}$

= **1.5 m/s^2**

The last section slopes **down**, so the acceleration there is **−1.5 m/s^2**.

WORKED EXAMPLE — TOTAL DISTANCE

Find the total distance travelled.

The shape is a **trapezium**

Area = $\frac{1}{2}(a + b)h$

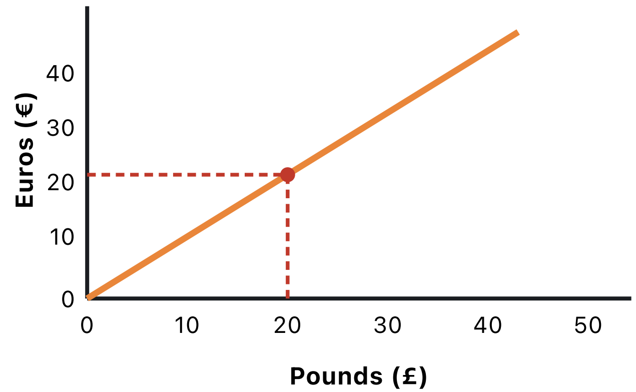
= $\frac{1}{2}(40 + 20) \times 15$

= **450 m**

CONVERSION GRAPHS

- A **straight line through the origin** — the two amounts are in direct proportion.
- Read **across then down**, or **up then across**.
- Draw **dashed lines** on the graph to show your reading.
- To convert a value **beyond** the graph, scale a reading up.

READING A CONVERSION GRAPH



£20 reads across to about **€23**. For £200, read £20 then **× 10**.

DESCRIBING A GRAPH IN WORDS

- **Steeper** line → faster rate of change.
- **Flat** line → nothing is changing.
- **Curved** line → the rate itself is changing.
- Say what is happening **in context**, not just “it goes up”.

⚠ COMMON EXAM MISTAKES (recurring errors flagged in examiners' reports)

- ✕ Reading a **downward** section of a distance–time graph as speeding up.
- ✕ Calling a flat section on a distance–time graph “constant speed” — it is **stopped**.
- ✕ Reading the **area** of a distance–time graph, which means nothing.
- ✕ Finding the gradient of a speed–time graph when the question wants the **distance**.
- ✕ Missing the **units**, especially m/s^2 for acceleration.
- ✕ Misreading a scale where one square is **not** one unit.
- ✕ Joining conversion-graph points with a **curve**.
- ✕ Describing a graph without ever mentioning the **context**.

Revise it: describe each section of a distance–time graph in words, with units.

6 Gradients & Areas of Graphs

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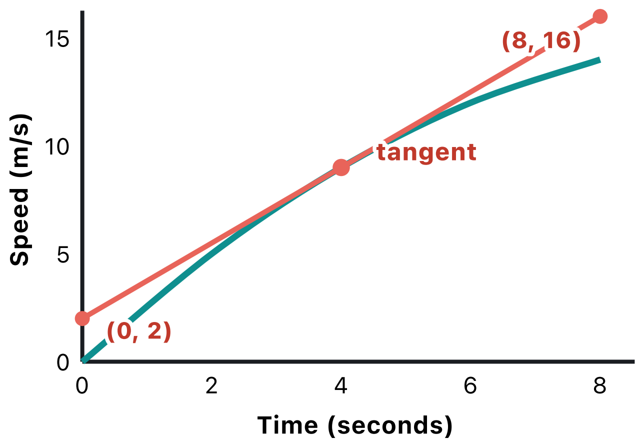
GRADIENT OF A CURVE

- A curve's gradient **changes** at every point.
- Draw a **tangent** — a straight line just touching the curve at that point.
- The gradient of the **tangent** is the gradient of the curve there.
- It is always an **estimate**, because the tangent is drawn by eye.

ESTIMATING THE GRADIENT AT T = 4



WORKED EXAMPLE — ACCELERATION AT T = 4

Draw the tangent at t = 4

It passes through (0, 2) and (8, 16)

gradient $= \frac{16 - 2}{8 - 0} = \frac{14}{8}$

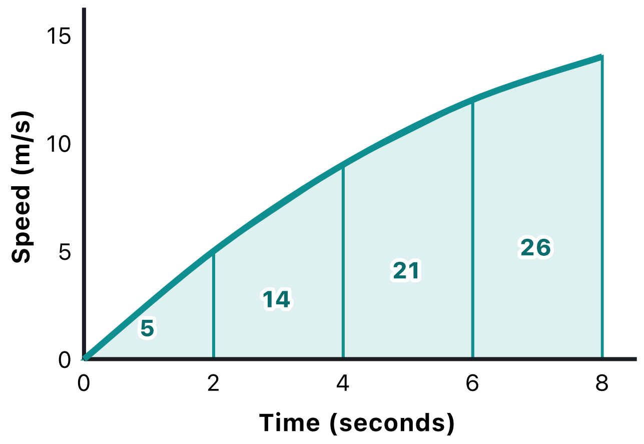
$=$ **1.75 m/s²**

Use **two points far apart** on the tangent to keep the reading accurate.

AREA UNDER A CURVE

- Split the region into **strips of equal width**.
- Each strip is a **trapezium**: area $= \frac{1}{2}(a + b)h$.
- **Add** the strip areas together.
- Under a speed–time graph the area is the **distance travelled**.

FOUR STRIPS OF EQUAL WIDTH



WORKED EXAMPLE — DISTANCE IN 8 SECONDS

Read the speeds: 0, 5, 9, 12, 14

$\frac{1}{2}(0 + 5) \times 2 = 5$

$\frac{1}{2}(5 + 9) \times 2 = 14$

$\frac{1}{2}(9 + 12) \times 2 = 21$

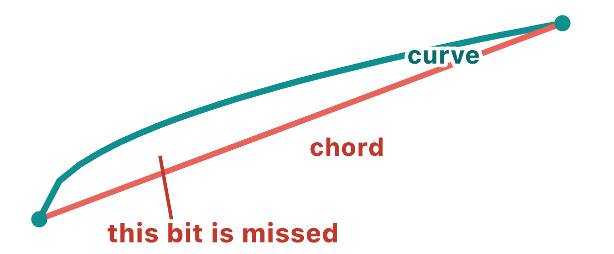
$\frac{1}{2}(12 + 14) \times 2 = 26$

Total = **66 m**

OVER- OR UNDERESTIMATE?

- Look at whether the **straight tops** sit above or below the curve.
- Curve bending **downwards** → tops sit **below** → **underestimate**.
- Curve bending **upwards** → tops sit **above** → **overestimate**.
- **More strips** always gives a better estimate.

WHY THIS ONE IS AN UNDERESTIMATE



The straight top cuts the corner, so the strips **miss** a sliver of area.

WHAT THE ANSWER MEANS

Graph	Gradient	Area
Distance–time	Speed	No meaning
Speed–time	Acceleration	Distance

Check the **axis labels** before you decide what to work out.

⚠ COMMON EXAM MISTAKES (recurring errors flagged in examiners' reports)

- ✕ Using two points on the **curve** instead of on the **tangent**.
- ✕ Drawing a **chord** and calling it a tangent.
- ✕ Picking two tangent points very **close together**, so a small slip ruins the gradient.
- ✕ Using strips of **unequal** width.
- ✕ Counting the **number** of strips as the width.
- ✕ Forgetting to **halve** in the trapezium area formula.
- ✕ Saying "overestimate" without looking at which way the curve **bends**.
- ✕ Giving the area of a **distance**–time graph as a distance.

Revise it: estimate the area under a curve with 4 strips, then say if it is an over- or underestimate.